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ON SPECTRAL PROPERTIES OF PSEUDO-RATIONAL SYSTEMS

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1. Introduction

There have been various attempts to extend the beautiful finite-dimensional linear system theory to other categories, for example, to linear infinite-dimensional systems, linear systems over commutative rings, etc. ([1], [6], [8], [13], [16]). Among these has been an approach, especially in the context of systems over rings, in which one considers transfer functions having an representation that is "quasi-rational" ([2], [10], [14]). It is only natural to speculate that such a class constitutes a nice generalization of the familiar finite-dimensional theory because

i) "rationality" somehow always corresponds to "finiteness";

ii) important systems, such as delay-differential systems, fall into this category; and

iii) such representations have already proved to be useful also in control of distributed parameter systems ([14]).

In this paper we shall see that such a representation is also very useful for establishing the relationship between input/output behavior and its canonical model - just like in the Fuhrmann realization ([5]) which has been very successful in finite-dimensional linear system theory (see, e.g., [4]).

Let us begin with the following informal definition of input/output maps. Let A be a $p \times m$ matrix whose entries are functions on $[0, \infty)$ which are locally square integrable on $[0, \infty)$. Our (zero-initial state) <u>input/output map</u> f_A, <u>associated to</u> A, is given by

$$f_A(u)(t) := \int_0^t A(t - \tau)u(\tau)d\tau. \tag{1.1}$$

For example, let $A(t)$ be the impulse response of the following retarded delay-differential system:

$$\frac{d}{dt} x(t) = x(t - 1) + u(t),$$

$$y(t) = x(t - 1).$$

Though its transfer function $W(s)$ (i.e., the Laplace transform $L[A]$) is certainly not rational, it still admits the following "quasi-rational" expression:

$$W(s) = 1/(se^s - 1). \tag{1.2}$$

In the language of distribution theory, this means that A can be written as $A = (\delta'_{-1} - \delta)^{-1} * \delta$, where δ_a is the Dirac distribution at point a, $(\delta := \delta_0)$, δ_a' its derivative, and $*$ denotes convolution as usual. Note also that the numerator and denominator of the above expression are coprime in the following sense:

$$(se^s - 1) \cdot 0 + 1 \cdot 1 = 1.$$

We shall then prove that the state space of the canonical realization is completely determined solely by the denominator $se^s - 1$ — as is precisely the case for the finite-dimensional theory. To be more precise, the canonical state space in this case is equal to the following space X^β $(\beta = \delta'_{-1} - \delta)$:

$$X^\beta = \{\gamma \in L^2_{loc}[0, \infty) : \pi((\delta'_{-1} - \delta) * \gamma) = 0\}, \tag{1.3}$$

where π denotes the truncation (in the sense of distributions) to $[0, \infty)$ (see Section 3 for details). One would readily notice that (1.3) is nothing but a generalization of Fuhrmann's state space described in the time domain.

We will then see that the spectrum of the canonical realization is fully determined by the denominator of the transfer function, i.e., it is the zeros of the denominator of the transfer function (namely, $se^s - 1$ in this case). Needless to say, this fact is quite useful for the study of internal stability of a canonical system via its transfer function.

An example is given, in terms of a retarded delay-differential system, to demonstrate the realization method.

2. Systems and Input/Output Maps

In the sequel, k will denote a fixed field, either $\mathbb{R}$ or $\mathbb{C}$, with the standard topology. Systems, functions, distributions etc. are considered over k.

