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Fixed point semantics of logical formulae

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Abstract:

In this paper the power space of finite trees is studied, on which logical formulae are interpreted as functions and self application of functions is permitted. It is shown that truth value of a logical formula can be considered as a fixed point set of a retraction map determined by the functional interpretation of the formula. This leads also a functorial interpretation of logical formula on the category of retracts.

Fixed point theorem and the fact, that functorial interpretation preserves the logical conjunction, are proved.

1. Introduction

The final objective of this research is to set a broader framework of logic in which logic programming language such as "prolog" can be formally described. In the rescent research of programming language, the famous $P(\omega)$ model of Scott turned out to be sufficient as a model of sequential computational language. For "prolog" only Herbrandt model is known which is too weak to descriminate the semantic deference of the evaluation of Horn sentences. So a model of logic programming language must have both logical aspects and computational aspects described in Scott model.

In $P(\omega)$ model, functions are interpreted as subsets of natural numbers. However to see list processing functions and unification are continuous operations it is suitable to take finite tree space as a basic data domain. In the power space of finite trees with Scott topology, a set of finite trees represents a state of information. For example, the set P(a), P(f(b)),... represents a state where P(a), P(f(b)),... hold. A continuous function on this space is an increasing function in the sense that output information increases if input information increases.

In this paper a space including negative information is considerd. The top element of this space, T, can be given a proper meaning of contradiction in the logical sense. To give universality to this space it is necessary to show self-application is possible in this space. It is also desierable to show fixed point theorem.

Next theme is to reconstrunct the formal logic on this space. Truth values are considered to give objects, informally considered as a logical formula, logical value in its true sense.

It is exemplified that truth values should be identified with the fixed point sets of retraction maps which are greater than the identity function. The category of set valued contravariant functors obtained from the category of such retracts becomes a model of higher order intuitionistic logic. Under this model interpretation the logical formulae informally considered are given truth-values. The logical meaning of conjunction is also preserved.

2. Space E

In the resolution theorem proving of first order logic, Herbrand model plays an essential role. For a given formula $\forall x \exists y\ Q(x,y)$ of first order logic a skolem function g(x) is assigned such that

$$\forall x \exists y\ Q(x,y) \Rightarrow Q(x,f(x)).$$

holds.

This means

$$Q(a, f(a)),\quad Q(b, f(b)),\quad Q(f(a), f(f(a))), \ldots .$$

for constants a,b,

holds in the Herbrand model of $\forall x \exists y\, Q(x, y)$. Thus the interpretation of logical formula is finite trees which are compositions of predicate symbols, skolem function symbols and constants. So to identify logical formula with a set of finite trees may be a step to construct a general model of theorem proving.

Let $\mathfrak{F}$ be a set of function symbols satisfying the following conditions.

1) $c_n \in \mathfrak{F}$ for $n \geq 1$

2) $f_m^n(x_1, \cdots, x_n) \in \mathfrak{F}$ for $n, m \geq 1$

The set of finite trees, FT, is defined as minimum solution of the following recursive equation.

$$FT = \{c_n \mid n \geq 1\} \cup \{f_n^m(\alpha_1, \ldots, \alpha_m) \mid f_n^m \in \mathfrak{F}, \alpha_1, \ldots, \alpha_m \in FT\}$$

The space to be considered is not FT itself but, P(FT), the power set of FT because the element of the latter represents the state of imformation. To make P(FT) a self applicable domain it is required to introduce Scott topology into the domain and to restrict the functions to be considered to continuous functions. The correspondence between $P(\omega)$ and

P(FT) is clealy obtained by identifying a natural number with the corresponding finite tree. So the topology to be introduced in P(FT) becomes as follows.

