数理解析研究所講究録
第 503 巻 1983 年 72-83


# Atoms and Molecules on Riemannian Symmetric Spaces

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In this announcement we shall describe a relation between atoms and molecules on a non-compact Riemannian symmetric space G/K, and consider a multiplier operator on the atomic Hardy space. This is continuous a line of study in [7]. The details will appear elsewhere.

§1. Introduction. Before to state the aim, we shall recall some results on the theory of Hardy space $H^p(\mathbf{R})$ $(0<p\leq\infty)$ on one dimensional Euclidean space $\mathbf{R}$. The classical Hardy space is the space of analytic functions f on the upper half plane $\{(x,t);x\in\mathbf{R},\ t>0\}$ with finite $H^p$-norm:

$$\|f\|_{H^p}=\sup_{t>0}\left(\int_{-\infty}^{+\infty}|f(x,t)|^p dx\right)^{1/p}<\infty. \tag{1.1}$$

Moreover, taking the limit as $t\to+0$, this space is identified with the subspace of $\mathcal{S}'(\mathbf{R})$ consisting of boundary distributions f(x,0). In this definition the concept of "analytic functions" is necessary. However, new characterizations of $H^p(\mathbf{R})$ are recently obtained without using the concept of analytic functions. That is, D.L.Burkholder-R.F.Gundy-M.L.Silverstein and C.Fefferman-E.M.Stein showed that $H^p(\mathbf{R})$ is chracterized by the tangential maximal functions:

$$f^*(x)=\sup_{(y,t)\in\Gamma(x)}|f(y,t)|, \tag{1.2}$$

where $\Gamma(x)=\{(y,t);\ y\in\mathbf{R},\ t>0,\ |x-y|<t\}$. They obtained the following

Theorem A ([1],[5]). $c_p\|f\|_{H^p}\le\|f^*\|_{L^p}\le C_p\|f\|_{H^p}$.

Moreover, R.Coifman showed that $H^p(\mathbf{R})$ $(0<p\le 1)$ can be characterized in terms of "atoms". Let $(p,q,s)$ be a triplet such that $0<p\le 1$, $1<q\le\infty$ and $s\in\mathbf{N}$, $s\ge[1/p-1]$. Then a $(p,q,s)$-atom is a mesurable function on $\mathbf{R}$ such that the support is contained in an interval $I$ and satisfies the following two conditions:

$$\begin{aligned}&\text{(i)}\quad \|f\|_q\le|I|^{1/q-1/p}\\ &\text{(ii)}\quad \int_{\mathbf{R}} f(t)t^k dt=0\ \ (0\le k\le s).\end{aligned} \tag{1.3}$$

Then the atomic Hardy space $H^p_{q,s}(\mathbf{R})$ is the space consisting of distributions of the form

$$f=\sum_{i=1}^{\infty}\lambda_i f_i, \tag{1.4}$$

where $f_i$'s are $(p,q,s)$-atoms and $\lambda_i\ge 0$, $\Sigma\lambda_i^p<\infty$. He obtained

Theorem B ([2]). $H^p(\mathbf{R})=H^p_{q,s}(\mathbf{R})$ and $c_p\|f\|^p_{H^p}\le\rho^p_{q,s}(f)\le C_p\|f\|^p_{H^p}$, where $\rho^p_{q,s}(f)$ is defined by the infimum of $\Sigma\lambda_i^p$ being taken over all decompositions (1.4).

Here let us define molecules corresponding to atoms. For a quartet $(p,q,s,\varepsilon)$ such that $(p,q,s)$ is as above and $\varepsilon>\max(s, 1/p-1)$, we put $a=1-1/p+\varepsilon$ and $b=1-1/q+\varepsilon$. Then a $(p,q,s,\varepsilon)$-

<u>molecule centerd at</u> $x_0$ is a function f on **R** such that f, $f|x|^b$ belong to $L^q(\mathbf{R})$ and satisfies the following two conditions:

$$\begin{aligned} &\text{(i)} \quad \|f\|_q^{a/b}\,\|f|x-x_0|^b\|_q^{1-a/b}=M(f) < \infty, \\ &\text{(ii)} \quad \int_{\mathbf{R}} f(x)x^k dx=0 \quad (0\le k\le s). \end{aligned} \tag{1.5}$$

Then M.H.Taibleson-G.Weiss showed the following

<u>Theorem C</u> ([10]).

(i) If f is a (p,q,s)-atom, then f is a (p,q,s,$\varepsilon$)-molecule for all $\varepsilon>0$ and $M(f)\le C$, where C is independent of the atom.