Let $L^2[a, b]$ denote the space of square-integrable functions on $(-\infty, \infty)$ having support contained in $[a, b]$. $L^2_{loc}[0, \infty)$ is the space of functions which are square-integrable on every compact interval in $[0, \infty)$. Suppose that the present time is 0 and the systems we deal with are constant (time-invariant). We then assume that each L^2 input having compact support in $(-\infty, 0]$ is applied to the system until time 0 and then we start observing the output of the system which is locally L^2. Expressing these hypotheses mathematically, we see that our input space Ω and output space Γ must be as follows:

(2.1) $$\Omega := (\bigcup_{n>0} L^2[-n, 0])^m;$$

$$\Gamma := (L^2_{loc}[0, \infty))^p.$$

These spaces are naturally endowed with the following shift operators which are strongly continuous semigroups:

(2.2) $$(\sigma_t \omega)(s) := \begin{cases} \omega(s+t), & s \leq -t, \\ 0, & s > -t, \quad s \leq 0, \quad t \geq 0, \quad \omega \in \Omega. \end{cases}$$

$$(\tilde{\sigma}_t \gamma)(s) := \gamma(s+t), \quad t, s \geq 0, \quad \gamma \in \Gamma.$$

For more details on these spaces, see [16]. Let π be the truncation mapping: $\pi\psi := \psi|_{[0,\infty)}$. The rigorous definition of our input/output maps is then given as follows:

(2.3) DEFINITION. Let A be a $p \times m$ matrix whose entries are functions belonging to $L^2_{loc}[0, \infty)$. Then the <u>constant linear input/output map</u> f_A <u>associated to</u> A is given by

(2.4) $$f_A(\omega) := \pi(A * \omega), \quad \omega \in \Omega.$$

A is called the <u>impulse response</u> or <u>weighting pattern</u> of f_A.

When input u is applied on $[0, t)$, (2.4) clearly agrees with (1.1).

For any such A, f_A gives a continuous linear map of Ω into Γ. Furthermore, f_A commutes with the shifts defined in (2.2), and hence the term "constant". For details, see [16].

The principal result in this framework is the existence and uniqueness of canonical realizations ([16]). Of course, the whole thing relies on how one interprets the notion of "canonical" here. Before giving it a clear meaning, let us specify the notion of systems.

(2.5) DEFINITION. A <u>constant linear (continuous-time) system</u> is a quadruple $\Sigma = (X, \Phi, g, h)$ such that

i) the state space X is a complete locally convex space;

ii) $g: \Omega \rightarrow X$, $h: X \rightarrow \Gamma$ are continuous linear maps;

iii) $\{\Phi(t)\}_{t \geq 0}$ is a strongly continuous semigroup in X;

iv) $g\sigma_t = \Phi(t)g$, $h\Phi(t) = \tilde{\sigma}_t h$ for all $t \geq 0$.

We understand that the <u>state-transition</u> is given by

$$(2.6) \qquad \varphi(t, x, u) := \Phi(t)x + g(\sigma_t^{\ell} u)$$

where $\varphi(t, x, u)$ denotes the state at time $t \geq 0$ starting from a given initial state x at $t = 0$ under the application of input u; and $(\sigma_t^{\ell} u)(s) := u(s + t)$. The linear map h gives the correspondence: initial states $\mapsto$ future outputs, under the hypothesis that the input is identically 0 during the observation (this conforms, of course, to our setting of input/output maps (2.1) and (2.4)). The instantaneous readout map H may be defined by

$$(2.7) \qquad Hx := h(x)(0)$$

provided that the right-hand side makes sense (for example, when $h(x)$ is continuous). Note, however, that this map H is often discontinuous for distributed parameter systems while h is often continuous in many applications.

A constant linear system Σ is said to be <u>quasi-reachable</u> if its reachable set is dense in the whole state space, i.e., $g(\Omega)$ is dense in X. It is <u>topologically observable</u> if the initial state can be determined continuously out of the observed data. In other words, there must exist a continuous inverse

$$h^{-1}: \text{im } h \rightarrow X.$$

Σ is said to be <u>canonical</u> if it is both quasi-reachable and topologically observable. Topological observability is evidently a much stronger property than observability since the latter merely requires that h be one to one. Though the notion of canonical here is hence highly special, what is important to recognize is that the existence and uniqueness theorem holds with this strong notion of canonicity ([16]). In fact, making the observability stronger as above is thekey to this result. Otherwise, we would only have the existence but not uniqueness ([1]).

