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Topology of Complex Webs of Codimension One and Geometry of Projective Space Curves

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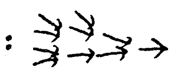
Abstract.

A b web of a manifold M of codimension 1 is a configuration $\mathcal{W}$ of b foliations $\mathcal{F}_1,\ldots,\mathcal{F}_b$ of M of codimension 1. In Chapter I, we prove that the topological and analytic classifications are the same for complex analytic webs of a complex manifold M under the condition $b \geqq \dim M + 1$ and a certain generic condition (Theorem I.4.1). This is a complex analytic version of Dufour's theorem for C^∞-webs [D_3,D_4]. In Chapter II, we apply our theorem for the d webs $\mathcal{W}_C$ of the dual projective space $\mathbb{P}_n^\vee$ of codimension 1 generated by the dual hyperplanes $x^\vee \in \mathbb{P}_n^\vee$ of $x \in C$ with algebraic curves $C \subset \mathbb{P}_n$ of degree d, and prove that the imbeddings $C \subset \mathbb{P}_n$ are determined by the topological structures of $\mathcal{W}_C$ up to projective transformations if $d \geqq n + 2$ (Theorem II.1.3). The singular locus $\Sigma(\mathcal{W}_C)$ of $\mathcal{W}_C$ is closely related with the projective geometry of C and the dual variety and curve of C. In the final two sections, we investigate the structure of $\mathcal{W}_C$ for the exceptional cases that $C \subset \mathbb{P}_n$ is of degree n, $n+1$, e.g., rational and elliptic normal curves, and singular plane curves.

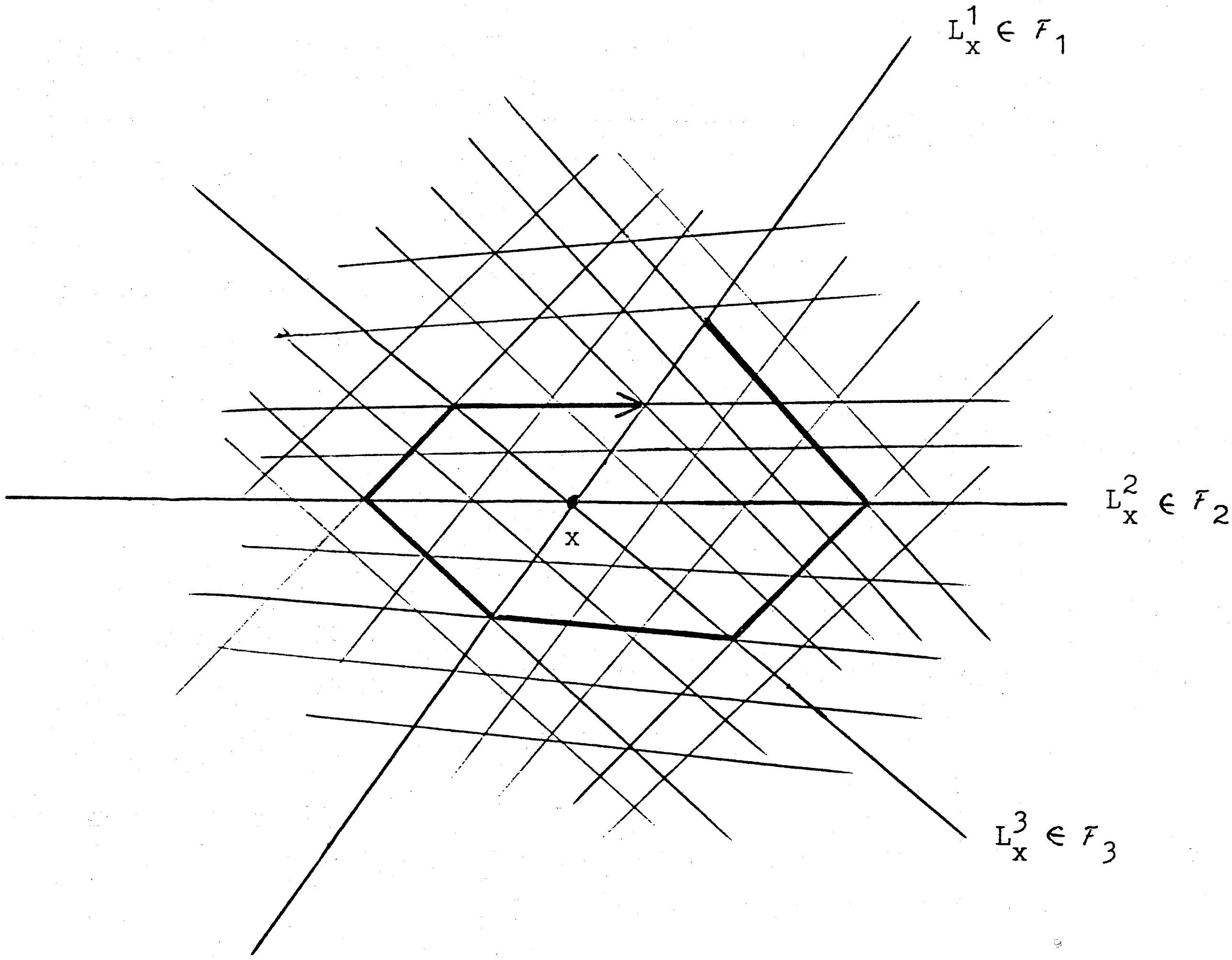
A foliation $\mathcal{F}_i$ of a manifold M is locally defined to be a family of level surfaces of non singular functions u_i on M, so the local study of b webs is equivalent to one of the diagrams of functions of the form: $M \xrightarrow{u_i} \mathbb{K}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$).
$\searrow \mathbb{K}$

The diagram of this type appears often in various areas of differential topology and its applications. Especially the envelope theory is reformulated by the diagram of this type which was studied by Thom, Arnol'd, Carneiro, Dufour, Bruce, Gibson [T,A,Ca,D_4,BG].

The problem of this diagram is the simplest and a very attractive part of the general theory of diagram of C^∞-mappings, for which Thom Mather theory does not work because of the fact that Malgramge's preparation theorem fails [D_1]. This difficulty seems not to be only on these appearence of the diagram: In fact Dufour proved in [D_1,D_2] that for non-degenerate diagrams of three functions $F, G : \mathbb{R}^2 \to \mathbb{R}, \mathbb{R}, \mathbb{R}$

(or $\mathbb{R}^2 \to \mathbb{R}^2$, $\searrow \mathbb{R}$), F, G are C^∞-equivalent if and only if topologically equivalent (Lemma I.0.1) using basically Lebesque's theorem, and consequently that the topological stability theorem does not hold for these divergent diagrams in contrast to the known result that for the convergent diagrams of C^∞-mappings:  , Thom-Mather theory works well and the topological stability theorem holds [B ,D_2,Da,N].

In Chapter I , we prove a Dufour type theorem for complex analytic case, namely if two 3-febs $\mathcal{W} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$,



The Poincaré map of a 3 web $W = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ of $\mathbb{R}^2$ with the center x .

Figure 1.

not hexagonal is essential. In fact any hexagonal 3 webs of foliations by parallel lines in $\mathbb{C}^2$ admits real linear but non complex linear automorphisms as $\mathbb{C}^2 = \mathbb{R}^4$.

In Chapter 2, we apply the theorem to the dual d webs W_C denerated by algebraic curves $C \subset \mathbb{P}_n$ of degree d. Of course W_C have the singular locus $\Sigma(W_C) = \mathrm{envl}(W_C)$ $\mathrm{degn}(W_C)$ (see Chapter II, section 2) biside which W_C form nondegenerate d webs.

The Graf-Sauer's theorem says that W_C is haexagonal beside $\Sigma(W_C)$ if and only if $C \subset \mathbb{P}_2$ is a cubic curve for $n = 2$ (Theorem II.3.1, or see [BB,GS]). This result was expanded by many authors [AG,Ba ,Ak,GC].

The restriction of W_C to an intersection $\check{x}_1 \cap \ldots \cap \check{x}_{n-2}$ $= \mathbb{P}_2$, $x_i \in C$ is the web generated by the image of C under the projection of $\mathbb{P}_n$ with the center $\mathbb{P}_{n-3}$ spanned by x_1, $\ldots, x_{n-2}$, which is a plane curve of degree $d-(n-2)$ if $C \cdot \mathbb{P}_{n-3} = x_1 + \ldots + x_{n-2}$ is non singular (Proposition II.1.4). Therefore we can apply the theorem to restrictions of W_C to generic planes $\mathbb{P}_2 = \check{x}_1 \cap \ldots \cap \check{x}_{n-2} \subset \mathbb{P}_n$ if the degree $d \geqq n+2$ and we get:

Theorem II.1.3. <u>Let</u> C , $C' \subset \mathbb{P}_n$ <u>be irreducible algebraic curves of degree</u> $\geq n+2$ <u>and</u> h <u>be a homeomorphism of the dual dual space</u> $\check{\mathbb{P}}_n$ <u>such that</u> $h(W_C) = W_{C'}$. <u>Then</u> h <u>or its complex conjugate</u> $\bar{h}$ <u>is a projective linear transformation of</u> $\check{\mathbb{P}}_n$.

