

Classification of T^2 -bundles over T^2 (summary)

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§1. Notations and definitions

Given $A, B \in GL(2, \mathbb{Z})$ such that $AB = BA$, and $m, n \in \mathbb{Z}$, we construct a T^2 -bundle over T^2 denoted by

$$\pi : M(A, B; m, n) \rightarrow S,$$

as follows.

Denote by $\begin{bmatrix} x \\ y \end{bmatrix}$ the point of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ corresponding to $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Let $F = T^2$, $S = T^2$, and we define

$$M(A, B; 0, 0) = F \times \mathbb{R}^2 / \sim$$

where

$$\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{pmatrix} x+1 \\ y \end{pmatrix} \right) \sim \left(\begin{bmatrix} A \begin{pmatrix} s \\ t \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} \end{bmatrix} \right),$$

and

$$\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{pmatrix} x \\ y+1 \end{pmatrix} \right) \sim \left(\begin{bmatrix} B \begin{pmatrix} s \\ t \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} \end{bmatrix} \right).$$

Denote the point of $M_0 = M(A, B; 0, 0)$ which corresponds to

$$\left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right), \text{ by } \left[\begin{pmatrix} s \\ t \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right] \text{ or } \begin{bmatrix} s, x \\ t, y \end{bmatrix}.$$

Then $\pi : M_0 \rightarrow S$ is a T^2 -bundle over T^2 , where π is

defined by $\pi \begin{bmatrix} s, x \\ t, y \end{bmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. Let D be a small disk in S

centered at $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ with radius ε , and let

$$M(A,B;m,n) = (M_0 - \pi^{-1}(\text{Int } D)) \cup (F \times D)$$

where $F \times \partial D$ is attached to $\pi^{-1}(\partial D)$ by the homeomorphism

$$h : \pi^{-1}(\partial D) \rightarrow F \times \partial D:$$

$$h \left[\begin{pmatrix} s \\ t \end{pmatrix}, \varepsilon(\theta) \right] = \left(\left[\begin{pmatrix} s \\ t \end{pmatrix} + (\theta/2\pi) \begin{pmatrix} m \\ n \end{pmatrix} \right], \left[\varepsilon(\theta) \right] \right)$$

$$\text{where } \varepsilon(\theta) = \begin{pmatrix} 1/2 + \varepsilon \cos \theta \\ 1/2 + \varepsilon \sin \theta \end{pmatrix}.$$

Define the map $\pi : M(A,B;m,n) \rightarrow S$

$$\pi \begin{bmatrix} s, x \\ t, y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{if } \begin{bmatrix} x \\ y \end{bmatrix} \notin D, \quad \text{and}$$

$$\pi \left(\begin{bmatrix} s \\ t \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{if } \begin{bmatrix} x \\ y \end{bmatrix} \in D.$$

Then this is a T^2 -bundle over T^2 .

Every T^2 -bundle over T^2 is isomorphic to this form, where the pair (A,B) represents the monodromy, and the pair (m,n) represents the obstruction for constructing a cross-section.

Corresponding to a T^2 -bundle over T^2 , $\pi : M \rightarrow S$, there is an exact sequence

$$1 \rightarrow \pi_1 F \rightarrow \pi_1 M \rightarrow \pi_1 S \rightarrow 1$$

where F is a fiber. We call this the associated exact sequence.

§2. Fundamental lemmas.

Proposition 1. $H_1(M(A,B;m,n))$ is isomorphic to $\mathbb{Z}^2 \oplus (\mathbb{Z}^2/K)$, where K is the subgroup of \mathbb{Z}^2 generated by

$\begin{pmatrix} m \\ n \end{pmatrix}$ and the column vectors of $A-E$ and $B-E$ (E stands for $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$).

In the following proposition, we study about typical bundle isomorphisms.

Proposition 2. Let $A, B, A', B' \in GL(2, \mathbb{Z})$ such that $AB = BA$ and $A'B' = B'A'$. Let $\alpha, \beta, \sigma, \tau$ and $\alpha', \beta', \sigma', \tau'$ are canonical generators of $\pi_1 M$ and $\pi_1 M'$ respectively, where $M = M(A, B; m, n)$ and $M' = M(A', B'; m', n')$.

(1) Assume $A' = A^P B^r$, $B' = A^q B^s$ and $\begin{pmatrix} m' \\ n' \end{pmatrix} = \delta \begin{pmatrix} m \\ n \end{pmatrix}$ for some $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL(2, \mathbb{Z})$ where $\delta = ps - qr = \pm 1$.

