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Asymptotic efficiency of $\{C_n\}$-consistent estimators

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$x_1, x_2, \cdots, x_n$ は，密度関数 $f(x,\theta)$ をもつ母集団からの任意標本とする ($\theta \in \Theta$)。

$f(x,\theta)$ は次の (i)〜(v) を満たす.

(i) $\{x \mid f(x,\theta)\}$ は θ と独立である

(ii) $f(x,\theta)$ は θ に関して 4 回連続偏微分可能で，その偏導関数は連続である.

(iii)

$$E_\theta(|\log f(x,\theta)|) < \infty$$

$$0 < I(\theta) = -E_\theta\left(\frac{\partial^2}{\partial\theta^2}\log f(x,\theta)\right)$$

(iv) 任意の $\theta \in \Theta$ に対して，θ の開近傍 Θ_0 が存在して，

$$\left|\frac{\partial^i}{\partial\theta^i}\log f(x,\theta)\right| < G(x) \quad (i = 1, 2, 3),$$

$$\left|\frac{\partial^i}{\partial\theta^i}\log f(x,\theta)\right| < H(x) \quad (i = 4) \quad (^\forall\theta \in \Theta_0)$$

かつ

$$E_\theta\left[\{G(x)\}^4\right] < \infty, \qquad E_\theta\{H(x)\} < \infty$$

(V) 最尤推定量 $\hat{\theta}_{ML}$ は $o(n^{-\frac{1}{2}})$ まで Θ_0 の中で一様に Edgeworth 展開可能である.

order $\{c_n\}$ の一致推定量 $\hat{\theta}_n$ が, 任意の θ の近傍 Θ_0 で一様に

$$\lim_{n\to\infty} c_n^{k-1} \left| P_\theta(\hat{\theta}_n \leqq \theta) - g(c_n^{-1}, \theta) \right|$$

$$= \lim_{n\to\infty} c_n^{k-1} \left| P_\theta(\hat{\theta}_n \geqq \theta) - 1 + g(c_n^{-1}, \theta) \right| = 0$$

を満たすとき, $\hat{\theta}_n$ は k-th order asymptotically $g(c_n^{-1}, \theta) - \frac{1}{2}$ biased estimator であるといい, そのような推定量の全体を $C(g(c_n^{-1}, \theta), k)$ で表わすことにする.

$\hat{\theta}_n \in C(g(c_n^{-1}, \theta), k)$ が, 任意の $a, b\ (>0)$ および任意の θ に対して,

$$\lim_{n\to\infty} c_n^{k-1} \{ P_\theta(-a \leqq c_n(\hat{\theta}_n - \theta) \leqq b) - \max_{\hat{\theta}_n \in C(g(c_n^{-1}, \theta), k)} P_\theta(-a \leqq c_n(\hat{\theta}_n - \theta) \leqq b) \}$$
$$\geqq 0$$

を満たすとき, $\hat{\theta}_n$ は k-th order asymptotically efficient in $C(g(c_n^{-1}, \theta), k)$ であるという. $g(c, \theta)$ は $c\ (|c| \leqq c_0)$, θ に関して偏微分可能で, その偏導関数は連続とする.

$\hat{\theta}_n \in C(g(n^{-\frac{1}{2}}, \theta), 2)$ ならば, 最強力検定の方法により任意の $t > 0$ に対して,

$$P_\theta(\sqrt{n}(\hat{\theta}_n - \theta) \leqq t) \leqq P_\theta(T_n \geqq a_n)$$

ただし

$$T_n = \sum \log \frac{f(x_i, \theta)}{f(x_i, \theta + \frac{t}{\sqrt{n}})} ,$$

a_nは，十分大きいnに対して，

$$P_{\theta + \frac{t}{\sqrt{n}}}(T_n \geqq a_n) = g(n^{-\frac{1}{2}}, \theta + \frac{t}{\sqrt{n}}) + o(n^{-\frac{1}{2}})$$