Corllary 2.1.5 says roughly that a complex structure of a line bundle $L \to C$ on a Riemann surface is determined by a topological structure of a net of effective divisors linearly equivalent to the divisor $D \subset C$ determining L.

In section 2, we investigate some results on the geometry of the singular locus $\Sigma(\mathcal{W}_C)$, some of which are classically known and can be found in [GH,P,We]. A point y is in $\Sigma(\mathcal{W}_C)$ if and only if $y^\vee \cdot C$ is singular or an n-tuple of points in $y^\vee \cdot C$ does not span $y^\vee = \mathbb{P}_{n-1}$. Corresponding to the mutiplicity or degeneracy of $y^\vee \cap C$, we define the filtration $P_n^\vee = \mathrm{envl}^0(\mathcal{W}_C) \supset \mathrm{envl}^1(\mathcal{W}_C) \supset \dots \supset \mathrm{envl}^{n-1}(\mathcal{W}_C) \supset \dots$ and $\mathrm{degn}(\mathcal{W}_C)$ so that $\mathrm{envl}^1(\mathcal{W}_C) \cup \mathrm{degn}(\mathcal{W}_C) = \Sigma(\mathcal{W}_C)$. Then $\mathrm{envl}^1(\mathcal{W}_C)$, $\mathrm{envl}^{n-1}(\mathcal{W}_C)$ $(= C^\vee)$ are called the dual variety and dual curve of C respectively, and $\mathrm{envl}^{i-1}(\mathcal{W}_C) = \mathrm{Tan}(\mathrm{envl}^i(\mathcal{W}_C))$ (Proposition II.2.2), $\mathrm{envl}^i(\mathcal{W}_C)$ is the union of the osculating $n-i-1$ planes of $C^\vee$ and form the duality of the osculating i bundle of C and n-i-1 bundle of $C^\vee$, it follows that $\mathrm{envl}^i(\mathcal{W}_C)$ and $\mathrm{envl}^{n-i-1}(\mathcal{W}_{C^\vee})$ are dual with each other (Proposition II.2.1).

The structure of the set $\mathrm{degn}(\mathcal{W}_C)$ is determined by the various secant varieties of C , but the structure of them seems to be less known even for simple space curves.

Section II .3 is devoted to an introduction of relations of the quasi group structure of C and the geometry of the web $\mathcal{W}_C$, and the Graf-Sauer's theorem.

In the last two sections Section II 4 , 5 , we report the web structure for the exceptional cases $d = n, n+1$ for Theorem II .1.3. First in Section 4, we consider for the case that $C \subset \mathbb{P}_n$ is a non singular curve of degree n , n+1, that is the rational or elliptic normal curve of degree n or n+1 , respectively.

The geometry of elliptic normal curve C_{n+1} of degree n+1 has been histrically studued by many mathematicians. We recall from the paper [H] the $\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}$ symmetry of C_{n+1} which is induced from the Shrodinger representation of Heisenberg group H_{n+1} on $\mathbb{C}^{n+1}$. Theorem II .1.3 suggests that $W_{C_{n+1}}$ may have a stronger topological symmetry than $\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}$. Using the group structure of the elliptic curve C_{n+1} and Abel's theorem, we prove the semi direct product $GL(2,\mathbb{Z}) \ltimes (\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1})$ acts on $W_{C_{n+1}}$ as homeomorphisms of $\mathbb{P}_n$ (Proposition II .4.4). The fact that n+1 torsion points of C_{n+1} are hyper osculating points is already known by Kato [K] and the degree of $\mathrm{envl}^i(W_{C_{n+1}})$ is presented, as a consequence of a general formula by Piene [P].

Any curve of genus 1 , 0 and degree n+1 , n in $\mathbb{P}_m$ is given by projecting the elliptic, rational normal curve of degree n+1 , n from a general center. This correspends, in turn to their webs, to the restrictions of W_C to n-m-1 plane dual to the center (c.f. Proposition II .1.4). This might be of some use for the study of those curves.

By the duality of curves C and $C^\vee = \mathrm{envl}^{n-1}(W_C)$, W_C is reproduced from $C^\vee$, so we can say $\mathrm{envl}^i(W_C)$ all have faithfull information of the original web W_C. So we are led to the geometry of $\mathrm{envl}^1(W_C)$. From another point of view, we can regard $\mathbb{P}_n^\vee$ as the parameter space of the deformation $C \cdot y^{\vee}$, $y \in \mathbb{P}_n^{\vee}$, and then $\mathrm{envl}^1(\mathcal{W}_C)$ is the discriminant (bifurcation) set.

In Section II . 5, we list a result for singular plane cubic curves.

Last of all the author would note that the motivation of this paper was originally a topological classification of non singular vector bundle mappings of bundles of rank n-1 to n . In another paper [N_2], the author proved that topological structure of generic involutive mappings $f:\bar{N} \to \bar{P}$ of involutive manifolds are determined by the differential $df:\mathcal{N}_N \to \mathcal{N}_P$ of the normal bundles of the fixed point sets

$$\begin{array}{ccc} \mathcal{N}_N & \to & \mathcal{N}_P \\ \downarrow & & \downarrow \\ f : N & \to & P \end{array}$$

$N \subset \bar{N}$, $P \subset \bar{P}$, under a certain condition. The results in Chapter II offer a partial answer for this problem.

Section 0. Preliminary in Web geometry.

Let M be a C^r manifold of dimension m, $r = 0,\dots,\infty$ or ω, i.e., real or complex analytic. We call a b-tuple $\mathcal{W} = (\mathcal{F}_1,\dots,\mathcal{F}_b)$ of C^r foliations of M of codimension 1 a b web of M of codimension 1, and we say $\mathcal{W}$ is non-degenerate if $\mathcal{F}_i$ are in a general position. We call a sub tuple $(\mathcal{F}_{i_1},\dots,\mathcal{F}_{i_c})$ a subweb of $\mathcal{W}$. Two b webs $\mathcal{W} = (\mathcal{F}_1,\dots,\mathcal{F}_b)$, $\mathcal{W}' = (\mathcal{F}_1',\dots,\mathcal{F}_b')$ are C^s equivalent if there is a C^s diffeomorphism h of M such that $h(\mathcal{F}_i) = \mathcal{F}_i'$ for $i = 1,\dots,b$. Then we denote $h(\mathcal{W}) = \mathcal{W}'$.

Lemma I.0.1. (Dufour [D_1,D_2]). Let $\mathcal{W}$, $\mathcal{W}'$ be non-degenerate C^r m+1 webs of a real C^r-m manifold M of codimension 1 and h be a homeomorphism of M such that $h(\mathcal{W}) = \mathcal{W}'$. Then h is a C^r diffeomorphism of M for $r = \infty, \omega$. This folds also for germs of m+1 webs.

A C^r-b-web $\mathcal{W}$ is octahedral (hexagonal for $m = 2$) if $\mathcal{W}$ is everywhere locally C^r equivalent to a b web of $\mathbb{R}^m$ or $\mathbb{C}^m$ by foliations with parallel hyperplanes as leaves.
In other words, we can say that $\mathcal{W}$ is octahedral if $\mathcal{W}$ is everywhere locally C^0 equivalent to the octahedral b web by hyperplanes, for the real case of $b \geq m+1$, $r = \infty$ by Dufour's theorem (Lemma I.0.1). Although this equivalence of definitions was already known in [BB]. In the following we introduce the classical results of web geometry along the book [BB] and we restrict ourselves to the case $n = 2$.

Let $\mathcal{W} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ be a nondegenerate C^r-3-web of C^r-2-manifold M defined by nonsingular C^r-1-forms $\omega_1, \omega_2, \omega_3$ with $\omega_1 + \omega_2 + \omega_3 = 0$ and $r = 3, \ldots, \infty, \omega$. Then we see

$$\Omega = \omega_1 \wedge \omega_2 = \omega_2 \wedge \omega_3 = \omega_3 \wedge \omega_1$$

holds and Ω is nonsingular. We define the functions h_i by

$$d\omega_i = h_i \ \Omega \quad ,$$

for $i = 1,2,3$. Then we see

$$\gamma = h_3 \ \omega_2 - h_2 \ \omega_3 = h_1 \ \omega_3 - h_3 \ \omega_1 = h_2 \ \omega_1 - h_1 \ \omega_2$$

and

$$d\omega_i = \gamma \wedge \omega_i \quad ,$$

for $i = 1,2,3$. We define the function k on M by

$$d\gamma = k \ \Omega \quad .$$

Then we see

$$k = h_{2,1} - h_{1,2} = h_{3,2} - h_{2,3} = h_{1,3} - h_{3,1} \quad ,$$

where $h_{i,j} = \partial/\partial x_j \ h_i$.