The definition of realization is as follows: Given an input/output map f, we say that a system $\Sigma = (X, \Phi, g, h)$ is a realization of f if f factors through Σ, i.e., $f = hg$.

Abstractly, the canonical realization of an input/output map f is given by the following construction: First take $\overline{\text{im } f}$, the closure of the image of Ω under f in Γ, as the state space. It is easy to see that $\overline{\text{im } f}$ is closed under the left shift operators $\{\tilde{\sigma}_t\}_{t\geq 0}$. So we can take $\tilde{\sigma}_t$, resricted to $\overline{\text{im } f}$, as the semigroup of the system. Then just take g to be f itself (with the difference of codomain which is now $\overline{\text{im } f}$ instead of Γ) and take h to be the natural inclusion $j: \overline{\text{im } f} \to \Gamma$. We denote this system by $\Sigma_f = (\overline{\text{im } f}, f, j)$. It is easy to see that Σ_f is quasi-reachable since the image of f is dense in $\overline{\text{im } f}$. Since the observability map h is just the inclusion map, Σ_f is trivially topologically observable. Hence this system is canonical.

Actually, we can derive a differential equation description for Σ_f ([17]). Indeed, define

(2.8) $$Fx := dx/dt \quad \text{for} \quad x \in \overline{\text{im } f} \cap (H^1_{loc}[0,\infty))^p;$$

$$(H^1_{loc}[0,\infty) = \{\gamma \in L^2_{loc}[0,\infty) : d\gamma/dt \in L^2_{loc}[0,\infty)\});$$

$$G_i := A_i \quad \text{(the } i\text{-th column of the impulse response } A)$$

$$Hx := x(0), \quad x \in \overline{\text{im } f} \cap (C[0,\infty))^p.$$

With these operators, the system Σ_f is described by the following functional differential equation:

(2.9)
$$\frac{d}{dt} x_t(\tau) = \frac{\partial}{\partial \tau} x_t(\tau) + \sum_{i=1}^{m} G_i u_i(t),$$
$$y(t) = Hx_t(\tau) = x_t(0),$$

where x_t is the state $\in \overline{\text{im } f}$ (as a function of τ) at time t.

Note however that this representation is generally of only abstract use and its practicality depends upon how we can find a concrete representation for $\overline{\text{im } f}$. This is precisely where pseudo-rationality comes into play.

3. Pseudo-Rational Input/Output Maps

Let us prepare some terminologies from distribution theory before giving the precise definition of pseudo-rationality.

As usual, $\mathcal{D}(\mathbb{R})$ is the space of C^{∞}-functions on $(-\infty, \infty)$ having compact support. Similarly, $\mathcal{D}(\mathbb{R}^+)$ is the space of C^{∞}-functions on $(-\infty, \infty)$ with compact support contained in $[0, \infty)$. $\mathcal{D}'(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R}^+)$ are the dual spaces of the above spaces respectively, and are spaces of distributions. $\mathcal{D}'_+(\mathbb{R})$ is the subspace of $\mathcal{D}'(\mathbb{R})$ that consists of distributions with support bounded on the left. $\mathcal{E}'(\mathbb{R}^-)$ denotes the subspace of $\mathcal{D}'_+(\mathbb{R})$ consisting of distributions having compact support contained in $(-\infty, 0]$. These spaces are each equipped with the standard topology based on duality ([12]).

We now want to generalize the truncation map π to distributions. Let $j: \mathcal{D}(\mathbb{R}^+) \to \mathcal{D}(\mathbb{R})$ be the natural inclusion. Then the desired truncation map π is defined to be the adjoint map of j, i.e., $\pi: \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}^+)$ is defined by

$$\langle \pi\alpha, \psi \rangle := \langle \alpha, j\psi \rangle.$$

This truncation π clearly agrees with the one defined in Section 2 when applied to functions. This map is surjective since the inclusion j is a topological isomorphism into $\mathcal{D}(\mathbb{R})$.

The following lemma is immediate from the definition.