を満たす．

一方(iv)により，T_nは$o(n^{-\frac{1}{2}})$まで，θに関して局所一様にEdgeworth展開可能であるから，

$$E_{\theta + \frac{t}{\sqrt{n}}}(T_n) = -\frac{t^2}{2} I(\theta) - \frac{t^3}{6\sqrt{n}}(3J(\theta) + 2K(\theta)) + o(n^{-\frac{1}{2}})$$

$$V_{\theta + \frac{t}{\sqrt{n}}}(T_n) = I(\theta)t^2 + \frac{t^3}{\sqrt{n}}(J(\theta) + K(\theta)) + o(n^{-\frac{1}{2}})$$

$$E_{\theta + \frac{t}{\sqrt{n}}}\left\{(T_n - E_{\theta + \frac{t}{\sqrt{n}}}(T_n))^3\right\} = -\frac{t^3}{\sqrt{n}} K(\theta) + o(n^{-\frac{1}{2}})$$

ただし

$$J(\theta) = E_\theta\left(\frac{\partial^2}{\partial\theta^2}\log f(x, \theta) \cdot \frac{\partial}{\partial\theta}\log f(x, \theta)\right)$$

$$K(\theta) = E_\theta\left\{\left(\frac{\partial}{\partial\theta}\log f(x, \theta)\right)^3\right\}$$

を用いて

$$b_n = \frac{a_n + \frac{1}{2}I(\theta)t^2}{\sqrt{I(\theta)}\,t}$$

とおくとき

$$-b_n = u(g_{00}(\theta)) + \frac{1}{\sqrt{n}} \frac{g_{10}(\theta) + g_{01}(\theta)t}{\phi(u(g_{00}(\theta)))} + \frac{t^2}{\sqrt{n}} \frac{3J(\theta) + K(\theta)}{I(\theta)}$$

$$+ \frac{t}{2\sqrt{n}} \frac{J(\theta) + K(\theta)}{I(\theta)} u(g_{00}(\theta)) + \frac{1}{\sqrt{n}} \frac{K(\theta)}{6\sqrt{I(\theta)}\,I(\theta)}\left[\{u(g_{00}(\theta))\}^2 - 1\right]$$

$$+ o(n^{-\frac{1}{2}})$$

ただし, $g_{00}(\theta) = g(0, \theta)$, $g_{10}(\theta) = \frac{\partial}{\partial c} g(0, \theta)$, $g_{01}(\theta) = \frac{\partial}{\partial \theta} g(0, \theta)$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt, \quad \phi(x) = \Phi'(x), \quad u(g) = \Phi^{-1}(g)$$

従って

$$E_\theta(T_n) = \frac{t^2}{2} I(\theta) + \frac{t^3}{6\sqrt{n}} (3J(\theta) + K(\theta)) + o(n^{-\frac{1}{2}})$$

$$V_\theta(T_n) = t^2 I(\theta) + \frac{t^3}{\sqrt{n}} J(\theta) + o(n^{-\frac{1}{2}})$$

$$E_\theta[\{T_n - E_\theta(T_n)\}^3] = -\frac{t^3}{\sqrt{n}} K(\theta) + o(n^{-\frac{1}{2}})$$

より

$$P_\theta(T_n \geqq a_n) = 1 - P_\theta\left(\frac{T_n - \frac{1}{2}I(\theta)t^2}{\sqrt{I(\theta)}\,t} \leqq b_n - \sqrt{I(\theta)}\,t\right)$$

$$= \Phi(u(g_{00}(\theta)) + \sqrt{I(\theta)}\,t)$$

$$+ \frac{1}{\sqrt{n}} \phi(u(g_{00}(\theta)) + \sqrt{I(\theta)}\,t) \left\{ \frac{g_{10}(\theta) + t g_{01}(\theta)}{\phi(u(g_{00}(\theta)))} \right.$$

$$\left. + \frac{tK(\theta)}{6I(\theta)} u(g_{00}(\theta)) + \frac{t^2(3J(\theta) + 2K(\theta))}{6\sqrt{I(\theta)}} \right\} + o(n^{-\frac{1}{2}})$$