It is easy to see that the 2-form $d\gamma = k\,\Omega$ is independent of the choice of 1-forms $\omega_1, \omega_2, \omega_3$ defining the foliations $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$, but dependent only on the web $\mathcal{W}$ (§6-8 in [Bl]). We call k, Ω, $d\gamma = k\,\Omega$ as follows:

k : <u>web curvature of</u> $\mathcal{W}$

Ω : <u>surface element of</u> $\mathcal{W}$

$d\gamma = k\,\Omega$: <u>normalized surface element of</u> $\mathcal{W}$.

Let x, y be a local coordinates of M and u_i be a local level C^r functions defining $\mathcal{F}_i$ and W be a C^r-function such that

$$W(u_1, u_2, u_3) = 0 .$$

Then we call W a <u>web function of</u> $\mathcal{W}$ (or u_1, u_2, u_3).

Let $W_{i,j,k} = \partial^3/\partial u_i \partial u_j \partial u_k\, W$ and $\omega_i = W_i \cdot du_i$ for $i,j,k = 1,2,3$. Then k, $d\gamma$, Ω are calculated as follows:

$$\Omega = W_1 W_2 \cdot du_1 \wedge du_2 = W_2 W_3 \cdot du_2 \wedge du_3 = W_3 W_1 \cdot du_3 \wedge du_1 ,$$

$$d\gamma = \frac{1}{2} \sum_{r,s=1}^{3} \frac{\partial^2}{\partial u_r \partial u_s} \cdot \log \frac{W_r}{W_s} \cdot du_r \wedge du_s ,$$

$$k = A_{2,3} + A_{3,1} + A_{1,2} ,$$

$$A_{i,j} = \frac{1}{W_r W_s} \cdot \frac{\partial^2}{\partial u_r \partial u_s} \log \frac{W_r}{W_s}$$

$$= \frac{W_{rrs}}{W_r^2 W_s} - \frac{W_{rss}}{W_r W_s^2} + \frac{W_{rs}}{W_r W_s}\left(\frac{W_{ss}}{W_s^2} - \frac{W_{rr}}{W_r^2}\right) .$$

Theorem I.0.2. Let $\mathcal{W}$ be a nondegenerate C^r-3-web of a C^r-2-manifold M and $r = 3,4,\ldots,\infty, \omega$ (real or complex analytic). Then $\mathcal{W}$ is hexagonal if and only if the normalized surface element $k\,\Omega$ (or the web curvature k) is identically zero on M.

For the proof of this theorem, see e,g, [Bl]. This result was expanded by many authors (see [BB,Ch]).

The geometric meaning of the web curvature k is explicitly explaned in the next section.

1

Section 2. Maps associated with webs: Poincaré map.

The geometric structure of a web is translated into the structure of the translation maps $T^{j,k}_{p,q}$ between two leaves along leaves possing through them transversally (see Fig. 2). These translation maps yeald manu topological invariants.

In this section we study nondegenerate analityc 3-webs of an open neighbourhood U of $0 \in \mathbb{C}^2$, $\mathcal{W} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ defined by level functions $u_i : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. We define $u_{i+3n} = u_i$, $n \in \mathbb{Z}$, $i = 1,2,3$ for a convention. Let $L^i_p = \{p' \in U \mid u_i(p') = u_i(p)\}$ denote the leaf of $\mathcal{F}_i$ passing through the point $p' \in U$. For a point $q \in L^i_p$, and $j,k \neq i$, the <u>translation map</u> $T^{j,k}_{p,q} : (L^j_p, p) \to (L^k_q, q)$ is defined by

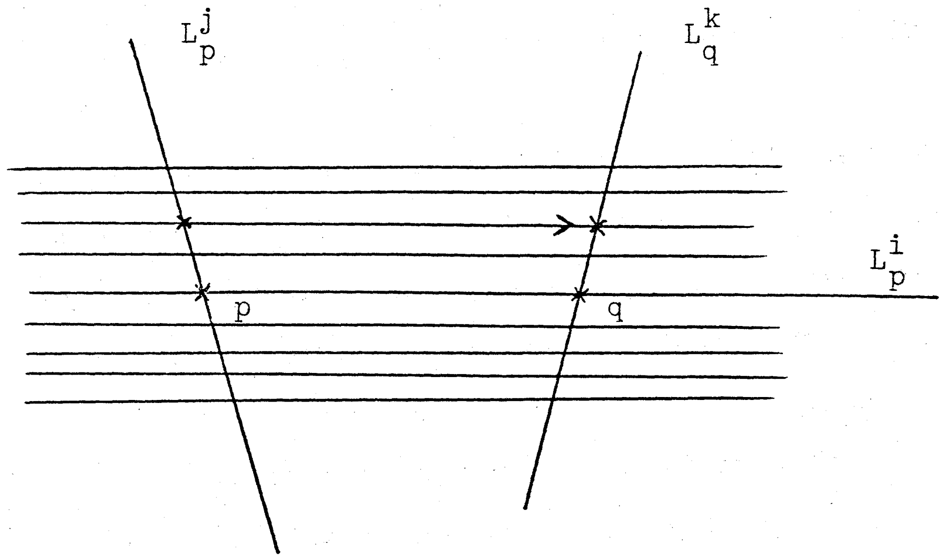
$$T^{j,k}_{p,q} = (u_i \mid L^k_q)^{-1} (u_i \mid L^j_p) \quad .$$

This construction of $T^{j,k}_{p,q}$ is recovered by the geometry of $\mathcal{W}$: $T^{j,k}_{p,q}(r) = s$ if $L^i_p \cap L^k_q = \{s\}$, and assuming $L^i_p \subset U$ is connected, the germ $T^{j,k}_{p,q}$ at p is independent of the choice of level functions u_i. We denote $T^{j.k}_{p,p}$ as $T^{j,k}_p$. Clearly we have

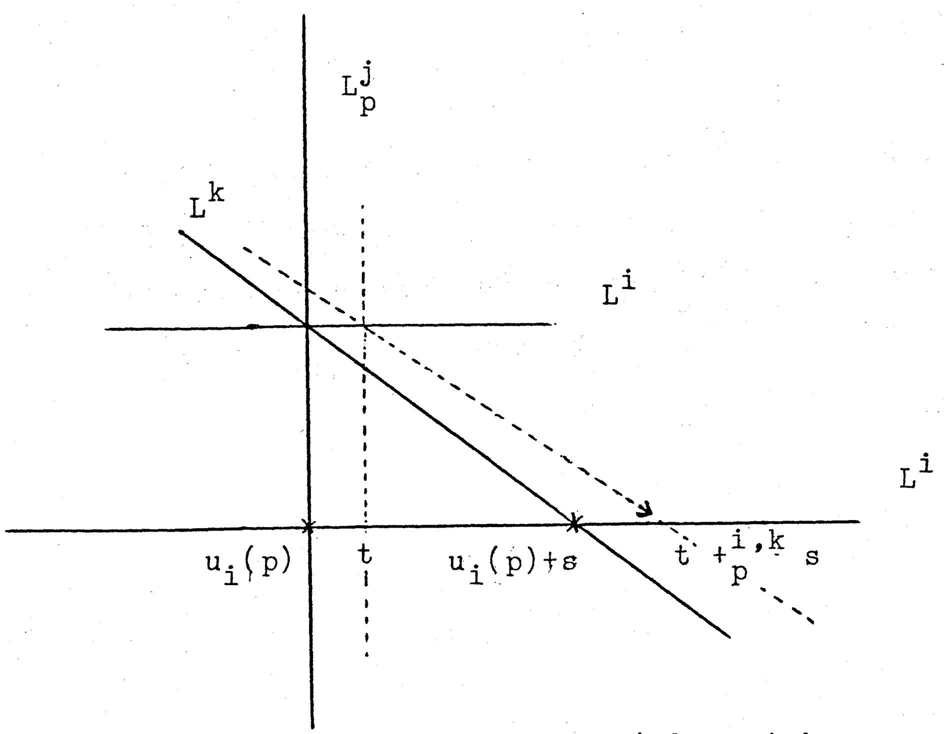
$$T^{j,k}_{p,q} \circ T^{k,j}_{q,p} = \mathrm{id}$$

and

$$u_i \circ T^{j,k}_{p,q} = u^i \quad .$$



The translation map $T_{p,q}^{j,k}$. Figure 2.