(ii) If f is a (p,q,s,$\varepsilon$)-molecule, then $f\varepsilon H^p_{q,s}(\mathbf{R})$ and $\rho^p_{q,s}(f)\le C'M(f)$, where C' is independent of the molecule.

By many people, these concepts: maximal functions, atoms and molecules on **R** were extended to $\mathbf{R}^n$ and moreover, to the general setting of spaces of <u>homogeneous type</u> (cf. [3],[6],[8]). But, our aim in this note is to extend these concepts to non-compact symmetric spaces G/K, which are <u>not</u> of homogeneous type. In §2, we shall give some notations about G, and in §3, define "radial maximal functions" and "atoms" on G/K and obtain a relation between them. In §4, we shall introduce "molecules" on G/K and obtain a theorem corresponding to Theorem C in **R**. Next we shall constract an atomic Hardy space by using the K-biinvariant, (p,q,s)-atoms on G centered at the unit element of G, and in §5, give a slightly simple characterization of this spce. In last §6, we shall consider convolution (or multiplier) operators on it.

We are grateful to Prof. H.Miyazaki for the many helpful discutions.

§2. Notations. Let G be a connected, real rank one semisimple Lie group with finite center, G=KAN an Iwasawa decomposition of G and $\underline{g}=\underline{k}+\underline{a}+\underline{n}$ the corresponding decomposition of the Lie algebra $\underline{g}$ of G. For any real vector space V let $V_c$ and $V^*$ denote the complexification and the dual space of V respectively. Let $\alpha$ be a reduced simple root of $(\underline{g}_c,\underline{a}_c)$ and $H_0$ the element of $\underline{a}$ such that $\alpha(H_0)=1$. In the following we identify A (resp. $\underline{a}_c^*$) with $\mathbf{R}$ by $a_t=\exp(tH_0)\leftrightarrow\alpha(\log(a_t))=t$ (resp. $\lambda\leftrightarrow\lambda(H_0)$) and moreover, by using the Cartan decomposition $G=KCL(A^+)K$ of G, we identify each K-biinvariant function f on G with the even function on $\mathbf{R}$ defined by, which we denote by the same letter, $f(t(x))=f(a_{t(x)})=f(x)$ for $x=k_1a_{t(x)}k_2\varepsilon KCL(A^+)K$. Let $m_1$ and $m_2$ denote the multiplicities of the root $\alpha$ and $2\alpha$ respectively and put $\rho=(m_1+2m_2)/2$. Then for any K-biinvariant functions f on G with compact support its integral on G is given by the integral on $\mathbf{R}^+$ with weight $\Delta(t)=(\mathrm{sh}t)^{m_1}(\mathrm{sh}2t)^{m_2}$:

$$\int_G f(x)\,dx=\int_0^\infty f(t)\Delta(t)\,dt. \tag{2.1}$$

Let B(r,x) denote the open ball with radius r and centerd at x and $|B(r,x)|$ the volume of it, i.e. $B(r,x)=\{y\varepsilon G;\ \sigma(x^{-1}y)<r\}$, where $\sigma(x)$ is the Riemannian distance between x and the unit element e of G, and $|B(r,x)|=\int_{B(r,x)}1dg=\int_0^r\Delta(t)\,dt$. For simplicity we put B(r)=B(r,e). Then the order of $|B(r)|$ with respect to r is given by

$$B(r)=\begin{cases} O(e^{2\rho r}) & (r\to\infty)\\ O(r^{m_1+m_2+1}) & (r\to 0).\end{cases} \tag{2.2}$$

This property means that G is <u>not of homogeneous type</u> in the sence of [3].

§3. Maximal functions and atoms on G/K. First we shall define maximal functions on G/K (see [7,§3]). Let $\phi$ be a K-biinvariant function on G with finite $L^1$-norm. Then for a positive number $\varepsilon>0$, we put

$$\phi_\varepsilon(x)=\frac{1}{\varepsilon}\frac{\Delta(t(x)/\varepsilon)}{\Delta(t(x))}\phi(t(x)/\varepsilon). \tag{3.1}$$

Now for any locally integrable functions f on G/K, we define the radial maximal function $M_\phi f$ of f as follows.

$$M_\phi f(x)=\sup_{\varepsilon>0}|f*\phi_\varepsilon(x)|, \tag{3.2}$$

where $*$ is the convolution on G. Then the following theorem is valid (see [7, Theroem 3.3]).

Theorem 3.1. If there exist a constant C and a positive number $\delta>0$ such that $|\phi(x)|\leq Ce^{-2\rho\sigma(x)/\delta}$ $(x\varepsilon G)$, the operator $M_\phi$ is of type $(L^p,L^p)$ $(1<p\leq\infty)$ and of weak type $(L^1,L^1)$.