(3.1) LEMMA. $\pi\alpha = 0$ iff $\operatorname{supp}\alpha \subset (-\infty, 0]$.

Let us now give the definition of pseudo-rationality.

(3.2) DEFINITION. An impulse response matrix A (or its associated input/output map f_A) is said to be pseudo-rational if it can be written as

$$A = Q^{-1}*P$$

for some $p\times p$ and $p\times m$ matrices Q and P with entries in $\mathcal{E}'(\mathbb{R}^-)$ such that

i) Q is invertible over $\mathcal{D}'_+(\mathbb{R})$ with respect to convolution;

ii) $\operatorname{supp} Q^{-1} \subset [0, \infty)$ and Q^{-1} is extendable to $(-\infty, \infty)$ by setting it to be zero on $(-\infty, 0]$.

(3.3) EXAMPLE. The transfer functions of many delay-differential systems (including the retarded ones, in particular) are expressible as the rational functions of variables $s, e^{h_1 s}, e^{h_2 s}, \ldots, e^{h_r s}$. According to Kamen [7], every nonzero polynomial in these variables (or its inverse Laplace transforms, to be more precise) satisfies the above conditions (i) and (ii). Hence the impulse responses of this type are pseudo-rational.

(3.4) EXAMPLE (Periodic case). Suppose that $A(t)$ satisfies $A(t + T) = A(t)$ for all $t \geq 0$. Then

$$A = (\delta_{-T} - \delta)^{-1}*(\delta_{-T}*(A|_{[0,T]}).$$

It is easy to check that $\delta_{-T} - \delta$ satisfies the above requirements. Hence A is pseudo-rational. In order to give our representation theorem for $\overline{\operatorname{im} f}$, let us first make the following definition.

(3.5) DEFINITION. Let (Q, P) be a pair which satisfies the conditions of Definition (3.2). The pair (Q, P) is called left coprime if there exist matirces R and S of suitable sizes with entries in $\mathcal{E}'(\mathbb{R}^-)$ such that

$$Q*R + P*S = \delta I_p$$

where I_p is the identity matrix of size p.

(3.6) REMARK. In other literature on systems over rings, the above condition is often referred to as the Bezout identity ([10], [14]).

(3.7) THEOREM. Let A be an impulse response matrix with the associated input/output map f. Suppose that A is pseudo-rational with a representation $A = Q^{-1}*P$. Then

$$\overline{\operatorname{im} f} \subset X^Q := \{\gamma \in \Gamma : \pi(Q*\gamma) = 0\}.$$

If, further, (Q, P) is a left coprime pair, then $\overline{\operatorname{im} f} = X^Q$.

[In the definition of X^Q, $Q*\gamma$ must be understood in the sense of distributions.]

We omit the proof (see [18]). Applying Theorem (3.7) to the realization (2.8), (2.9), we have the following theorem.

(3.8) THEOREM. Under the same hypotheses on A as in Theorem (3.7), the following system Σ^Q is a topologically observable realization of f:

1) State space $= X^Q$.

2) State transition:

(3.9)
$$\frac{d}{dt} x_t(\tau) = \frac{\partial}{\partial \tau} x_t(\tau) + Au(t),$$
$$x_t(\cdot) \in X^Q \cap (H^1_{loc}[0, \infty))^p.$$

3) Output equation:

(3.10)
$$y(t) = x_t(0).$$

<u>If the pair</u> (Q, P) <u>is left coprime, the above system is canonical</u>.

PROOF. The last statement follows from teh first half and Theorem (3.8). So we need only to prove the first half.