The local translation map $A_{p+s}^{i,k} = +_p^{i,k}\ s$. Figure 3.

Next we define the germ (rotation map or Poincaré map) as

$$P_p^{i,j,k} = T_p^{k,j} \circ T_p^{j,k}\, T_p^{i,j} : (L_p^i, p) \to (L_p^i, p) ,$$

for a distinct triple (i,j,k). We denote $P_p^{i,i+1,i+2}$ simply as P_p^i. By the definition we see that

$$P_p^{i,j,k} \circ P_p^{i,k,j} = \mathrm{id}$$

and

$$P_p^{i+1} \circ T_p^{i,i+1} = T_p^{i+3,i+4} \circ P_p^i ,$$

from which we have

$$(u_{i+2}|L_p^{i+1}) \circ P_p^{i+1} \circ (u_{i+2}|L_p^{i+1})^{-1} = u_{i+2} \circ P_p^i \circ (u_{i+2}|L_p^i)^{-1}$$

which we denote simply by

$$\overline{P}_p^{i+2} : (\mathbb{C}, u_{i+2}(p)) \to (\mathbb{C}, u_{i+2}(p)) .$$

We define $\overline{U}_p^{i,j} : (\mathbb{C}, u_i(p)) \to (\mathbb{C}, u_j(p))$ by

$$\overline{U}_p^{i,j} = (u_j|L_p^k) \circ T_p^{j,k} \circ (u_i|L_p^j)^{-1} .$$

Then we have

$$\bar{P}_p^i = \bar{T}_p^{i+2,i+3} \circ \bar{T}_p^{i+1,i+2} \circ \bar{T}_p^{i,i+1} \quad .$$

Next we introduce a local translation map in $\mathbb{C}$ the range of u_i . Let $s \in \mathbb{C}$ be small. Then leaves L_p^j and $u_i^{-1}(u_i(p)+s)$ have a unique intersection q close to p for $i \neq j$. We define the local translation map $A_{p+s}^{i,k} : (\mathbb{C}, u_i(p)) \to (\mathbb{C}, u_i(p)+s)$ by

$$A_{p+s}^{i,k}(t) = u_i \circ T_{p.q}^{k,k} \circ (u_i, u_j)^{-1}(t, u_j(p)) \quad ,$$

for $k \neq i, j$. (see Figure 3). We denote sometimes as

$$A_{p+s}^{i,k}(t) = t +_p^{i,k} s \quad .$$

By an infinitesimal calculation we see that

$$A_{p+s}^{j,k}(u_i(p)+t) = u_i(p) + t + s + R_2(t) \quad ,$$

where R_2 denotes the remainder terms of order $\geqq 2$. Note that

$$A_{p+s}^{i,k} \circ A_{p-s}^{i,k} = \mathrm{id} \quad .$$

For a point $p \in U$, we define the mapping C_p^i : $(\mathbb{C}, u_i(p)) \to (\mathbb{C}, u_i(p))$ by

$$C_p^i(t+u_i(p)) = u_i \circ (u_j,u_k)^{-1}(u_j(u_i,u_k)^{-1}(t+u_i(p),0) ,$$

$$u_k(u_j,u_i)^{-1}(0,u_i(p)+t))$$

and $C_p^{i,j}:(L_p^j,p) \to (L_p^j.p)$ by

$$C_p^{i,j} = (u_i \mid L_p^j)^{-1} \circ C_p^i \circ (u_i \mid L_p^j) \quad .$$

It is easy to see that $C_p^{i,j}$ is independent of the choice of the level functions u_i .

By an infinitesimal calculation, we see that C_p^i is of the form:

$$C_p^i(u_i(p)+t) = u_i(p) + 2t + R_2'(t) \quad ,$$

where R_2' denotes the remainder terms of order $\geqq 2$.

Section 3. Calculation of Poincaré map.

We study a non degenerate analytic 3-web of an open neighbourhood U of $0 \in \mathbb{C}^2$. Fiest we assume that the level functions and the web function are of the following form:

$$(*) \qquad u_1 = x \quad ,$$

$$u_2 = y \quad ,$$

$$u_3(x,y) = \omega(x,y) = x + y + a(x^2y - xy^2) + R_4(x,y)$$

and

$$W(t_1,t_2,t_3) \doteq \omega(t_1,t_2) - t_3 \quad ,$$

where R_4 denotes the remainder terms of order $\geqq 4$ such that $R_4(t,0) = R_4(0,t) = t$ and $R_4(t,t) = 0$. Then

$$L_0^1 = y\text{-axis} \; , \; L_0^2 = x\text{-axis} \; \text{ and } \; L_0^3 = \{\omega(x,y)=0\} \; .$$

Let $(0,y) \in L_0^1$. By the normal form (*), we can easily see that

$$T_0^{1,2}(0,y) = (y,0) \; .$$

Let $T_0^{2,3}(y,0) = (y,y_1)$. Then, by the equality

$$W(u_1,u_2,u_3) = \omega(u_1,u_2) - 0$$

$$= y + y_1 + a(y^2y_1 - yy_1{}^2) + R_4(y,y_1) = 0 \ ,$$

we have

$$y_1 = -y - a(y^2(-y) - y(-y)^2) + R_4'(y)$$

$$= -y + 2ay^2 + R_4'(y) \quad .$$

Clearly

$$T_0^{3,1}(y,y_1) = (0,y_1) \quad ,$$

so we have

(a) $$P_0^{1,2,3}(0,y) = (0,y_1) \quad .$$

and similarly we have

$$P^{2,3,1}(x,0) = (x_1,0) \quad ,$$

$$x_1 = -x + 2ax^3 + R_4'(x) \quad ,$$

hence we have

$$\overline{P}_0^3(t) = -t + 2at^t + R_4''(t) \quad .$$

Next we consider for a general u_i and W .
Let $W_i(0,0,0) = a_i$, $i = 1.2.3$ and $\{W=0\} = \{\omega(t_1,t_2)+a_3t_3 = 0\}$ with an analytic function ω on $\mathbb{C}^2$ such that $\omega_i(0,0) = a_i$, $i = 1.2.$ Then

$$(**) \qquad W(t_1,t_2,t_3) = f(t_1,t_2,t_3)\cdot(\omega(t_1,t_2) + a_3t_3)$$

with an analityc function f with $f(0,0,0) = 1$. Let $u_1'(t) = \omega(t,0)$, $u_2'(t) = \omega(0,t)$ and define the functions f' and ω' by the next commutative diagram:

$$\begin{array}{rcl} f\ ,\ \omega : \mathbb{C}^3 & \longrightarrow & \mathbb{C} \\ (u_1',u_2',\mathrm{id})\ \downarrow & & \| \\ f',\ \omega' : \mathbb{C}^3 & \longrightarrow & \mathbb{C} \end{array} .$$

Then $f'(0,0,0) = 1$ and $\omega'(t,0) = \omega'(0,t) = t$.
Applying Poincaré's lemma to the function

$$(t,t) \longmapsto \omega'(t,t) ,$$

we see that there is an analytic function germ $h:(\mathbb{C},0) \to (\mathbb{C},0)$ such that $h'(0) = 1$ and

$$\omega'(h(t),h(t)) = h(2t) .$$

Define the functions f'' and ω'' by the next commutative diagram:

$$\begin{array}{ccc} f',\omega' : \mathbb{C}^3 & \longrightarrow & \mathbb{C} \\ (h,h,h) \downarrow & & \| \\ f'',\omega'' : \mathbb{C}^3 & \longrightarrow & \mathbb{C} \end{array} \quad .$$

Then $f''(0,0,0) = 1$, $\omega''(t,0) = \omega'(0,t) = t$ and $\omega''(t,t) = 2t$. We can replace the level functions u_i with

$$u_i'' = h \circ u_i' \circ u_i \quad \text{for } i = 1.2$$

$$u_3'' = -h(a_3 u_3) \quad ,$$

and the web function $W = f\cdot(\omega + a_3 t_3)$ with

$$W''(t_1,t_2,t_3) = (\omega''(t_1,t_2) - t_3 \quad .$$

Then

(b) $$W''(u_1'',u_2'',u_3'') = \omega''(u_1'',u_2'') - u_3'' = 0 \quad ,$$

$$\partial u_i''/\partial u_i(0) = a_i \quad , i = 1,2 \quad ,$$

$$\partial u_3''/\partial u_3(0) = - a_3$$

and

$$\omega''(t_1,t_2) = t_1 + t_2 + a(t_1^2 t_2 - t_1 t_2^2) + R_4(t_1,t_2) \quad ,$$

with a number $a \in \mathbb{C}$. By (a), we have

$$(u''_\ell \mid L_0^i) \circ P_0^{i,i+1,i+2} \circ (u'' \mid L_0^i)^{-1}(t)$$

$$= -t + 2at^3 + R'_{4,\ell}(t) \qquad ,$$

$i = 1,2,3,\ \ell \neq i$. By this together with (b), we have

$$(u_\ell \mid L_0^i) \circ P_0^{i,i+1,i+2} \circ (u_\ell \mid L_0^i)^{-1}(t)$$

$$= -t + 2a\ a_\ell^2\ t^3 + R''_{4,\ell}(t) \qquad .$$

Therefore we proved that for any point $(x,y) \in U$,

(c) $$(u_\ell | L^i_{(x,y)}) \circ P^{i,i+1,i+2}_{(x,y)} \circ (u_\ell | L^i_{(x,y)})^{-1}(u_\ell(x,y)+t)$$

$$= u_\ell(x,y) - t + k_\ell(x,y)\ t^3 + R'''_4(t) \qquad ,$$

where k_ℓ is a function on U , $\ell \neq i$. In the following we are calculate the function k_ℓ .