Next we shall define atoms on G/K (see [7,§4]). Let (p,q,s) be a triplet such that $0<p\leq 1$, $2(\alpha+1)/3<q\leq\infty$ and $s\varepsilon\mathbf{N}$, $s\geq[2(\alpha+1)/(1/p-1)]$, where $\alpha$ and $\beta$ are defined by $m_1=2(\alpha-\beta)$ and $m_2=2\beta$. Then we say that a function f on G/K is a (p,q,s)-atom centered at x if the support is contained in an open ball B(r,x) and satisfies the following conditions:

$$\begin{aligned}&\text{(i)}\quad \|f\|_q\leq|B(r)|^{1/q-1/p},\\&\text{(ii)}\quad \text{if } r<r_p=(\alpha+1)p/\rho(1-p),\text{ then}\\&\qquad\int_0^\infty f_{x,K}(t)t^k\Delta(t)dt=0\ (0\leq k\leq s),\end{aligned} \tag{3.3}$$

where $f_{x,K}$ is the K-biinvariant function on G defined by $f_{x,K}(g)=$

$\int_K f(xkg)dk$. Of course, if we put $\alpha=\beta=-1/2$, i.e., $\Delta=1$ and $\rho=0$, this definition of atoms on G/K coincides with one on $\mathbb{R}$. If f satisfies the condition (ii) of (3.3), we say that f has <u>vanishing monents</u>. Here we define the <u>modified</u> radial maximal function $M'_\phi f$ for $f\in L^q(G/K)$ $(1\leq q\leq\infty)$ as follows.

$$M'_\phi f(x)=\sup_{0<\varepsilon<\varepsilon_p}|f*\phi_\varepsilon(x)|, \tag{3.4}$$

where $\varepsilon_p=(1-1/\delta)/(1-1/p)$ if $|\phi(t)|\leq Ce^{-2\rho|t|/\delta}$. Then the following theorem was obtained in [7,Theroem 4.1].

<u>Theroem 3.2</u>. *Let* $G\neq SL(2,\mathbf{R})$ *and* (p,q,s) *be as above. If there exist a constant* C, $0<\delta<1$ *and* $\lambda>1/p$ $(0<p<1)$ *such that* $|((\frac{d}{dt})^\ell\phi(t))(1+|t|)^\ell|\leq Ce^{-2\rho|t|/\delta}(1+|t|)^{-\lambda}$ *for all* $0\leq\ell\leq s+1$, *then there exists a constant* $c=c(C,p,q,s,\delta,\lambda)$ *such that* $\|M'_\phi f_{x,K}\|_p\leq c$ *for all* (p,q,s)*-atoms* f *on* G/K.

<u>§4. Molecules on G/K</u>. In the following we shall restrict our attension to K-biinvariant functions on G. Then the natural extension to G/K of the definition of molecules centered at 0 in **R** is given as follows. Let $(p,q,s,\varepsilon)$ be a quartet such that (p,q,s) is as above , $\varepsilon>1/p-1$ and put $a=1-1/p+\varepsilon$, $b=1-1/q+\varepsilon$. Let B(x) denote the K-biinvariant function on G defined by $B(x)=|B(\sigma(x))|$. Then we say that a function f is a K-biinvariant, $(p,q,s,\varepsilon)$-<u>molecule centered at</u> e if it satisfies the following two conditions:

$$\text{(i)}\quad \|f\|_q^{a/b}\|fB^b\|_q^{1-a/b}=M(f)<\infty,$$
$$\text{(ii)}\quad \int_0^\infty f(t)t^k\Delta(t)dt=0\ (0\leq k\leq s) \tag{4.1}$$
$$\text{or}\ \|f\|_q\leq|B(r_p)|^{a-b}.$$

Of course, if we put $\alpha=\beta=-1/2$, this definition coincides with one of $(p,q,s,\varepsilon)$-molecules centered at 0 in **R**. Then we can obtain the following

Theorem 4.1.

(i) *If* $f$ *is a K-biinvariant,* $(p,q,s)$*-atom centered at* $e$*, then* $f$ *is a K-biinvariant, molecule centered at* $e$ *for all* $\varepsilon>0$ *and* $M(f)\leq C$*, where* $C$ *is independent of the atom.*

(ii) *If* $f$ *is a K-biinvariant,* $(p,q,s,\varepsilon)$*-molecule centered at* $e$ *with vanishing moments, then* $f$ *has an atomic decomposition* $f=\Sigma\lambda_i f_i$ *such that* $f_i$*'s are K-biinvariant,* $(p,q,s)$*-atoms centered at* $e$ *with vanishing moments and* $\lambda_i\geq 0$, $(\Sigma\lambda_i^p)^{1/p}\leq C' M(f)(1+N(f))^s$, *where* $C'$ *is independent of the molecule and* $N(f)$ *is defined by* $\|f\|_q=M(f)|B(N(f))|^{a-b}$.