We must give the semigroup, reachability map g, and the observability map h of Σ^Q. The semigroup generated by $(d/d\tau)$ is the shift operator $\tilde{\sigma}_\tau$ restricted to X^Q. This can be easily checked by computing the infinitesimal generator of $\tilde{\sigma}_\tau|_{X^Q}$. Then the solution of (3.9) is given by

$$x_t(\tau) = \tilde{\sigma}_t x_0(\tau) + \int_0^t (\tilde{\sigma}_{t-s}A(\tau))u(s)ds$$
$$= x_0(\tau + t) + \int_0^t A(\tau + t - s)u(s)ds.$$

Hence the reachability map g is given by

$$(g(\omega))(\tau) = \int_{-\infty}^0 \sigma_{-s}A(\tau)\omega(s)ds$$
$$= \int_{-\infty}^0 A(\tau - s)\omega(s)ds.$$

The observability map induced by (3.10) is simply the inclusion map $j: X^Q \to \Gamma$. Therefore, the topological observability of Σ^Q is obvious. It remains only to prove that Σ^Q is a realization. But we have

$$(jg(\omega))(t) = (g(\omega))(t)$$
$$= \int_{-\infty}^0 A(t - \tau)\omega(\tau)d\tau$$
$$= f(\omega)(t).$$

This completes the proof. □

In order to illustrate the method above, we now give an example which realizes a retarded delay-differential system. For more general results, the reader is referred to [18].

(3.11) EXAMPLE. We consider the case $m = 1$ and $p = 2$. Let $W(s)$ be the following transfer function matrix:

(3.12) $$W(s) = \begin{bmatrix} \dfrac{1}{e^s(s-1)} \\ \\ \dfrac{1}{e^s(s-1)(se^s-1)} \end{bmatrix}.$$

The corresponding impulse response matrix $A = (A_1, A_2)'$ for $0 \le t \le 2$ is given by

(3.13) $$A_1(t) = \begin{cases} 0, & 0 \le t < 1, \\ 1, & 1 \le t \le 2; \text{ and} \end{cases}$$

$$A_2(t) = 0 \quad \text{for} \quad 0 \le t \le 2.$$

It is easy to find a factorization of $W(s)$:

(3.14) $$W(s) = \begin{bmatrix} sz - z & 0 \\ -1 & sz - 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =: Q^{-1}P,$$

where z denotes e^s. Now take R and S as follows:

(3.15) $$R := \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \qquad S := [1, sz - z].$$

It is easy to check $QR + PS = I$. [To find R and S, consider the matrix $[Q, P]$ and convert it to $[I, 0]$ via fundamental column operations over $\mathbb{R}[s, z]$.] Hence $L^{-1}[Q]$ and $L^{-1}[P]$ are left coprime. For brevity of notation, denote the inverse Laplace transforms of Q and P by the same symbols.

It is readily seen that A is pseudo-rational. Hence we can apply Theorem (3.8). For a smooth $\gamma = (\gamma_1, \gamma_2)'$, the equation $\pi(Q*\gamma) = 0$ means

(3.16) $$\gamma_1'(t+1) - \gamma_1(t+1) = 0 \quad \text{for all} \quad t \ge 0;$$

(3.17) $$\gamma_2'(t+1) - \gamma_2(t) - \gamma_1(t) = 0 \quad \text{for all} \quad t \ge 0.$$

Solving (3.16), we obtain

(3.18) $\gamma_1(t) = e^t\gamma_1(1)$ for all $t \geq 1$.

Hence $\gamma_1(t)$ for $t \geq 1$ is completely determined by specifying the value $\gamma_1(1)$. On the other hand, $\gamma_1|_{[0,1]}$ can be arbitrarily chosen without violating the rule (3.16). Hence $(\gamma_1|_{[0,1]}, \gamma_1(1))$ completely determines the values of $\gamma_1(t)$ for all $t \geq 0$. Once $\gamma_1(t)$ is determined, it is easy to solve (3.17) as follows:

(3.19) $\gamma_2(t) = \gamma_2(1) + \int_1^t \gamma_2(\tau - 1)d\tau + \int_1^t \gamma_1(\tau - 1)d\tau, \quad 1 \leq t \leq 2.$