By a direct calculation with the form (**) , we see that the web curvature of W (see Section 0) is

$$k(W)(0,0\ ) = k(\omega + a_3 t_3)(0,0\ )$$

$$= k(\omega\ + a_3 t_3)(0,0\ )$$

$$= k(\omega' + a_3 t_3)(0,0\ )$$

$$= 2a \qquad .$$

Therefore we have, by (c),

$$k_\ell(0,0) = 2a\, a_\ell^2 = k(W)(0,0) \cdot W_\ell^2(0,0,0) \quad .$$

Summaryzing these results above, we have

Proposition I.2.1. _Let_ u_1 , u_2 , u_3 _be level functions of a 3-web_ $\mathcal{W}$ _of_ $\mathbb{C}^2$ _and_ W _be a web function. Then_

$$(u_\ell \mid L^i_{(x,y)}) \circ P^{i,i+1,i+2}_{(x,y)} \circ (u_\ell \mid L^i_{(x,y)})^{-1}(u_\ell(x,y)+t)$$

$$= u_\ell(x,y) - t + k(W)(x,y) \cdot W_\ell^2(u_1,u_2,u_3) \cdot t^3 + R_{4,\ell}(t) \ ,$$

for $\ell \neq i$, _where_ $R_{4,\ell}$ _denotes the remainder terms of order_ ≥ 4 .

Next we prove

Proposition I.2.2. _If a nondegenerate 3-web of an open neighbourhood_ U _of_ $0 \in \mathbb{C}^2$ _with level functions_ u_i _and_ a _web function_ W _is not hexagonal_, i.e. _the web curvature_ $k(W)$ _is not identitcally zero on_ U, _then_ $k(W) \cdot W_i^2(u_1,u_2,u_3)$ _is not constant restricted on a leaf_ L^i_p _for an_ $i = 1,2,3$.

Proof. For simplicity we suppose u_i and W are of the normal form (*) and $k(W)(0,0) = a = 1$ and $k(W) \cdot W_i^2(u_1,u_2,u_3)$ is constant restricted on each leaf L^i_p for $i = 1,2,3$.

Then we have, on the leaf $\{u_1=x_0\}$,

$$k(W)(x_0,y)\cdot W_1^2(x_0,y,\omega(x_0,y)) = k(W)(x_0,0)\cdot W_1^2(x_0,0,x_0)$$

$$= k(W)(x_0,0) \quad ,$$

and on the leaf $\{u_2=y_0\}$,

$$k(W)(x,y_0)\cdot W_2^2(x,y_0,\omega(x,y_0)) = k(W)(0,y_0)\cdot W_2^2(0,y_0,y_0)$$

$$= k(W)(0,y_0) \quad ,$$

from which we have

$$k(W) = \frac{1}{W_1W_2}\cdot\frac{\partial}{\partial x\partial y}\log\left(\frac{W_2}{W_1}\right) \equiv 0 \quad .$$

This is a contradiction to the supposition $k(W)(0,0) = a = 1$.

Therefore we have proven the proposition.

Section 3. Characteristic sets of two function germs on $(\mathbb{C},0)$: stable and unstable sets.

Let P , $C : (\mathbb{C},0) \to (\mathbb{C},0)$ be germs of analytic functions with Taylor expansion $P(z) = z - k z^3 + \dots$ and $C(z) = 2z$. We define the germ $S(P,C)$, $U(P,C)$ at $0 \in \mathbb{C}$ to be the direct limits of the cruster sets as follows;

$$S(P,C) = \varinjlim_{0 \in U:\text{open}} \left\{ \lim_{i \to \infty} \overline{C}^{j(i)}\overline{P}^{i}(z) \,\middle|\, \overline{C}^{j(i)}\overline{P}^{i}(z) \in U \ , \ z \quad U \atop j(i) \to \infty \text{ as } i \to \infty \right\} ,$$

$$U(P,C) = \varinjlim_{0 \in U:\text{open}} \left\{ \lim_{i \to \infty} \overline{C}^{j(i)}\overline{P}^{-i}(z) \,\middle|\, \overline{C}^{j(i)}\overline{P}^{-i}(z) \in U \ , \ z \quad U \atop j(i) \to \infty \text{ as } i \to \infty \right\} ,$$

where $\overline{C}$, $\overline{P}$ are respresentatives of C , P and j runs over the set of all sequences of positive integers such that $j(i) \to \infty$ as $i \to \infty$ and the limit exists. Clearly this is we defined and we have $C(S(P,C)) \equiv S(P,C)$, $C(U(P,C)) = U(P,C)$ and $S(P^{-1},C) = U(P,C)$, $U(P^{-1},C) = S(P,C)$.

The purpose of this section is to prove

Proposition I .3.1. <u>Let</u> $P(z) = z - k z^3 + \dots$, $C(z) = 2z$ <u>be as above and assume</u> $k \neq 0$. <u>Then</u>

$$S(P,C) = \frac{1}{\sqrt{k}} \mathbb{R} \subset \mathbb{C} ,$$

$$U(P,C) = \frac{\sqrt{-1}}{\sqrt{k}} \mathbb{R} \subset \mathbb{C} ,$$

<u>where</u> $\mathbb{R} \subset \mathbb{C}$ <u>denotes the real number field</u>.

In the following we shall analyze the germs $S(P,C)$ and $U(P,C)$.

First we suppose $k = 1$, i.e., $P(z) = z - z^3 + \ldots$ and we analyze in the domain of convergence.

By the Taylor expansion

$$\frac{z}{\sqrt{1+2az^2}} = z - a z^3 + 3a z^5 - 3\cdot 5 a z^7 + 3\cdot 5\cdot 7 a z^9 - \ldots$$

we have

$$\frac{z}{\sqrt{1+2az^2}} < z - z^3 + \ldots < \frac{z}{\sqrt{1+2bz^2}} ,$$

for any sufficiently small real number $z > 0$ and a , b with $0 < b < 1 < a$. Define sequences of real numbers a_i , b_i and c_i by

$$a_{i+1} = \frac{a_i}{\sqrt{1+2aa_i^2}} , \quad b_{i+1} = \frac{b_i}{\sqrt{1+2bb_i^2}} ,$$

$$c_{i+1} = P(c_i) = c_i - c_i^3 + \ldots ,$$

with sufficiently small $a_0 = b_0 = c_0 > 0$. It is easy to see that

$$\frac{1}{a_i^2} = \frac{1}{a_0^2} + 2ai , \quad \frac{1}{b_i^2} = \frac{1}{b_0^2} + 2bi ,$$

by which with the inequality above, we have

(1) $$\frac{1}{\sqrt{2ai+\frac{1}{a_0^2}}} = a_i \leq c_i \leq b_i = \frac{1}{\sqrt{2bi+\frac{1}{b_0^2}}} \quad .$$

Next we claim

(2) <u>Let</u> $z_0 \in \mathbb{C} - 0$, $z_{i+1} = P(z_i)$ <u>and suppose</u> $z_i \to 0$. <u>Then</u>

$$|z_i|^2 > \frac{1}{2ai} \quad ,$$

<u>for</u> $i = 0,1,\ldots$ <u>with some real number</u> $a > 1$.

Proof. By the definition of z_i , $z_{i+1} = z_i - z_i^3 + O(z_i^4)$, we have

$$|z_{i+1}| \geq |z_i| - |z_i|^3 - O'(|z_i|^4) \quad .$$

Applying (1), we have

$$|z_i| \geq \frac{1}{\sqrt{2a'(i-i_0)+\frac{1}{|z_{i_0}|^2}}} \quad ,$$

for an $a' > 1$ and a sufficiently large i_0 and $i = i_0$, ..., from which we have (2).