[Definition 1]

Let α be a subset of FT then we define above(α) as

$$\text{above}(\alpha) = \{\beta \mid \beta \supset \alpha\}$$

The topology $\mathcal{I}$ of P(FT) then becomes

$$\mathcal{I} = \{\bigcup_{i \in G} \text{above}(\alpha_i) \mid \alpha_i \text{ is a finite subset of FT for any } i \in G \text{ and } G \text{ is an arbitrary index set}\}$$

Trivially P(FT) is a continuous lattice. To consider both negative and positive informations $P(FT)^2$ is considered. Further to identify states with contradicted information, function r defined as

$$r : P(FT)^2 \rightarrow P(FT)^2 ; r(\langle x,y \rangle) = \begin{cases} T & \text{if } x \cap y \neq \emptyset \\ \langle x, y \rangle & \text{else} \end{cases}$$

should be considered.

[Proposition 1]

r is a retraction map of P(FT) .

(Proof)

The relation $r = r \circ r$ trivialy holds. So it is only necessary to prove that r is continuous. To show the continuity of r it is only necessary to show

$$r(\sqcup D) = \bigsqcup_{b \in D} r(b)$$

for any directed set D.

In case

$$r(\sqcup D) \neq T \text{ and } r(\sqcup D) = \sqcup D.$$

For any $x \in D$

$$x = r(x) \sqsubseteq r(\sqcup D)$$

holds.

So

$$\sqcup D = \sqcup r(D) \sqsubseteq r(\sqcup D) = \sqcup D$$

This shows

$$r(\sqcup D) = \sqcup r(D).$$

In case

$$r(\sqcup D) = T.$$

and $\exists x \in D \quad r(x) = T$ holds

Then

$$r(\sqcup D) = \sqcup r(D) = T$$

is trivial.

In case

$$r(\sqcup D) = T$$

and $\forall x \in D \quad r(x) \neq T$ holds.

For certain α, β

$$\sqcup D = \langle \alpha, \beta \rangle \quad \text{and} \quad \alpha \cap \beta \neq \emptyset$$

So there exists $a \in \alpha \cap \beta$.

From

$$\langle \{a\}, \{a\} \rangle \sqsubseteq \langle \alpha, \beta \rangle = \sqcup D$$

there must exist $\langle \gamma, \delta \rangle \in D$ such that

$$\langle \{a\}, \{a\} \rangle \sqsubseteq \langle \gamma, \delta \rangle .$$

This shows

$$r(\langle \gamma, \delta \rangle) \sqsupseteq r(\langle \{a\}, \{a\} \rangle) \sqsupseteq T.$$

This contradicts to the assumption.

(Q.E.D)

From Scott[1], the image of r is a continuous lattice. So we define the space E as in the following.

[Definition 2]

$$E = r(P(FT)^2)$$

3. Graph model of E

In the previous section, an element of space E is logical information such as

$$P(a,b) \wedge \forall x Q(x) \wedge \neg R(a).$$

However another logical imformation such as

$$(P(a,b) \Rightarrow Q(a)) \wedge (\neg Q(x) \Rightarrow R(x))$$

should be considered.

In this section, it is shown that a formula such as

$$P(a,b) \Rightarrow Q(a)$$

can be interpreted as a continuous function on the space E and that it can be identified with the element of E by proving that E has a graph model.

[Proposition 2]

The function

$$\tilde{P}_1(f_1(x)) \wedge \cdots \wedge \tilde{P}_n(f_n(x)) \Rightarrow \tilde{Q}(g(x))$$

defined as

$$(\tilde{P}_1(f_1(x)) \wedge \cdots \wedge \tilde{P}_n(f_n(x)) \Rightarrow \tilde{Q}(g(x))(y)$$

$$Q \circ g \; ((P_1 \circ f)^{-1} \sqcap \cdots \sqcap (P_n \circ f_n)^{-1})(y)$$

where $\tilde{P}$ denotes P or $\neg$P

is a continuous function.

(Proof)

1) The function f from FT to FT is a distributive function on E.

So also is f^{-1}. This shows f and f^{-1} are continuous.

2) $\sqcap$ is known to be continuous.

3) The function $\neg$ defined as

$$\neg \langle x,y\rangle = \langle y,x\rangle$$

is trivially continuous.