Iterating this formula successively while satisfying (3.17), we see that $\gamma_2|_{[0,1]}$ and $\gamma_2(1)$ entirely determine the values of $\gamma_2(t)$ (note that $\gamma_1(t)$ is already known). Hence the paris $(\gamma_1|_{[0,1]}, \gamma_1(1))$ and $(\gamma_2|_{[0,1]}$ $\gamma_2(1))$ completely determine the values of $\gamma(t)$ for all $t \geq 0$. Taking the closure of all such pairs in Γ, we have

(3.20) $\overline{\text{im } f} \cong (L^2[0, 1] \times \mathbb{R})^2.$

This is clearly isomorphic to the so-called M_2-space which is well known for the study of retarded systems ([3]). (Note that while M_2-spaces are usually intorduced by associating $L^2[0, h]$ to the delay of h seconds, in our present approach, it arises naturally as a result of our canonical construction.)

Let us compute the functional differential equation description (3.9, 10) for this case. To compute F, we need to (i) shift γ by ε, (ii) divide $\tilde{\sigma}_\varepsilon\gamma - \gamma$ by ε, and (iii) take the limit as $\varepsilon \to 0$. Note that

$$(\tilde{\sigma}_\varepsilon\gamma_1)(1) = \gamma_1(1 + \varepsilon) = e^\varepsilon\gamma_1(1),$$

$$(\tilde{\sigma}_\varepsilon\gamma_2)(1) = \gamma_2(1 + \varepsilon) = \gamma_2(1) + \int_0^\varepsilon \{\gamma_1(t) + \gamma_2(t)\}dt.$$

So we have

$$(\tilde{\sigma}_\varepsilon \gamma_1 - \gamma_1)/\varepsilon \to \gamma_1(1),$$

$$(\tilde{\sigma}_\varepsilon \gamma_2 - \gamma_2)/\varepsilon \to \gamma_1(0) + \gamma_2(0),$$

provided that $(\gamma_1(t), \gamma_2(t))$ belongs to $D(F) = \overline{\text{im } f} \cap (H^1_{loc}[0,\infty))^2$ as required in (3.9). For $0 \leq t \leq 1$, $\tilde{\sigma}_\varepsilon$ acts as a simple left shift operator and no reference to (3.16, 17) is necessary to compute $\lim (\tilde{\sigma}_\varepsilon \gamma - \gamma)/\varepsilon$ here; the infinitesimal generator is the differential operator d/dt. In conformity with the notation in the usual M_2-space models, we write $(z_1(\theta), z_2(\theta), x_1, x_2)$ in place of $(\gamma_1(t), \gamma_2(t), \gamma_1(1), \gamma_2(1))$. Then we have the following differential equation description of the state transition:

(3.22)
$$\frac{d}{dt}\begin{bmatrix} z_1(\theta) \\ z_2(\theta) \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (\partial/\partial\theta) z_1(\theta) \\ (\partial/\partial\theta) z_2(\theta) \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = (z_1(0), z_2(0))'.$$

Note again that (3.22) is nothing but the M_2-space model of the following delay-differential system.

$$\frac{d}{dt} x_1(t) = x_1(t) + u(t),$$

$$\frac{d}{dt} x_2(t) = x_1(t-1) + x_2(t-1),$$

$$y(t) = (x_1(t-1), x_2(t-1))'.$$

4. Spectrum of Σ^Q

In this section we study some spectral properties of the system Σ^Q given in Theorem (3.8). Our main target is the characterization of the spectrum of the infinitesimal generator F of the transition semigroup.

This has of course a lot of bearing on the stability properties of canonical realizations.

For simplicity of arguments, we assume that our field k is $\mathbb{C}$. But the results carry over to the case $k = \mathbb{R}$, <u>mutatis mutandis</u>.