Sketch of the proof: As in **R**, (i) is obvious from the definition. To prove (ii), without loss of generality, we may assume that $M(f)=1$. We define the number $N=N(f)$ by $\|f\|_q=|B(N)|^{a-b}$ and $k_0$ by the smallest integer such that $2^{k_0}N\geq 1$. Then we put

$$(4.2)\qquad G=\bigcup_{k=0}^{\infty} B_k,\qquad B_k=\begin{cases} B(0,N) & (k=0)\\ B(2^{k-1}N,2^kN) & (0<k\leq k_0)\\ B(N_0+k-k_0-1,N_0+k-k_0) & (k_0<k),\end{cases}$$

where $B(r,r')=B(r)^c\cap B(r')$ and $N_0=2^{k_0}N$. Let $f_k$ denote the restriction of $f$ to $B_k$. Obviously, $f=\Sigma f_k$ and $f_k$'s are K-biinvariant functions on G. To obtain the desired decomposition, we modify this to the desired one as in **R** (see [10,Theorem 2.9]). In this step we use the following lemma.

Lemma 4.2. For each k, there exist K-biinvariant functions $h_k^i$ $(0\leq i\leq s)$ satisfying the following conditions:

(i) $\mathrm{supp}(h_k^i) \subset B_k$ $(0\leq i\leq s)$,

(ii) $\int_0^\infty h_k^i(t)t^j\Delta(t)dt=\delta_{ij}$ $(0\leq i,j\leq s)$,

(iii)

$$\|h_k^i\|_\infty \leq C \begin{cases} N^{-(i+2\alpha+2)} & (k=0,\ N<1) \\ N^{s-i}|B(N)|^{-1} & (k=0,\ N\geq 1) \\ (2^{k-1}N)^{-(i+2\alpha+2)} & (0<k\leq k_0) \\ N_0^{s-i}(k-k_0)^{s-i}|B(N_0+k-k_0+1)|^{-1} & (k_0<k). \end{cases}$$

Remark 1. If we use the decomposition of G such that $G=\bigcup_{k=0}^\infty B_k'$, $B_k'=B(2^{k-1}N,2^kN)$ instead of (4.2) (this corresponds to the case of **R**), we have an atomic decomposition $f=\Sigma\lambda_i f_i$ such that $(\Sigma\lambda_i^p)^{1/p}\leq C'M(f)e^{2\rho N(f)}$.

Remark 2. When f is a K-biinvariant $(p,q,s,\varepsilon)$-molecule $(0<p<1)$ which satisfies the latter condition of (ii) in (4.1), the similar result is valid. In this case f has an atomic decomposition consisting of atoms which satisfy (i) in (3.3) only.

§5. Atomic Hardy space on G/K. Let (p,q,s) be as above. Now let $L_+^p=L_+^p(G//K)$ denote the space of all K-biinvariant functions f on G having a non-increasing, K-biinvariant function $f^+\varepsilon L^p(G)$ such that $|f|\leq f^+$. We call such a $f^+$ the $L^p$ non-increasing dominator ($L^p$ n.i.d.). In this section we shall consider the following three spaces:

$$^\circ L_+^p=\{f\varepsilon L_+^p;\ f \text{ has a } L^p \text{ n.i.d. } f^+ \text{ such that } |B(r)|^{-1}\int_{B(r)_c} f(x)dx \leq f^+(r)\}. \quad (5.1)$$

$H^p=\{f\in L^1_{loc}(G//K);\ M'_\phi f\in L^p_+$ for all $\phi$ satisfying the condition in Theorem 3.2$\}$.

$H^p_{q,s}=\{f=\Sigma\lambda_i f_i;$ all $f_i$'s are K-biinvariant, (p,q,s)-atoms centered at e, $\lambda_i\geq 0$ and $\Sigma\lambda_i^p<\infty\}$.