Furthermore, under the the same condition as (2), we claim

(3) $\arg z_i \to 0$ <u>or</u> π

To prove the claim (3) we prove the following statements (4) - (6).

(4) <u>If</u> $\frac{1}{3}\pi < \arg z_i < \frac{2}{3}\pi$ <u>or</u> $\frac{4}{3}\pi < \arg z_i < \frac{5}{3}\pi$ <u>and</u> $|z_i| \neq 0$ <u>is sufficiently small</u> (i <u>is sufficiently large</u>) <u>then</u> $|z_i| < |z_{i+1}|$.

Proof. By the equality $z_{i+1} = z_i - z_i^3 + O(z_i^4)$, we have

$$|z_{i+1}| \geqq |z_i| + \cos\frac{\pi}{6} \cdot |z_i|^3 + O'(|z_i|^4)$$

$$\geqq |z_i| + \frac{1}{2}\cos\frac{\pi}{6} \cdot |z_i|^3$$

$$= |z_i| + \frac{\sqrt{3}}{4}|z_i|^3$$

$$> |z_i| \quad ,$$

for sufficiently large i .

(5) <u>If</u> $0 < \theta <$ are $z_i < \frac{\pi}{3}$ <u>and</u> $|z_i| \neq 0$ <u>is sufficiently small, then</u>

$$|\arg z_{i+1}| < |\arg z_i| - \frac{1}{4\pi}\sin\theta \cdot |z_i|^2 \quad .$$

Proof. By the definition of z_i , we have

$$\log z_{i+1} = \log (z_i - z_i^3 - 0(z_i^4))$$

$$= \log z_i + \log (1 - z_i^2 + 0'(z_i^3))$$

$$= \log z_i - z_i^2 + 0''(z_i^3) \quad .$$

from which we have

$$|\arg z_{i+1}| = |\frac{1}{2\pi\sqrt{-1}} \operatorname{Im} \log z_{i+1}|$$

$$= |\frac{1}{2\pi\sqrt{-1}} \operatorname{Im} \log z_i - \frac{}{2\pi\sqrt{-1}} \operatorname{Im} z_i^2 + \frac{1}{2\pi\sqrt{-1}} \operatorname{Im} 0'' (z_i^3)|$$

$$< |\arg z_i - \frac{1}{2\pi} |\sin \theta| \; |z_i|^2 + \frac{1}{2\pi\sqrt{-1}} \operatorname{Im} 0''(z_i^3)|$$

$$< |\arg z_i| - \frac{1}{4\pi} \sin \theta \cdot |z_i|^2$$

for sufficiently small z_i , where we take the blanch of $\log z$ such that $\log 1 = 0$.

Similarly to (5) above, we can prove

(6) <u>If</u> $0 < \theta < |\arg z_i - \pi| < \frac{\pi}{3}$ <u>and</u> $z_i \neq 0$ <u>is sufficiently small</u>, <u>then</u>

$$|\arg z_{i+1} - \pi| < |\arg z_i - \pi| - \frac{1}{4\pi} |\sin \theta| \cdot |z_i|^2 \quad .$$

By (4) and (5),(6), we see that if $z_i \to 0$ then

$$0 \leqq |\arg z_i| < \frac{\pi}{3} \quad \text{or} \quad 0 \leqq |\arg z_i - \pi| < \frac{\pi}{3} \quad ,$$

for any sufficiently large i , and

$$\lim_{i \to \infty} \arg z_i \in [-\tfrac{\pi}{3},\tfrac{\pi}{3}] \cup [\tfrac{2}{3}\pi \ \tfrac{4}{3}\pi] \quad .$$

Now we prove the claim (3). We suppose $\arg z_i \to \theta$, $0 < \theta < \frac{\pi}{3}$. Then by (5) and (2), we have

$$|\arg z_{i+1}| < |\arg z_i| - \frac{1}{4\pi} \sin\theta \cdot |z_i|^2 \quad ,$$

$$< |\arg z_i| - \frac{1}{4\pi} \sin\theta \cdot \frac{1}{2ai} \quad .$$

Since $\sum_{k=i_0}^{i} \frac{1}{2ak} \to \infty$ as $i \to \infty$, it then follows that

$$|\arg z_i| \to \infty \quad \text{as} \quad i \to \infty \quad .$$

But this is a contradiction, so we have proven that $\theta = 0$.

Similarly we can prove that if $|\arg z_i| \to \theta$, $0 \leqq |\theta - \pi| < \frac{\pi}{3}$, then $\theta = \pi$. This completes the proof of the claim (3).

Proof of Proposition I.3.1. First we assume that $k = 1$. By the claim (3)

$$\arg P^i(z) \to 0 , \pi ,$$

if $P^i(z) \to 0$. Since the expansion C preserves $\arg P^i(z)$, we see that $S(P,C) \subset \mathbb{R} \subset \mathbb{C}$. To see that $S(P,C) = \mathbb{R} \subset \mathbb{C}$ is an easy exercise with the order of the convergence (1).

Using the coordinate $z' = \sqrt{-1}\, z$, P, C are of the forms:

$$P^{-1}(z') = z' + z'^3 + \dots$$

$$= \sqrt{-1}\,(z - z^3 + \dots) = \sqrt{-1}\, P'(z)$$

$$C(z') = \sqrt{-1}\, C(z) .$$

By this and the statement for $k = 1$, we have

$$U(P,C) = S(P^{-1},C) = \sqrt{-1}\, S(P',C) = \sqrt{-1}\, \mathbb{R} \subset \mathbb{C} .$$

For $k \neq 1, 0$, by the linear coordinate change $h: \mathbb{C} \to \mathbb{C}$, $h(z) = \sqrt{k}\, t$, we can normalize $k = 1$, i.e., $h \circ P \circ h^{-1}(z) = z - z^3 + \dots$. Then by the ststement above for $k = 1$, we have

$$h(S(P,C)) = S(h \circ P \circ h^{-1} , h \circ C \circ h^{-1})$$

$$= S(h \circ P \circ h^{-1}, C)$$

$$= \mathbb{R} \subset \mathbb{C} \qquad ,$$

and similarly we have

$$h(U(P,C)) = \sqrt{-1}\ \mathbb{R} \qquad ,$$

from which we have

$$S(P,C) = \frac{1}{\sqrt{k}}\ \mathbb{R} \ , \quad U(P,C) = \frac{\sqrt{-1}}{\sqrt{k}}\ \mathbb{R} \qquad .$$

This completes the proof of Proposition I.3.1.

Section 4. Proof of Theorem I.4.1.

In this section, we prove the theorem:

Theorem I.4.1. Let $\mathcal{W} = (\mathcal{F}_1, \ldots, \mathcal{F}_{n+1})$, $\mathcal{W}' = (\mathcal{F}'_1, \ldots, \mathcal{F}'_{n+1})$ be germs of nondegenerate analytic $n+1$ -webs of $\mathbb{C}^n$ at 0 of codimension 1 , and assume that for any $i = 1, \ldots, n+1$, there are j , $k \neq i$ such that the restriction of the subwebs $(\mathcal{F}_i, \mathcal{F}_j, \mathcal{F}_k)$, $(\mathcal{F}'_i, \mathcal{F}'_j, \mathcal{F}'_k)$ to the intersections of leaves $\bigcap_{\ell \neq i,j,k} L^\ell_0$, $\bigcap_{\ell \neq i,j,k} L'^\ell_0$, $0 \in L^\ell_0, L'^\ell_0$, $L^\ell_0 \in \mathcal{F}_\ell$, $L'^\ell \in \mathcal{F}'^\ell$ are not hexagonal. Let h be a germ of homeomorphism of $(\mathbb{C}^n, 0)$ such that

$$h(\mathcal{W}) = \mathcal{W}' \quad , \text{ i.e., } \quad h(\mathcal{F}_i) = \mathcal{F}'_i \ , \ i = 1, \ldots, n+1 \quad .$$

Then h or the complex conjugate $\bar{h}$ is a complex analytic diffeomorphism of $(\mathbb{C}^n, 0)$.

Remark. The condition for $\mathcal{W}$, $\mathcal{W}'$ in the theorem is too strong. This is used only for the resuction for the case $n = 2$.

Proof of the theorem. The ststement is for germs of mappings and subsets at the origin 0 in $\mathbb{C}^n$. But throughout the proof, we suppose all mappings and subsets are given by their representatives in an open neighbourhoods cf the origin, and we shall analyze the germs by those representatives. For simplicity we denote sometimes the germs ambiguously as subsets and mappings of $\mathbb{C}^n$ when no confusions occurs by the notations.