Let $A = Q^{-1} * P$ be pseudo-rational, and let Σ^Q be the system given by (3.9), (3.10). Suppose that $\lambda \in \mathbb{C}$ belongs to the resolvent set $\rho(F)$ of the infinitesimal generator F of the semigroup $\tilde{\sigma}_t$ in X^Q. Then for any $y(t) \in X^Q$ there exists a unique x in $X^Q \cap (H^1_{loc}[0, \infty))^p$ such that

(4.1) $$(\lambda - \frac{d}{dt})x(t) = y(t) \quad \text{for almost all } t \geq 0.$$

Solving (4.1) for x in Γ, we have

(4.2) $$x(t) = e^{\lambda t}x(0) - \int_0^t e^{\lambda(t-\tau)}y(\tau)d\tau.$$

If λ belongs to $\rho(F)$, $y = 0$ must imply $x(0) = 0$. Hence we have

(4.3) LEMMA. <u>If a complex number</u> λ <u>belongs to</u> $\rho(F)$, <u>then the following statement holds</u>:

(4.4) $$e^{\lambda t}v \in X^Q \Rightarrow v = 0 \quad (v \in \mathbb{C}^p).$$

PROOF. Suppose otherwise. Then λ becomes an eigenvalue with eigenvector $x(t) = e^{\lambda t}v$. This is a contradiction. □

We need the following lemma from distribution theory.

(4.5) LEMMA. (Paley-Wiener-Schwartz Theorem for Laplace Transforms - A Special Case) <u>A distribution</u> β <u>has compact support contained in</u> $[-a, 0]$ <u>iff its Laplace transform</u> $\hat{\beta}$ <u>is an entire function satisfying the following estimate for some</u> $C > 0$ <u>and a positive integer</u> n.

(4.6) $$|\hat{\beta}(s)| \leq C(1 + |s|)^n \exp(n\mathrm{Re}\, s) \quad \text{if } \mathrm{Re}\, s \geq 0,$$
$$\leq C(1 + |s|)^n \quad \text{if } \mathrm{Re}\, s < 0.$$

PROOF. Omitted. See Kaneko [9]. □

We can now prove the following proposition.

(4.7) PROPOSITION. <u>A function</u> $e^{\lambda t}v$ <u>belongs to</u> X^Q <u>for some nonzero</u> $v \in \mathbb{C}^p$ <u>iff</u> $(\det \hat{Q})(\lambda) = 0$.

PROOF. Recall that $e^{\lambda t}v$ belongs to X^Q iff $\pi(Q*e^{\lambda t}v) = 0$, i.e., the support of $Q*e^{\lambda t}v$ is contained in $[-a, 0]$ for some $a \geq 0$. Then $L[Q*e^{\lambda t}v]$ is an entire function by the preceding lemma. But we have

$$(4.8) \qquad L[Q*e^{\lambda t}v] = \frac{\hat{Q}(s)v}{s - \lambda} .$$

Since $\hat{Q}(s)$ is entire, expand $\hat{Q}(s)$ in the powers of $(s - \lambda)$. Then the constant term is $\hat{Q}(\lambda)$. It follows that the right-hand side of (4.8) is entire iff $\hat{Q}(\lambda)v = 0$. Since $v \neq 0$, $(\det \hat{Q})(\lambda) = 0$ follows.

Conversely, suppose that $(\det \hat{Q})(\lambda) = 0$. Then by the same argument as above, $L[Q*e^{\lambda t}v]$ is an entire function for some $v \neq 0$. Since each entry of $\hat{Q}(s)$ satisfies the estimate (4.6) for some a, so does each entry of $\hat{Q}(s)v$. Since $1/|s - \lambda|$ is bounded for large enough s, $|\hat{Q}(s)v/(s - \lambda)|$ satisfies the same type of estimate as (4.6) for large enough s. Also, since $\hat{Q}(s)v/(s - \lambda)$ is entire, it is bounded in a neighborhood of λ. Therefore, $\hat{Q}(s)v/(s - \lambda)$ satisfies the same type of estimate as (4.6). This implies $\pi(Q*e^{\lambda t}v) = 0$, i.e., $e^{\lambda t}v$ belongs to X^Q. □

Combining Proposition (4.7) with Lemma (4.4), we readily obtain

(4.9) COROLLARY. <u>A complex number</u> λ <u>is an eigenvalue of</u> F <u>if and only if</u> $(\det \hat{Q})(\lambda) = 0$.