よって $t > 0$ のとき

$$P_\theta(\sqrt{n}(\hat{\theta}_n - \theta) \leqq t) \leqq \Phi(u(g_{00}(\theta)) + \sqrt{I(\theta)}\,t)$$

$$+ \frac{1}{\sqrt{n}} \phi(u(g_{00}(\theta)) + \sqrt{I(\theta)}\,t) \left\{ \frac{g_{10}(\theta) + t g_{01}(\theta)}{\phi(u(g_{00}(\theta)))} + \frac{tK(\theta)}{6I(\theta)} u(g_{00}(\theta)) \right.$$

$$\left. + \frac{t^2(3J(\theta) + K(\theta))}{6\sqrt{I(\theta)}} \right\} + o(n^{-\frac{1}{2}})$$

同様にして，$t>0$ のとき

$$P_\theta(\sqrt{n}(\hat{\theta}_n-\theta)\leqq -t)\geqq \Phi(u(g_{00}(\theta))-\sqrt{I(\theta)}\,t)$$

$$+\frac{1}{\sqrt{n}}\phi(u(g_{00}(\theta))-\sqrt{I(\theta)}\,t)\left\{\frac{g_{10}(\theta)-g_{01}(\theta)t}{\phi(u(g_{00}(\theta)))}-\frac{tK(\theta)}{6I(\theta)}u(g_{00}(\theta))\right.$$

$$\left.+\frac{t^2(3J(\theta)+2K(\theta))}{6\sqrt{I(\theta)}}\right\}+o(n^{-\frac{1}{2}})$$

<u>定理1</u>. θ の推定量 $\hat{\theta}_n$ が，任意の t に対して，

$$P_\theta(\sqrt{n}(\hat{\theta}_n-\theta)\leqq t)=\Phi(u(g_{00}(\theta))+\sqrt{I(\theta)}\,t)$$

$$+\frac{1}{\sqrt{n}}\phi(u(g_{00}(\theta))+\sqrt{I(\theta)}\,t)\left\{\frac{g_{10}(\theta)+g_{01}(\theta)t}{\phi(u(g_{00}(\theta)))}+\frac{tK(\theta)}{6I(\theta)}u(g_{00}(\theta))\right.$$

$$\left.+\frac{t^2(3J(\theta)+2K(\theta))}{6\sqrt{I(\theta)}}\right\}+o(n^{-\frac{1}{2}})$$

を満たすならば，$\hat{\theta}_n$ は，second order asymptotically efficient in $C(g(n^{-\frac{1}{2}},\theta),2)$

$\hat{\theta}_n$ が second order asymptotically efficient in $C(g(n^{-\frac{1}{2}},\theta),2)$ ならば，

$$Z_n=\sqrt{nI(\theta)}\left(\hat{\theta}_n+\frac{u(g_{00}(\theta))}{\sqrt{nI(\theta)}}-\theta\right)$$

とおくとき，

$$E_\theta(Z_n) = \frac{1}{\sqrt{n}}\left\{\frac{g_{01}(\theta)\ u(g_{00}(\theta))}{\phi(u(g_{00}(\theta)))\ \sqrt{I(\theta)}} - \frac{g_{10}(\theta)}{\phi(u(g_{00}(\theta)))}\right.$$

$$\left. - \frac{3J(\theta)+K(\theta)}{6I(\theta)\sqrt{I(\theta)}}\{u(g_{00}(\theta))\}^2 - \frac{3J(\theta)+2K(\theta)}{6I(\theta)\sqrt{I(\theta)}}\right\} + o(n^{-\frac{1}{2}})$$

$$V_\theta(Z_n) = 1 + \frac{2}{\sqrt{n}}\left\{-\frac{g_{01}(\theta)}{\phi(u(g_{00}(\theta)))\sqrt{I(\theta)}} + \frac{2J(\theta)+K(\theta)}{2I(\theta)\sqrt{I(\theta)}}u(g_{00}(\theta))\right\} + o(n^{-\frac{1}{2}})$$

$$E_\theta\left[\{Z_n - E_\theta(Z_n)\}^3\right] = -\frac{1}{\sqrt{n}}\frac{3J(\theta)+2K(\theta)}{I(\theta)\sqrt{I(\theta)}} + o(n^{-\frac{1}{2}})$$