The theorem is clear from these facts.

(Q.E.D)

[Definition 3]

Let β and α be as

$$\beta = \langle F,G\rangle, \qquad \alpha = \langle \bigsqcup_{n=1}^{l} F_n, \bigsqcup_{n=1}^{m} G_n \rangle$$

for F, G, Fn, Gn $\in$ FT.

Pair function $g(\beta,\alpha)$ on such pair (β,α) s is defined as follows.

$$g(\beta,\alpha) = \langle f_{l+m+1}(F, b_1(F_1), \cdots, b_l(F_l), c_1(G_1), \cdots, c_m(Gm)), f_{l+m+1}(G, b_1(F_1), \cdots, b_l(F_l), c_1(G_1), \cdots, c_m(Gm))\rangle$$

where $b_i = f^1_{2i}$, $c_i = f^1_{2i+1}$ for $1 \leq i \leq \max(l,m$

To assure the uniqueness of the function values, the order of arrangement of Fi , Gi must be determined.

Let $n(fij(\gamma_1, \ldots, \gamma_i))$ be as

$$n(fij(\gamma_1, \ldots, \gamma_i))$$

$$=n(fij)^{Pr(1)^{n(\gamma_1)} \cdots Pr(i)^{n(\gamma_i)}}$$

where $n(fij) = Pr(\frac{1}{2}(i+j)\cdot(i+j-1)-i+1)$.

Then the order is defined as

$$n(F_1) \leq n(F_2) \leq \cdots \leq n(F_\ell)$$

and $$n(G_1) \leq n(G_2) \leq \cdots \leq n(G_m)$$

Using this pair function g the graph of a continuous function of E can be defined as in the following.

[Definition 4]

Let $\beta = \langle F, G \rangle$ and $\alpha = \bigsqcup_{i=1}^{\ell} \langle Fi, Gi \rangle$.

Define the function χ_α^β as follows.

$$\chi_\alpha^\beta(x) = \begin{cases} \beta & \text{if } \alpha \sqsubseteq x \\ \perp & \text{else} \end{cases}$$

Then the graph of arbitray continuous function f is defined as

$$\lambda x f(x) = \bigsqcup_{\chi_\alpha^\beta \sqsubseteq f} g(\beta, \alpha)$$

[Proposition 3]

Define function γ as

$$\gamma(u) = \bigsqcup_{\lambda z \chi_{\alpha}^{\beta}(z) \sqsubseteq \gamma} \chi_{\alpha}^{\beta}(u)$$

for any $u \in E$.

Then

1) γ is continuous

2) $[\lambda z f(z)](u) = f(u)$

hold.

To assure the soundness of this graph interpretation, we will show the fixed point theorem.

[Proposition 4]

The paradoxical combinatior Y defined as

$$Y = \lambda u\ \lambda x\ u(x(x))(\lambda x u(x))$$

coincides with the fixed point operator fix defined as

$$fix(x) = \bigsqcup^{n} f^{n}(x)$$

in the graph model of E.

(Proof)

Let N be a function on $D = \{\langle F,G\rangle \mid F,\ G \in FT,\ F \neq G\}$ defined as

$N(\langle F,G\rangle)$ = the number of nodes in the tree F + the number of nodes in the tree G.

Then trivialy

$$N(g(\langle F,G\rangle, \langle \sqcup Fn, \sqcup Gn\rangle)) > N(Fn),\ N(Gn)$$

holds.

From Scott[3], this is a sufficient condition for the theorem.

(Q.E.D)

4. The logic of space E

The purpose of this section is to construct the model of higher order logic including the space E in which the logical formulae discussed somewhat informally in the previous sections acquires proper truth values.