Then we put $\rho^p_+(f)=\inf_{f^+}\|f^+\|^p_p$ for $f\in{}^\circ L^p_+$, where the infimum being taken over all $L^p$ n.i.d. $f^+$ of f satisfying (5.1), $\rho^p(f)=\sup_\phi\inf\|(M'f)^+\|^p_p$ for $f\in H^p$, where the supremum (resp. the infimum) being taken over all $\phi$ satisfying the condition in Theorem 3.2. with C=1 (resp. all $L^p$ n.i.d. of M'f) and $\rho^p_{q,s}(f)=\inf\Sigma\lambda_i^p$ for f $\in H^p_{q,s}$, where the infimum being taken over all K-biinvariant (p,q,s) -atomic decompositions of f. Obviously, $H^p_{q,s}\subset H^p_{q',s}$ $(q\geq q')$. The following proposition was obtained in [7,Proposition 5.1].

<u>Proposition 5.1</u>. $H^p_{q,s}\subset H^p$.

Moreover we can prove

<u>Theorem 5.2</u>. $H^p_{\infty,0}={}^\circ L^p_+$ <u>and</u> $\rho^p_{\infty,0}\sim\rho^p_+$.

Sketch of the proof: Let f be in $H^p_{\infty,0}$. Then f has an atomic decomposition $f=\Sigma\lambda_i f_i$ such that all $f_i$'s are K-biinvariant, $(p,\infty,0)$ -atoms centered at e. That is, $supp(f_i)\subset B(r_i)$ and $\|f_i\|_\infty\leq |B(r_i)|^{-1/p}$. Therefore, $|f|\leq\Sigma\lambda_i|f_i|\leq\sum_{\sigma(x)<r_i}\lambda_i|B(r_i)|^{-1/p}$. Here we define $f^+$ by the right hand side. Then we can show that $f^+$ is a $L^p$ n.i.d. of f satisfying the condition (5.1). To prove the converse, we use Theorem 4.1 (ii) and the similar argument in the proof of the theorem.

<u>Corollary 5.3</u>. $H^p_{\infty,0}$ <u>is complete</u>.

Conjecture. $H^p_{\infty,s} = {}^\circ L^p_+ = H^p$.

Remark. As in $\mathbf{R}$, if the integral: $|B(r)|^{-1} \int_{B(r)_c} f(x)dx$ can be expressed suitably in terms of the convolutions on G and be bounded by the maximal functions of f, this conjecture is valid.

§6. Multiplier operators on $H^p_{q,s}$. In this section we shall consider convolution (or multiplier) operators on $H^p_{q,s}$. First, as in $\mathbf{R}$, we see that

Proposition 6.1. If a linear operator T maps each K-biinvariant, (p,q,s)-atoms centered at e into a K-biinvariant, (p,q,s)-molecule T(f) centered at e and $M(T(f)) \leq C$, where C is independent of the atom f, then T is a bounded operator on $H^p_{q,s}$.

By using this proposition we can obtain the following results. For a K-biinvariant function f on G (resp. an even function $\mu$ on $\underline{a}^*$) with a suitable condition, the Spherical Fourier transform $\hat{f}$ of f (resp. the inverse Fourier transform $\check{\mu}$ of $\mu$) is defined as follows (cf. [11,Chap.9.2]).

$$\hat{f}(\nu) = \int_G f(x)\phi_\nu(x)dx$$
$$\text{(resp. } \check{\mu}(x) = \int_{\underline{a}^*} \mu(\nu)\phi_\nu(x)|C(\nu)|^{-2}d\nu\text{)}.$$

Now we put $F(\xi) = \{\nu \in \underline{a}^*_c;\ |\mathrm{Im}(\xi)| < \xi\rho\}$. Then we have the following

Theorem 6.1. Suppose that $\mu$ is an even function on $\underline{a}^*$ such that $\mu$ is bounded and holomorphic on $F(\xi)$ $(\xi > 2/p-1)$ and $\mu(\nu)(1+|\nu|)^{1-[p]} C(-\nu)^{-1} \in L^1(\mathbf{R}+\sqrt{-1}\xi\rho)$. Then if the multiplier operator $T_\mu$, i.e., $T_\mu(f) = (\mu\hat{f})^\vee$, is of type $(L^\infty, L^\infty)$, $T_\mu$ is also of type $(H^p_{\infty,0}, H^p_{\infty,0})$ for $2\alpha+2/2\alpha+3 \leq p \leq 1$.

Moreover, using this theorem and Corollary 5.3, we can obtain

Corollary 6.3. *Suppose that* $m$ *is a K-biinvariant function on* $G$ *with finite* $L^1$*-norm and* $\hat{m}(\nu)\ C(-\nu)^{-1} \varepsilon L^1(\mathbf{R}+\sqrt{-1}\rho)$. *Then the convolution operator* $T_m$, i.e., $T_m(f)=m*f$, *is of type* $(H^1_{\infty,0}, H^1_{\infty,0})$.

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