Reduction for the case $n = 2$. Let u_i, u_i' be analytic level functions for $\mathcal{F}_i$, $\mathcal{F}_i'$. Then $h(\bigcap_{\ell\neq i,j,k} L_0^\ell) = \bigcap_{\ell\neq i,j,k} L_0^{\ell\prime}$ and h is a homeomorphism of nondegenerate 3-webs $(\mathcal{F}_i,\mathcal{F}_j,\mathcal{F}_k)$, $(\mathcal{F}_i',\mathcal{F}_j',\mathcal{F}_k')$ to the intersections. Applying the statement for $n = 2$ to these 3-webs, we see that the restriction of h or $\bar{h}$ to $\bigcap_{\ell\neq i,j,k} L_0^\ell$ is an analytic diffeomorphism and in particular this induces the diffeomorphisms h_ℓ of $(\mathbb{C},0)$ so that the level functions u_ℓ, u_ℓ' are conjugate:

$$\begin{array}{ccc} u_\ell : (\bigcap_{\ell\neq i,j,k} L_0^\ell ,0) & \longrightarrow & (\mathbb{C},0) \\ \downarrow & & \downarrow\ h_\ell \\ u_\ell' : (\bigcap_{\ell\neq i,j,k} L_0^{\ell\prime},0) & \longrightarrow & (\mathbb{C},0) \end{array} \qquad \text{commutes,}$$

and h_ℓ or $\bar{h}_\ell$ are analitic, for $\ell = i,j,k$.

Since h maps a leaf of $\mathcal{F}_\ell$ to a leaf of $\mathcal{F}_\ell'$ for ℓ $\ell = 1,\ldots,n+1$, the level functions u_ℓ, u_ℓ' are cunjugate by h and h_ℓ :

$$(*)\qquad \begin{array}{ccc} u_\ell : (\mathbb{C}^n,0) & \longrightarrow & (\mathbb{C},0) \\ h\ \downarrow & & \downarrow\ h_\ell \\ u_\ell' : (\mathbb{C}^n,0) & \longrightarrow & (\mathbb{C},0) \end{array}$$

commutes for $\ell = 1,\ldots,n+1$. The result that h_i, h_j, h_k or their conjugate are analytic holds for any choice of i,j,k we see that h_i or $\bar{h}_i$ are uniformly analytic. By the diagram (*), we have

$$h = (u_1',\dots,u_n')^{-1}(h_1,\dots,h_n)(u_1,\dots,u_n) \quad ,$$

so h or $\bar{h}$ is analytic. This proves the implication of $n = 2 \Rightarrow n \geqq 3$. Next we prove for the case $n = 2$.

Proof for the case $n = 2$. First we suppose that the web $\mathcal{W}$, $\mathcal{W}'$ are of the normal form defined by the following level functions with web functions:

(a) $$u_1 = u_1' = x \quad , \quad u_2 = u_2' = y \quad ,$$

$$u_3 = \omega(x,y) = x + y + k(x^2y - xy^2) + R_4(x,y) \quad ,$$

$$u_3' = \omega'(x,y) = x + y + k'(x^2y - xy^2) + R_4'(x,y) \quad ,$$

and

$$W(t_1,t_2,t_3) = \omega(t_1,t_2) - t_3 \quad ,$$

$$W'(t_1,t_2,t_3) = \omega'(t_1,t_2) - t_3 \quad ,$$

where R_4 , R_4' are the remainder terms of order $\geqq 4$ such that $R_4(t,t) = R_4'(t,t) = 0$ and $R_4(t,0) = R_4(0,t) = t$, $R_4'(t,0) = R_4'(0,t) = t$. Then the leaves are $L_0^1 = L_0'^1 =$ y-axis and $L_0^2 = L_0'^2 =$ x-axis in $\mathbb{C}^2$.

We introduce two invariant germs of subsets associated with the web $\mathcal{W}$:

$$S_p^i(\mathcal{W}) = S(((u_{3-i}|L_0^i)\circ P_{0,p}^i(\mathcal{W})\circ(u_{3-i}|L_0^i)^{-1})^2 \quad ,$$

$$(u_{3-i}|L_0^i)\circ C_0^{3,i}(\mathcal{W})\circ(u_{3-i}|L_0^i)^{-1} \quad)$$

$$= S((\overline{P}_p^{3-i}(\mathcal{W})^2 \ , \ C)$$

and

$$U_p^i(\mathcal{W}) = U((\overline{P}_p^{3-i}(\mathcal{W})^2 \ , \ C) \qquad ,$$

for $p \in L_0^i$, $i = 1,2$, where S , U are the stable and unstable sets, $P_{0,p}^i(\mathcal{W})$, $\overline{P}_p^{3-i}(\mathcal{W})$, $C_0^{3,i}(\mathcal{W})$ are the mappings associated with the web $\mathcal{W}$ and $C(z) = 2z$ (Note that $C_0^3(z) = 2z$ by the form of $\mathcal{W}$, so $C_0^{3,1}(\mathcal{W})(0,z) = (0,2z)$ and $C_0^{3,1}(\mathcal{W})(z,0) = (2z,0)$, For the definitions, see Section 2). Note here that $S_0^1(\mathcal{W}) = S_0^2(\mathcal{W})$ and $U_0^1(\mathcal{W}) = U_0^2(\mathcal{W})$ by the definition.

First we assume the following condition (G):

(G) k^- , $k' \neq 0$ and the function $k(W)\cdot W_2(x,0,x)$ restricted on L_0^2 = x axis is non singular at 0 .

Then $\arg k(W)\cdot W_2(x,0,x)$ is non singular at 0 restricted on the real lines S_0^2 or $U_0^2 = \sqrt{-1}\ S_0^2$ as a real valued analytic function. Here we assume the first case (for the other case, the argument goes the same).

By Proposition I.2.1, we have

$$(\bar{P}^2_{(x,0)})^2(t) = t - 2k(W)(x,0)\cdot W_2^2(x,0,x)\cdot t^3 + R_4(x,0)(t),$$

where $k(W)$ is the web curvature of u_i, W and $k(W)(0,0)\cdot W_2(0,0,0) = k(W)(0,0) = k$, $W_i = \partial W/\partial t_i$.
By Proposition I.3.1, we have

$$S^1_{(x,0)} = \frac{1}{\sqrt{2k(W)(x,0)\cdot W_2^2(x,0,x)}} \quad \mathbb{R} \subset \mathbb{C} ,$$

$$U^1_{(x,0)} = \frac{\sqrt{-1}}{\sqrt{2k(W)(x,0)\cdot W_2^2(x,0,x)}} \quad \mathbb{R} \subset \mathbb{C} ,$$

and similarly we have

$$S^2_{(0,y)} = \frac{1}{\sqrt{2k(W)(o,y)\cdot W_1^2(0,y,y)}} \quad \mathbb{R} \subset \mathbb{C} ,$$

$$U^2_{(0,y)} = \frac{\sqrt{-1}}{\sqrt{2k(W)(0,y)\cdot W_1^2(0,y,y)}} \quad \mathbb{R} \subset \mathbb{C} .$$

We define the following real analytic mappings.

$$M_1 : (\mathbb{R} \times S_0^2,(0,0)) \to (\mathbb{C},0) ,$$

$$M_2 : (S_0^1 \times U_0^1,(0,0)) \to (\mathbb{C},0) ,$$

$$M_3 : (U_0^1 \times S_0^1,(0,0)) \to (\mathbb{C},0) ,$$

by

$$M_1(\lambda,x) = \frac{\lambda}{\sqrt{2k(W)(x,0)\ W_2^2(x,0,x)}} , \lambda \in \mathbb{R} , \quad x \in S_0^2 ,$$

$$M_2(x,y) = x +_0^{2,3} y \ , (x,0) \in S_0^1 , \quad y \in U_0^1 ,$$

$$M_3(x,y) = x +_0^{2,3} y \ , (x,0) \in U_0^1 , \quad y \in S_0^1 ,$$

where $+_0^{2,3}$ is the local translation map of the range of u_2 defined in Section 1, and we denote

$$C_x^1 = M_1(\mathbb{R} ,x) \ , \ C_y^2 = M_2(S_0^1,y) \ , \ C_y^3 = M_3(U_0^1,y)$$

and G_1 , G_2 , G_3 be the collections of manifolds

$$G_1 = \{C_x^1 \mid x \in S_0^2\} , \ G_2 = \{C_y^2 \mid y \in U_0^1\} , \ G_3 = \{C_y^3 \mid y \in S_0^1\} .$$

Proposition I.4.2. Assume that the real valued function $\arg k(W)(x,0)\cdot W_2^2(x,0,x)$ restricted to the real line $S_0^2(W)$ is topologically non singular at $0 \in \mathbb{C}^2$. Then G_2 , G_3 are germs of real analytic foliations of $\mathbb{C} = \mathbb{R}^2$ of codimension 1 , and G_1 forms a real analytic foliation of codimension 1 on a germ of deleted neighbourhood U of $S_0^1 - 0$ in $\mathbb{C}$ at the origin $0 \in \mathbb{C}^2$, on which (G_1,G_2,G_3) forms a nondegenerate 3-web, where we mean by a germ of deleted neighbourhood U a germ of a subset at $0 \in \mathbb{C}^2$ represented by a set of the form $U' - (S_0^1-0)$ such that U' is an open neighbourhood of $S_0^1 - 0$ in $\mathbb{C}$ at the origin.

