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Partial Regularity and the Navier-Stokes equations

Robert V. Kohn (ロバート V. コーン)
Courant Institute of Mathematics
251 Mercer Street
New York, NY 10012

I wish to report on recent joint work with L. Nirenberg and L. Caffarelli [1], in which we prove

Theorem: The singular set of a "suitable weak solution" of the Navier-Stokes equations in three space dimensions has "parabolic one-dimensional measure zero" in space-time.

This theorem strengthens the results V. Scheffer [3-7]. The "suitable weak solutions" we study are like Leray-Hopf weak solutions, but they satisfy a generalization of the usual energy inequality: if $\phi > 0$ is C^{∞} and compactly supported in space time then

$$(1) \qquad 2\iint |\nabla u|^2 \phi < \iint |u|^2 (\phi_t + \Delta\phi) + \iint (|u|^2 + 2p) u \cdot \nabla\phi + 2\iint (u \cdot f)\phi,$$

where u is the velocity, p is pressure, and f is the external force:

$$(2)\qquad u_t + u\cdot\nabla u - \Delta u + \nabla p = f$$
$$\nabla\cdot u = 0 .$$

The singular set of u is

$$S = \{(x,t)\colon u \text{ is not } L^{\infty}_{loc} \text{ in any neighborhood of } (x,t)\}.$$

To say that "S has parabolic one-dimensional measure zero" means that for any $\varepsilon > 0$ there is a finite family of parabolic cylinders

$$Q_{r_i}(x_i,t_i) = \{(y,\tau)\colon |y-x_i| < r_i,\ |\tau-t_i| < r_i^{\,2}\}$$

satisfying $S \subseteq \bigcup_i Q_{r_i}(x_i,t_i)$ and $\Sigma r_i < \varepsilon$.

The proof of the theorem draws heavily from Scheffer's method in [4]. There are three main steps:

<u>Step 1</u>: <u>There is a minimum rate at which singularities can develop</u>. The precise statement of what we prove is somewhat technical, and I do not repeat it here. Heuristically, however, it says that if

$$R(r;x,t) = \frac{1}{\mathrm{vol}(Q_r)} \iint_{Q_r(x,t)} (|u|r)^3 \,dxdt$$

is small enough, then $|u|$ is bounded on $Q_{r/2}(x,t)$. Since we have set viscosity = 1 in (2), $R(r;x,t)$ is dimensionless; one should think of it as a _local_ _Reynolds_ _number_.

Step 2: $|\nabla u| \to \infty$ _as_ $1/r^2$ _near a singular point_. Step 1 suggests that as $r \to 0$

$$|u|(x,t) > C/r, \quad r = |x-x_0| + |t-t_0|^2$$

if (x_0,t_0) is a singular point. One expects, then, on dimensional grounds, that $|\nabla u|^2 > C/r^2$. We have proved the following estimate: _if_ $\limsup_{r \to 0} r^{-1} \iint_{Q_r(x,t)} |\nabla u|^2 \, dxdt$ _is small enough, then (x,t) is a regular point_.

Step 3. _S has parabolic one-dimensional measure zero_. This follows easily from step 2, using a Vitali-type covering lemma.

An expository discussion of this work will appear in [2]; the mathematical details are in [1].

References

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