数理解析研究所講究録
第 537 巻 1984 年 29-34

A Sum related to the Eratosthenes Sieve

By Yoichi Motohashi

1. In their recent work "Sur une somme liée à la fonction de Möbius" Dress, Iwaniec and Tenenbaum proved the asymptotic formula

$$(1) \qquad \sum_{d_1, d_2 \le Z} \frac{\mu(d_1)\mu(d_2)}{[d_1, d_2]} = (1+o(1))\frac{1}{\pi}\int_0^\infty \prod_p \left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}\left(|1-p^{it}|^2-1\right)\right)\frac{dt}{t^2}$$

as Z tends to infinity. This has some interest in its connection with the Eratosthenes sieve through the relation

$$\sum_{n \le X}\Big(\sum_{\substack{d|n \\ d \le Z}} \mu(d)\Big)^2 = X \sum_{d_1, d_2 \le Z} \frac{\mu(d_1)\mu(d_2)}{[d_1, d_2]} + O(Z^2).$$

To show (1) they made use of a real functional method. But, as I remarked in a letter to Dress, a routine application of Perron's inversion formula to the function

$$\sum_{d_1, d_2} \frac{\mu(d_1)\mu(d_2)}{[d_1, d_2]\, d_1^{s_1} d_2^{s_2}} \qquad (\mathrm{Re}(s_j) > 0,\ j=1,2\ )$$

yields (1) with an explicit error of the order of $\exp(-c(\log Z/\log\log Z)^{1/3})$

Here, employing the same argument I will give a brief proof of a little more difficult

<u>THEOREM</u>

$$(2) \qquad \sum_{d_1, d_2, d_3, d_4 \le Z} \frac{\mu(d_1)\mu(d_2)\mu(d_3)\mu(d_4)}{[d_1, d_2, d_3, d_4]} = c_0 \log^2 Z + O(\log Z),$$

$$c_0 = \frac{3}{2\pi}\int_0^\infty \prod_p \left(1-\frac{1}{p}\right)^5\left(1+\frac{1}{p}\left(4|1-p^{it}|^2 - |1-p^{2it}|^2 - 1\right)\right)\frac{dt}{t^4}.$$

Remark

(i) Our sum comes from

$$\sum_{n\le X}\Big(\sum_{\substack{d|n\\ d\le Z}}\mu(d)\Big)^4 .$$

(ii) As is apparent from our proof the right side of (2) can be put more precisely in the form $c_0(\log Z)^2 + c_1\log Z + c_2 + O(\exp(-c(\frac{\log Z}{\log\log Z})^{\frac{1}{3}})))$, though the explicit computation of c_1 and c_2 is quite tedious.

2. Let s_j $(1 \le j \le 4)$ be complex variables, and put

$$I_j = \{ s_j \,;\, \mathrm{Re}(s_j) = j\eta \,,\ |\mathrm{Im}(s_j)| \le jZ^2 \} \quad (1 \le j \le 3),$$

$$I_4 = \{ s_4 \,;\, \mathrm{Re}(s_4) = 7\eta \,,\ |\mathrm{Im}(s_4)| \le 7Z^2 \},$$

where $\eta = c(\log Z)^{-\frac{2}{3}}(\log\log Z)^{-\frac{1}{3}}$ (cf. Vinogradov's zero-free region). Also we assume that $Z = [Z] + 1/2$, without losing any generality. Then, applying Perron's inversion formula fourtimes we see that the sum to be estimated is equal to

$$\frac{1}{(2\pi i)^4}\iiiint\limits_{\substack{s_j\in I_j\\ (1\le j\le 4)}} \sum_{d_1,d_2,d_3,d_4=1}^{\infty} \frac{\mu(d_1)\mu(d_2)\mu(d_3)\mu(d_4)}{[d_1,d_2,d_3,d_4]\, d_1^{s_1} d_2^{s_2} d_3^{s_3} d_4^{s_4}} \frac{Z^{s_1+s_2+s_3+s_4}}{s_1 s_2 s_3 s_4}\, ds_1 ds_2 ds_3 ds_4 + O(Z^{-\frac{1}{2}})$$

$$= J + O(Z^{-\frac{1}{2}}),$$

say. The sum in the integrand is equal to

$$\prod_p \Big[1 + \frac{1}{p}\sum_{l=1}^{4}(-1)^l \sum_{1\le j_1< \dots <j_l\le 4} p^{-s_{j_1}-\dots-s_{j_l}} \Big]$$

$$= \prod_{l=1}^{4}\ \prod_{1\le j_1<\dots<j_l\le 4} \zeta(1+s_{j_1}+\dots+s_{j_l})^{(-1)^l}\, W(s_1,s_2,s_3,s_4) ;$$

$W(s_1, s_2, s_3, s_4)$ is regular and bounded in the region $\mathrm{Re}(s_j) > -c$ $(1 \leq j \leq 4,\ c > 0)$.