Then there is a bundle isomorphism $f : M' \rightarrow M$ such that

$$f_{\#}(\sigma') = \sigma, \quad f_{\#}(\tau') = \tau \quad \text{and}$$

$$f_{\#}(\alpha') = \alpha^P \beta^r, \quad f_{\#}(\beta') = \alpha^q \beta^s$$

where $f : S' \rightarrow S$ is a corresponding homeomorphism between base spaces and $\alpha = \pi_{\#}(\alpha)$ etc.

(2) Assume $A' = P^{-1}AP$, $B' = P^{-1}BP$ and $\begin{pmatrix} m \\ n \end{pmatrix} = P \begin{pmatrix} m' \\ n' \end{pmatrix}$ for some $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in GL(2, \mathbb{Z})$. then there is a bundle isomorphism $f : M' \rightarrow M$ such that

$$f_{\#}(\alpha') = \alpha, \quad f_{\#}(\beta') = \beta \quad \text{and}$$

$$f_{\#}(\sigma') = \sigma^p \tau^r, \quad f_{\#}(\tau') = \sigma^q \tau^s.$$

(3) Assume $A' = A$, $B' = B$ and $\begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix}$
 $= (A-E) \begin{pmatrix} p \\ q \end{pmatrix} + (B-E) \begin{pmatrix} k \\ l \end{pmatrix}$ for some $p, q, k, l \in \mathbb{Z}$. Then there is
 a bundle isomorphism $f : M' \rightarrow M$ such that

$$f_{\#}(\alpha') = \sigma^{k'} \tau^{l'} \alpha, \quad f_{\#}(\beta') = \sigma^{p'} \tau^{q'} \beta \quad \text{and}$$

$$f_{\#}(\sigma') = \sigma, \quad f_{\#}(\tau') = \tau,$$

where $\begin{pmatrix} k' \\ l' \end{pmatrix} = B \begin{pmatrix} k \\ l \end{pmatrix}$ and $\begin{pmatrix} p' \\ q' \end{pmatrix} = A \begin{pmatrix} p \\ q \end{pmatrix}$.

Remark. The last result (3) of the above proposition corresponds to the fact that the obstruction class to constructing a cross section lies in $H^2(S, \widetilde{\pi}_1(F))$ ($\widetilde{\pi}_1(F)$ is the locally constant sheaf whose stalk at $x \in S$ is naturally isomorphic to $\pi_1 F_x$, where $F_x = \pi^{-1}(x)$), and that $H^2(S, \widetilde{\pi}_1(F))$ is isomorphic to the quotient group $\mathbb{Z}^2 / \langle A-E, B-E \rangle$, where $\langle A-E, B-E \rangle$ is the subgroup generated by the column vector of $A-E$ and $B-E$.

§3. Main results.

The problem of bundle isomorphisms is reduced to the group theory of the associated exact sequences by the following theorem.

Theorem 1. Let $\pi : M \rightarrow S$ and $\pi' : M' \rightarrow S'$ be T^2 -bundles over T^2 . Then the following statements are equivalent.

- 1) They are bundle isomorphic to each other.
- 2) The associated exact sequences of them are isomorphic to each other, that is, there exist isomorphism of groups $\psi : \pi_1 M' \rightarrow \pi_1 M$ and $\bar{\psi} : \pi_1 S' \rightarrow \pi_1 S$ such that $\pi_{\#} \circ \psi = \bar{\psi} \circ (\pi')_{\#}$.

Corollary. Two fibrations $M(A, B; m, n)$ and $M(A', B'; m', n')$ are isomorphic if and only if there exist $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and $P \in GL(2, \mathbb{Z})$ as follows:

$$APB^r = PA'P^{-1}, \quad A^q B^s = PB'P^{-1} \quad \text{and}$$

$$P \begin{pmatrix} m' \\ n' \end{pmatrix} - \begin{pmatrix} m \\ n \end{pmatrix} \in \langle A-E, B-E \rangle,$$

where $\langle A-E, B-E \rangle$ is the subgroup of \mathbb{Z}^2 generated by the column vectors of $A-E$ and $B-E$.

Theorem 2. Let $\pi : M \rightarrow S$ and $\pi' : M' \rightarrow S'$ be T^2 -bundles over T^2 .

- (1) $\text{rank}(H_1 M) = 4$ if and only if $M = M(E, E; 0, 0)$, which is a 4-dimensional torus.
- (2) Assume $\text{rank}(H_1 M) \leq 3$. Then the above fibrations are

isomorphic if and only if $\pi_1 M$ and $\pi_1 M'$ are isomorphic.