Consider the set of invariant points of f, x f(x) x . This set is characterized as the image of a retraction map J(f) defined as

$$J(f)(x) = \bigsqcup_{n=0}^{\infty} f^{n}(x)$$

This retraction map J(f) further satisfies the relation that

$$J(f)(x) \sqsupseteq x.$$

If s E is an invariant point of

$$f = P_1(x) \wedge \cdots \wedge P_n(x) \Rightarrow Q(x)$$

then s satisfies

$$s \sqsupseteq P_1(x) \sqcup \cdots \sqcup P_n(x) \quad \Rightarrow \quad s \sqsupseteq Q(x).$$

This means f holds at s.

So it is natural to think that the truth value of f is the retract J(f). We shall assume that truth values are retracts of retraction maps which satisfy $r(x) \sqsupseteq x$ for any $x \in E$.

[Definition 5]

$\mathcal{C}_E$ is defined to be a category whose objects are retracts of retraction maps which satisfy $r(x) \sqsupseteq x$, and whose morphisms are inclusion maps.

[Proposition 5]

Category $\mathcal{C}_E$ has products as

$$r_1 \times r_2 = \bigsqcup (r_1 \sqcup r_2)^n .$$

(Proof)

$$\bigsqcup_{n=0}^{\infty} (r_1 \sqcup r_2)^n (x) = x$$

implies $x \sqsupseteq r_1(x) \sqsupseteq x$.

So

$$r_1 \times r_2 \to r_1 \quad \text{and} \quad r_1 \times r_2 \to r_2$$

Assume

$r \to r_1$ and $r \to r_2$

Then

$r \subset r_1 \cap r_2$

So

$r_1(x) = r_2(x) = x$

holds for any $x \in r$

This shows

$\bigsqcup_{n=0}^{\infty} (r_1 \sqcup r_2)^n (x) = x$ for any $x \in r$

That is

$r \to r_1 \times r_2$

and the diagram

$$\begin{tikzcd} & & r_1 \\ r \arrow[urr] \arrow[r] \arrow[drr] & r_1 \times r_2 \arrow[ur] \arrow[dr] & \\ & & r_2 \end{tikzcd}$$

commutes

(Q.E.D)

[Proposition 6]

In the category of set valued contravariant functors of $\mathcal{C}_E$, $\mathrm{Sets}^{\mathcal{C}_E^{op}}$

$\mathcal{C}_E(-, r_1) \times \mathcal{C}_E(-, r_2)$

$= \mathcal{C}_E(-, r_1 \times r_2)$

holds,

where $\mathcal{C}_E(-,r)$ is a representable functor of r.

(Proof)

The proof is easily obtained using the pulback property of the product $r_1 \times r_2$ along E.

(Q.E.D)

Thus products are preserved under this model extension. It is easy to see that the product in $\mathcal{C}_E$ corresponds to the logical conjunction. The product in Sets also corresponds to the logical conjunction of the higher order intuitionistic logic which has $\mathrm{Sets}^{\mathcal{C}_E^{op}}$ as a model. So logical meaning is also preserved under this model extention. Further logical formulae, $p(a)$, $\forall x\, p(x) \Rightarrow q(x)$, $\neg p(x)$ can be given logical meaning in the model $\mathrm{Sets}^{\mathcal{C}_E^{op}}$.

5. Application

The unit resolution is a complete proof procedure for the smallest model of Horn sentence. However the application of unit resolution is not restricted to Horn sentences, because it permits negative terms. Assume the following sentence

$$
\begin{aligned}
&\wedge\ \tilde{P}_1(f_{11}(x)) \wedge \cdots \wedge \tilde{P}_n(f_{1n}(x)) \Rightarrow Q_1(g_1(x)) \\
&\qquad \vdots \\
&\wedge\ \tilde{P}_1(f_{n1}(x)) \wedge \cdots \wedge \tilde{P}_n(f_{nn}(x)) \Rightarrow Q_n(g_n(x))
\end{aligned}
$$

is given. We define the function f as

$$f = \bigsqcup_{i=1}^{n} di$$

where di denotes a function determined by the i-th clause.

Then it is easy to see that unit resolution is a complete proof procedure for the minimum fixed point of f.

Reference

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