Actually, this completely characterizes the spectrum of F: they are all eigenvalues. In other words, we have

(4.10) THEOREM. <u>The resolvent set</u> $\rho(F)$ <u>is given by</u>

$$\rho(F) := \{\lambda \in \mathbb{C}: (\det \hat{Q})(\lambda) \neq 0\}.$$

Therefore, the spectrum $\sigma(F)$ consists only of eigenvalues and is given by

$$\sigma(F) = \{\lambda \in \mathbb{C}: (\det \hat{Q})(\lambda) = 0\}$$

$$= \sigma_p(F).$$

We omit the proof since it is rather involved to be given here. See [18]

Since $(\det \hat{Q})(s)$ is an entire function, Theorem (4.10) yields a lot of interesting facts. For example, each eigenvalue has finite multiplicity, and the set of all eigenvalues is a discrete set. Also, the dimension of the generalized eigenspace M_λ corresponding to λ is equal to the order of λ as a zero of $\det \hat{Q}(s)$. Summarizing, we have

(4.11) THEOREM. Let F, Σ^Q be as above. For each eigenvalue λ of F, the dimension of the generalized eigenspace M_λ of F corresponding to λ is m = order of λ as a zero of $\det \hat{Q}(s)$. Furthermore, we have the direct sum decomposition:

$$X^Q = \operatorname{im} (\lambda - F)^m \oplus \ker (\lambda - F)^m.$$

And the resolvent operator $(s - F)^{-1}$ has the Laurent series expansion

$$(s - F)^{-1} = \sum_{n=-m}^{\infty} (s - \lambda)^n P_n$$

where

$$P_n := \frac{1}{2\pi j} \oint_C (s - \lambda)^{-n-1} (s - F)^{-1} ds.$$

Here C is a closed rectifiable curve that encircles λ and no other zeros of $\det \hat{Q}(s)$ once in the positive direction.

PROOF. Omitted. See [18] and [19]. □

Let us finally mention the relationship of the above facts to the so-called spectral reachability. We say that Σ^Q is <u>spectrally reachable</u> if the finite-dimensional subsystem Σ^Q_λ, obtained by projecting it to the generalized eigenspace M_λ, is reachable for every λ. The definition is made primarily for delay-differential systems ([20]), but carries over to the present context with noessential alterations since M_λ is finite-dimensional. Let us make the following definition.

(4.12) DEFINITION. Let (Q, P) be a pair of $p \times p$ and $p \times m$ matrices with entries in $\mathcal{E}'(\mathbb{R}^-)$. The pair (Q, P) is said to be <u>spectrally left coprime</u> if

$$\operatorname{rank}\,[\hat{Q}(\lambda),\ \hat{P}(\lambda)] = p$$

for every $\lambda \in \mathbb{C}$.

We then have the following theorem.

(4.13) THEOREM. <u>Let</u> $A = Q^{-1} * P$ <u>be pseudo-rational. Then</u> Σ^Q <u>is spectrally reachable if and only if</u> (Q, P) <u>is spectrally left coprime</u>.

5. Conclusion

Pseudo-rational input/output maps and the associated systems Σ^Q as above comprise a natural extension of the classical finite-dimensional linear systems in the following sense:

i) The transfer functions admit a fractional representation.

ii) Such fractional representations play a crucial role in determination of the canonical realization.

iii) The spectrum of Σ^Q is precisely the zeros of $\det \hat{Q}(s)$; and hence, if Σ^Q is canonical, the spectrum is precisely the "zeros of the denominator of the transfer function."

iv) Several other spectral properties hold as in teh finite-dimensional case.

Note again that retarded delay-differential systems fall into this category, and that the realization Σ^Q in this case often agrees with the M_2-space model for such systems, as exemplified in Example (3.11). However, our approach has an added advantage that we can reduce redundancy often exercised in the M_2-space approach by suitably choosing the representation $A = Q^{-1}*P$. For details, see [18].

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