Any fibration has a simple expression as follows:

Theorem 3. Any T^2 -bundle over T^2 is isomorphic to one of the following types:

$$M(A, B; m, n) \text{ where } B = \pm E.$$

Furthermore, we may assume that A satisfies the following conditions:

- (1) if $\det A = -1$, $\text{trace } A \geq 0$,
- (2) if $\det A = 1$ and $B = -E$, $\text{trace } A \geq 2$ and $A = E$,

Remark. Under the above assumption (2), $B = E$ if and only if the subgroup of $GL(2, \mathbb{Z})$ generated by A and B is a cyclic group. The conjugacy class of this group in $GL(2, \mathbb{Z})$ is an invariant of the associated exact sequence. In fact, if $\rho : \pi_1 S \rightarrow \text{Aut}(\pi_1 F)$ is the homomorphism defined by

$$\rho(\pi_{\#}(x))(y) = x^{-1}yx \quad (x \in \pi_1 M, y \in \pi_1 F \subset \pi_1 M),$$

then $\text{Im } \rho$ is mapped onto the above group by a global

isomorphism from $\text{Aut}(\pi_1 F)$ to $\text{GL}(2, \mathbb{Z})$.

Theorem 4. Assume $M = M(A, B; m, n)$ and $M' = M(A', B'; m', n')$ satisfy the condition of Theorem 3. Denote by $\langle A-E \rangle$ the subgroup of \mathbb{Z}^2 generated by the vectors of $A-E$, and similarly for $\langle A-E, 2E \rangle$.

(0) If M and M' are bundle isomorphic to each other, then $B = B'$.

(1) Assume, $B = B' = E$. Then M is bundle isomorphic to M' , if and only if there exists a matrix $P \in \text{GL}(2, \mathbb{Z})$ such that

i) $PA'P^{-1} = A$ or $PA'P^{-1} = A^{-1}$ and

ii) $\begin{pmatrix} m \\ n \end{pmatrix} - P \begin{pmatrix} m' \\ n' \end{pmatrix} \in \langle A-E \rangle$.

(2) Assume, $B = B' = -E$. Then M is bundle isomorphic to M' , if and only if there exists a matrix $P \in \text{GL}(2, \mathbb{Z})$ such that

i) $PA'P^{-1} = \pm A$ or $PA'P^{-1} = \pm A^{-1}$ and

ii) $\begin{pmatrix} m \\ n \end{pmatrix} - P \begin{pmatrix} m' \\ n' \end{pmatrix} \in \langle A-E, 2E \rangle$.

§4. Homeomorphism types.

Let $\pi : M \rightarrow S$ be a T^2 -bundle over T^2 . If

$\text{rank}(H_1M) \neq 3$, the bundle isomorphism type is determined by π_1M (Theorem 2).

Now we consider the case when $\text{rank}(H_1M) = 3$. According to proposition 1, $\text{rank}(H_1(M(A,B;m,n))) = 3$ if and only if the rank of the 2×5 matrix $\begin{pmatrix} A-E & B-E & \frac{m}{n} \end{pmatrix}$ is equal to 1.

Hence in view of Theorem 3, M is isomorphic to one of the following forms:

- 1) $M\left(\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, E; m, 0\right)$ ($k \geq 0$)
- 2) $M\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E; 0, n\right)$ or
- 3) $M\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, E; m, -m\right)$.

Furthermore, we have:

Proposition 3. If $\text{rank}(H_1M) = 3$, M is homeomorphic to

one and only one of the following forms:

- 1) $M\left(\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, E; 0, 0\right)$ ($d > 0$)
- 2) $M\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E; 0, n\right)$ ($n = 0$ or 1)

Corollary. Let $\pi : M \rightarrow S$ and $\pi' : M' \rightarrow S'$ be T^2 -bundle over T^2 . Assume that M and M' are both orientable or both non-orientable, and $\text{rank } H_1M = \text{rank } H_1M' = 3$. Then M is homeomorphic to M' if and only if $H_1M \cong H_1M'$.

Remark. The orientability of M is an invariant of $\pi_1 M$. In fact, let $\rho : H_1 M \rightarrow \text{Aut}([\pi_1, \pi_1])$, where $[\pi_1, \pi_1]$ is the commutator subgroup of $\pi_1 M$ and ρ is the homomorphism which is defined similarly to the remark to Theorem 3. When $\text{rank } H_1 M = 3$, by the above proposition, we see that ρ is a trivial map if and only if M is orientable.

This remark and Theorem 2 imply:

Theorem 5. Let $\pi : M \rightarrow S$ and $\pi' : M' \rightarrow S'$ be T^2 -bundles over T^2 . Then M is homeomorphic to M' if and only if $\pi_1 M$ is isomorphic to $\pi_1 M'$.