3. Shifting the line of integration I_4 of the variable s_4 to $\{\mathrm{Re}(s_4) = -7\eta\ ,\ |\mathrm{Im}(s_4)| \leq 7Z^2\}$, we encounter sigularities only at $s_4 = -s_3\ ,\ -s_2\ ,\ -s_1\ ,\ -(s_3 + s_2 + s_1)$, which are all simple poles. Then computing the residues we get

$$J =$$

$$- \frac{1}{(2\pi i)^3} \iiint\limits_{\substack{s_j \in I_j \\ (1 \leq j \leq 3)}} \frac{\zeta(1+s_1+s_2)^2 \zeta(1+s_1+s_3)\zeta(1+s_1-s_3)\zeta(1+s_2+s_3)\zeta(1+s_2-s_3) W(s_1,s_2,s_3,-s_3) Z^{s_1+s_2}}{\zeta(1+s_1)^2 \zeta(1+s_2)^2 \zeta(1+s_3)\zeta(1-s_3)\zeta(1+s_1+s_2+s_3)\zeta(1+s_1+s_2-s_3) s_1 s_2 s_3^2} ds_1 ds_2 d$$

$$- \frac{1}{(2\pi i)^3} \iiint\limits_{\substack{s_j \in I_j \\ (1 \leq j \leq 3)}} \frac{\zeta(1+s_1+s_2)\zeta(1+s_1+s_3)^2 \zeta(1+s_1-s_2)\zeta(1+s_2+s_3)\zeta(1+s_3-s_2) W(s_1,s_2,s_3,-s_2) Z^{s_1+s_3}}{\zeta(1+s_1)^2 \zeta(1+s_2)\zeta(1+s_3)^2 \zeta(1-s_2)\zeta(1+s_1+s_2+s_3)\zeta(1+s_1+s_3-s_2) s_1 s_2^2 s_3} ds_1 ds_2 .$$

$$- \frac{1}{(2\pi i)^3} \iiint\limits_{\substack{s_j \in I_j \\ (1 \leq j \leq 3)}} \frac{\zeta(1+s_1+s_2)\zeta(1+s_1+s_3)\zeta(1+s_2+s_3)^2 \zeta(1+s_2-s_1)\zeta(1+s_3-s_1) W(s_1,s_2,s_3,-s_1) Z^{s_2+s_3}}{\zeta(1+s_1)\zeta(1+s_2)^2 \zeta(1+s_3)^2 \zeta(1-s_1)\zeta(1+s_1+s_2+s_3)\zeta(1+s_2+s_3-s_1) s_1^2 s_2 s_3} ds_1 ds_2$$

$$+ O(1)$$

$= J_1 + J_2 + J_3 + O(1)$, say.

Next, in the integral J_1 we shift the line of integration I_2 of the variable s_2 to the line $I_2^* = \{s_2;\ \mathrm{Re}(s_2) = -2\eta\ ,\ |\mathrm{Im}(s_2)| \leq 2Z^2\}$.

The singularity to be accounted is only at $s_2 = -s_1$ which is a double pole, and a simple estimation gives

$$J_1 = O(\log Z)$$

As for the integral J_2 we shift the line of integration I_3 of the variable s_3 to the line $\{ s_3 \,;\, \mathrm{Re}(s_3) = -3\eta \,,\, |\mathrm{Im}(s_3)| \leq 3Z^2 \}$. Then we encounter singularities only at $s_3 = s_2, -s_2, -s_1$. The first two are simple poles, and the last is a double pole with the residue of the order of $\log Z$. Also it is easy to see that the residue at $s_3 = -s_2$ is $O(Z^{-c\eta})$. Hence we have

$$J_2 = -\frac{1}{(2\pi i)^2} \iint_{s_1 \in I_1, s_2 \in I_2} \frac{\zeta(1+s_1+s_2)^3 \zeta(1+s_1-s_2)\zeta(1+2s_2) W(s_1,s_2,s_2,-s_2) Z^{s_1+s_2}}{\zeta(1+s_1)^3 \zeta(1+s_2)^3 \zeta(1-s_2)\zeta(1+s_1+2s_2)\, s_1 s_2^3} \, ds_1 \, ds_2$$

$$+ O(\log Z)$$

$= J_{21} + O(\log Z)$, say.