最尤推定量 $\hat{\theta}_{ML}$ に対して

$$E_\theta\{\sqrt{nI(\theta)}(\hat{\theta}_{ML}-\theta)\} = -\frac{J(\theta)+K(\theta)}{2\sqrt{n}\ I(\theta)\sqrt{I(\theta)}} + o(n^{-\frac{1}{2}})$$

$$V_\theta\{\sqrt{nI(\theta)}(\hat{\theta}_{ML}-\theta)\} = 1 + o(n^{-\frac{1}{2}})$$

$$E_\theta\left[\{\sqrt{nI(\theta)}(\hat{\theta}_{ML}-\theta) - E_\theta(\sqrt{nI(\theta)}(\theta_{ML}-\theta))\}^3\right]$$

$$= -\frac{1}{\sqrt{n}}\frac{3J(\theta)+2K(\theta)}{I(\theta)\sqrt{I(\theta)}} + o(n^{-\frac{1}{2}})$$

従って

$$Z_n = \left\{1 + \frac{1}{\sqrt{n}}\left(-\frac{g_{01}(\theta)}{\phi(u(g_{00}(\theta)))\sqrt{I(\theta)}} + \frac{2J(\theta)+K(\theta)}{2I(\theta)\sqrt{I(\theta)}}u(g_{00}(\theta))\right)\right\}$$

$$\times\sqrt{nI(\theta)}(\hat{\theta}_{ML}-\theta) + \frac{1}{\sqrt{n}}\left\{\frac{g_{01}(\theta)}{\phi(u(g_{00}(\theta)))\sqrt{I(\theta)}}u(g_{00}(\theta))\right.$$

$$\left.-\frac{g_{10}(\theta)}{\phi(u(g_{00}(\theta)))} - \frac{3J(\theta)+K(\theta)}{6I(\theta)\sqrt{I(\theta)}}\{u(g_{00}(\theta))\}^2 + \frac{K(\theta)}{6I(\theta)\sqrt{I(\theta)}}\right\} + o_p(n^{-\frac{1}{2}})$$

よって

$$\hat{\theta}_n = \hat{\theta}_{ML} + \frac{1}{\sqrt{n}}\left(-\frac{g_{01}(\theta)}{\phi(u(g_{00}(\theta)))\sqrt{I(\theta)}} + \frac{2J(\theta)+K(\theta)}{2I(\theta)\sqrt{I(\theta)}} u(g_{00}(\theta)) \right)(\hat{\theta}_{ML}-\theta)$$

$$- \frac{u(g_{00}(\theta))}{\sqrt{nI(\theta)}} + \frac{1}{n}\left\{ \frac{g_{01}(\theta)}{\phi(u(g_{00}(\theta)))\sqrt{I(\theta)}} u(g_{00}(\theta)) - \frac{g_{10}(\theta)}{\phi(u(g_{00}(\theta)))\sqrt{I(\theta)}} \right.$$

$$\left. - \frac{3J(\theta)+K(\theta)}{6\{I(\theta)\}^2}\{u(g_{00}(\theta))\}^2 + \frac{K(\theta)}{6\{I(\theta)\}^2} \right\} + o_p(n^{-1})$$

より

$$\hat{\theta}_n = \hat{\theta}_{ML} - \frac{1}{\sqrt{n}} \frac{u(g(\hat{\theta}_{ML}))}{\sqrt{I(\hat{\theta}_{ML})}} + \frac{1}{n}\left\{ \frac{g_{01}(\hat{\theta}_{ML})\, u(g_{00}(\hat{\theta}_{ML}))}{\phi(u(g_{00}(\hat{\theta}_{ML})))I(\hat{\theta}_{ML})} \right.$$

$$- \frac{g_{10}(\hat{\theta}_{ML})}{\phi(u(g(\hat{\theta}_{ML})))\sqrt{I(\hat{\theta}_{ML})}} - \frac{3J(\hat{\theta}_{ML})+K(\hat{\theta}_{ML})}{6\{I(\hat{\theta}_{ML})\}^2}\{u(g_{00}(\theta_{ML}))\}^2$$

$$\left. + \frac{K(\hat{\theta}_{ML})}{6\{I(\hat{\theta}_{ML})\}^2} \right\} + o_p(n^{-1})$$

<u>定理2.</u>

$$\hat{\theta}_{ML} - \frac{1}{\sqrt{n}} \frac{u(g(\hat{\theta}_{ML}))}{\sqrt{I(\hat{\theta}_{ML})}} + \frac{1}{n}\left\{ - \frac{g_{10}(\hat{\theta}_{ML})}{\phi(u(g_{00}(\hat{\theta}_{ML})))\sqrt{I(\hat{\theta}_{ML})}} + \frac{g_{01}(\hat{\theta}_{ML})\, u(g_{00}(\hat{\theta}_{ML}))}{\phi(u(g_{00}(\hat{\theta}_{ML})))I(\theta_{ML})} \right.$$

$$\left. - \frac{3J(\hat{\theta}_{ML})+K(\hat{\theta}_{ML})}{6\{I(\hat{\theta}_{ML})\}^2}\{u(g_{00}(\theta_{ML}))\}^2 + \frac{K(\hat{\theta}_{ML})}{6\{I(\theta_{ML})\}^2} \right\}$$

は, *second order asymptotic efficient in* $C(g(n^{-\frac{1}{2}}\theta), 2)$.

定理1より次の結果が得られる.

<u>定理3</u>. θ の推定量 $\hat{\theta}_n$ に対して, 連続関数 $h(\theta)$ が存在して, 任意の t に対して,

$$P_\theta(\sqrt{n}(\hat{\theta}_n-\theta)\leqq t)=\Phi(\sqrt{I(\theta)}\,t)$$

$$+\phi(\sqrt{I(\theta)}\,t)\left\{\sqrt{2\pi}\,g_{10}(\theta)+\frac{t^2(2K(\theta)+3J(\theta))}{6\sqrt{I(\theta)}}\right\}+o(n^{-\frac{1}{2}})$$

が成り立つならば, $\hat{\theta}_n$ は, second order two-sided asymptotically efficient in $\mathcal{S}(G(n^{-\frac{1}{2}},\theta),2)$ である.

<u>定理4</u>. 任意の連続関数 $h(\theta)$ に対して,

$$\hat{\theta}_{ML}+\frac{1}{n}\left\{\frac{\sqrt{2\pi}\,h(\hat{\theta}_{ML})}{\sqrt{I(\hat{\theta}_{ML})}}+\frac{K(\hat{\theta}_{ML})}{6\{I(\hat{\theta}_{ML})\}^2}\right\}$$

は, second order two-sided asymptotically efficient in $\mathcal{S}(G(n^{-\frac{1}{2}},\theta),2)$ である.

<u>系1</u>

$$\hat{\theta}_{ML}+\frac{1}{n}\frac{K(\hat{\theta}_{ML})}{6\{I(\theta_{ML})\}^2}$$

は, second order two-sided asymptotically efficient in $\mathcal{S}(G(n^{-\frac{1}{2}},\theta),2)$ である.

M. Akahira & K. Takeuchi : Asymptotic Efficiency of Statistical Estimators (Springer)