In much the same way we have

$$J_3 = -\frac{1}{(2\pi i)^2} \iint_{s_1 \in I_1, s_2 \in I_2} \frac{\zeta(1+s_1+s_2)^3 \zeta(1+s_2-s_1)\zeta(1+2s_1) W(s_1,s_2,s_1,-s_1) Z^{s_1+s_2}}{\zeta(1+s_1)^3 \zeta(1+s_2)^3 \zeta(1-s_1)\zeta(1+2s_1+s_2)\, s_1^3 s_2} \, ds_1 \, ds_2$$

$$+ \frac{1}{(2\pi i)^2} \iint_{s_1 \in I_1, s_2 \in I_2} \frac{\zeta(1+s_1+s_2)\zeta(1+s_2-s_1)^3 \zeta(1-2s_1) W(s_1,s_2,-s_1,-s_1) Z^{s_2-s_1}}{\zeta(1+s_1)\zeta(1+s_2)^3 \zeta(1-s_1)^3 \zeta(1-2s_1+s_2)\, s_1^3 s_2} \, ds_1 \, ds_2$$

$$+ O(\log Z)$$

$= J_{31} + J_{32} + O(\log Z)$, say.

Then in the integral J_{21} we shift the line of integration I_2 to I_2^* defined above. The simple pole at $s_2 = s_1$ has the residue

$$\frac{1}{2\pi i}\int_{I_1} \frac{\zeta(1+2s_1)^4\, W(s_1,s_1,s_1,-s_1)}{\zeta(1+s_1)^6\,\zeta(1-s_1)\zeta(1+3s_1)} \cdot \frac{Z^{2s_1}}{s_1^4}\, ds_1$$

and, shifting I_1 to $\{ s_1 \ ; \ \mathrm{Re}(s_1) = -\eta \ , \ |\mathrm{Im}(s_1)| \le Z^2 \}$ we see that this is $O(Z^{-c\eta})$. On the other hand the triple pole at $s_2 = -s_1$ has the residue

$$\frac{(\log Z)^2}{4\pi i}\int_{I_1} \frac{\zeta(1+2s_1)\zeta(1-2s_1)W(s_1,-s_1,-s_1,s_1)}{\zeta(1+s_1)^4\,\zeta(1-s_1)^4\ \ s_1^4}\, ds_1 + O(\log Z).$$

Hence we have

$$J_{21} = \frac{(\log Z)^2}{4\pi}\int_{-\infty}^{\infty} \frac{|\zeta(1+2it)|^2}{|\zeta(1+it)|^8} W(it,-it,-it,it)t^{-4}dt + O(\log Z).$$

Similarly we have

$$J_{31} = \frac{(\log Z)^2}{4\pi}\int_{-\infty}^{\infty} \frac{|\zeta(1+2it)|^2}{|\zeta(1+it)|^8} W(it,-it,it,-it)t^{-4}dt + O(\log Z),$$

$$J_{32} = \frac{(\log Z)^2}{4\pi}\int_{-\infty}^{\infty} \frac{|\zeta(1+2it)|^2}{|\zeta(1+it)|^8} W(it,it,-it,-it)t^{-4}dt + O(\log Z).$$

Finally on noting that W is a symmetric function of its variables and that for real t

$$\frac{|\zeta(1+2it)|^2}{|\zeta(1+it)|^8} W(it,it,-it,-it)$$

$$= \prod_p \left(1-\frac{1}{p}\right)^5\left(1+\frac{1}{p}\left(4|1-p^{it}|^2-|1-p^{2it}|^2-1\right)\right)$$

3

we end the proof of the theorem.

4. Readers are urged to consider more generally the estimation of the sum

$$\sum_{d_1,\cdots,d_r} \frac{\mu(d_1)\mu(d_2)\cdots\mu(d_r)}{[d_1,d_2,\cdots,d_r]} .$$

Addendum

$$C_0 = \frac{3}{2\pi}\int_0^\infty \prod_p \left(1-\frac{1}{p}\right)^5 \left(1+\frac{1}{p}\left(|1-p^{it}|^4-1\right)\right)\frac{dt}{t^4}$$