Proof. Since S_0^1, $U_0^1 = \sqrt{-1}\, S_0^1$ are real lines and $dM_i(0) = id : T_0\mathbb{C} \to T_0\mathbb{C}$ for $i = 2,3$, M_2, M_2 are germs of real analytic diffeomorphisms and G_2, G_3 are germs of nonsingular real analytic foliations of codimension 1, and clearly G_1 and G_3 are in general position. So we consider for G_1 and G_2.

Since G_2 is real analytic, the singular point set

$$\Sigma = \{(\lambda,x) \in \mathbb{R}\times S_0^2 \mid \text{the leaf } C_x^1 \text{ is not transversal to the foliation } G_2^2 \text{ at } M_1(\lambda,x)\}$$

is real analytic. It is easy to see that if $\Sigma = \mathbb{R} \times S_0^2$ then $C_x^1 = S_0^1$ for any $x \in S_0^2$. However $\arg k(W)(x,0)\cdot W_2^2(x,0,x)$ is topologically nonsingular at 0 S_0^2, so M_1 is an open map beside the subset $0 \times S_0^2$ by the form of M_1. Therefore we see that Σ is a proper real analytic subset and there is a germ of deleted neighbcurhood U of $(\mathbb{R} - 0) \times 0$ in $(\mathbb{R} - 0) \times S_0^2 - \Sigma$ at 0×0 and the foliations G_1, G_2 are in general position on the germ of deleted neighbourhood $M_1(U)$ of $S_0^1 - 0$ in $\mathbb{C}$ at 0. This proves Proposition I.4.2.

Now we prove Theorem I.4.1. The following is a part the theorem.

Proposition I.4.3. <u>Assume the condition</u> G <u>and other assumptions above</u>. <u>Let</u> h , h_i <u>be germs of homeomorphisms such that the following diagram commutes</u>:

$$(**) \qquad \begin{array}{ccc} u_i : (\mathbb{C}^2,0) & \longrightarrow & (\mathbb{C}^1,0) \\ h \downarrow & & \downarrow h_i \\ u_i' : (\mathbb{C}^2,0) & \longrightarrow & (\mathbb{C}^1,0) \end{array} ,$$

<u>for</u> $i = 1,2,3$. <u>Then</u> $h_1 = h_2 = h_3$ <u>and</u> $h = (h_1,h_2)$ <u>and</u> h , h_i <u>or</u> <u>their conjugates</u> $\bar{h}$, $\bar{h}_i$ <u>are complex analytic diffeomorphisms</u>.

Proof. It is clear that $h_1 = h_2 = h_3$ and $h = (h_1,h_2)$ hold by the normal form (a). We have only to prove h_2 is complex analytic diffeomorphism at 0 $\mathbb{C}$.

Recall that the real analytic 3-webs $G = (G_1,G_2,G_3)$, $G' = (G_1',G_2',G_3')$ of codimension 1 of $\mathbb{C}$ are constructed by purely topological structure of the webs $\mathcal{W}$, $\mathcal{W}'$, so we see that

$$h_2(G_i) = G_i' \quad ,$$

for $i = 1,2,3$ and especially we have

$$h_2(S^1_{(x,0)}(W)) = S^1_{(h_1(x),0)}(W') \quad ,$$

$$h_2(U^1_{(x,0)}(W)) = U^1_{(h_1(x),0)}(W')$$

and

$$h_1(S_0^2(W)) = S_0^2(W') \quad ,$$

from which, with Proposition I.3.1, we have

$$h_2\left(\frac{\mathbb{R}}{\sqrt{2k(W)(x,0)\cdot W_2^2(x,0,x)}}\right) = \frac{\mathbb{R}}{\sqrt{2k(W')(h_1(x),0)\cdot W_2'^2(h_1(x),0,h_1(x))}}$$

$$h_2\left(\frac{\sqrt{-1}\ \mathbb{R}}{2k(W)(x,0)\cdot W_2^2(x,0,x)}\right) = \frac{\sqrt{-1}\ \mathbb{R}}{2k(W')(h_1(x),0)\cdot W_2'^2(h_1(x),0,h_1(x))}$$

for $x \in S_0^2(W) \subset \mathbb{C}$. Since the real valued function $\arg\ k(W)(x,0)\cdot W_2^2(x,0,x)$ restricted to the real line $S_0^2(W) \subset \mathbb{C}$ is nonsingular at $0 \in \mathbb{C}$, the function $\arg\ k(W')(x,0)\cdot W_2'^2(x,0,x)$ restricted to $S_0^2(W') \subset \mathbb{C}$ is also topologically nonsingular at $0 \in \mathbb{C}$ for $h_2(G_1) = G_1'$. So, by Proposition I.4.2, (G^1,G^2,G^3), (G_1',G_2',G_3') form nondegenerate real analytic 3-webs of codimension 1 on germs of deleted neighbourhood U, U' of $S_0^1(W) - 0$, $S_0^1(W') - 0$ in $\mathbb{C}$ at the origin. By Dufour's theorem (Lemma I.0.1), h_2 is a real analytic diffeomorphism restricted on the non-empty set $U \cap h_2^{-1}(U')$.

By the diagram (**), we have the following commutative diagram:

$$(***) \qquad \begin{array}{cccc} +_0^{2,3}(W) & x & : (\mathbb{C},0) \longrightarrow & (\mathbb{C},x) \\ & & h_2 \downarrow & \downarrow h_2 \\ +_0^{2,3}(W') & h_2(x) & : (\mathbb{C},0) \longrightarrow & (\mathbb{C},h_2(x)) \end{array} \quad ,$$

for any point $x \in \mathbb{C}$. By this diagram, we see that

h_2 <u>is a real analytic diffeomorphism of</u> $(\mathbb{C},0)$.

Since the homeomorphism h_2 carries all right angles in $\mathbb{C}$ formed by real lines $S^1_{(x,0)}(W)$ and $U^1_{(x,0)}(W) = \sqrt{-1}\, S^1_{(x,0)}(W)$ passing through $0 \in \mathbb{C}$ for $x \in S^2_0$ to the right angles of $S^1_{(h_1(x),0)}(W')$ and $U^1_{(h_1(x),0)}(W')$ at $0 \in \mathbb{C}$, we see that h_2 or the complex conjugate $\overline{h}_2$ is conformal at $0 \in \mathbb{C}$, respectively whether h_2 is orientation preserving or not, and again by the diagram (***), we see that h_2 or $\overline{h}_2$ is conformal on a neighbourhood of $0 \in \mathbb{C}$. Then Reimann's theorem says that h_2 or $\overline{h}_2$ is complex analytic at $0 \in \mathbb{C}$.

This completes the proof of Proposition I.4.3, which is a particular case of Theorem I.4.1 for $n = 2$ and the condition (G) holds. Next we consider for general case of $n = 2$.

Since the curvatures $k(W)$, $k(W')$ of $\mathcal{W}$, $\mathcal{W}'$ are not identitically zero on a neighbourhood of the origin $0 \in \mathbb{C}^2$, by Theorem I.o.1 and the assumption of Theore I.4.1, there is a point $p \in \mathbb{C}^2$ sufficiently close to the origin such that $k(W)(p)$, $k(W')(h(p)) \neq 0$. By Proposition I.2.2, $k(W)\cdot W^2_i(u_1,u_2,u_3)$ is not constant on a leaf $L^i_q = u_i^{-1}(u_i(q))$ for a sufficiently small $u_i(q)$ and an $i = 1,2,3$. So we may assume, in addition, that $k(W)\cdot W^2_2(u_1,u_2,u_3)$ is nonsingular at p restricted on L^i_p. This property inherits after reforming the functions u_i to the normal form (a). Applying Proposition I.4.3 for the reformed webs, we see that h $(\overline{h})$ is complex analytic at $p \in \mathbb{C}^2$ and h_j $(\overline{h}_j)$ is complex analytic at $u_j(p) \in \mathbb{C}$ for $j = 1,2,3$. Again by the diagram (***):

$$(***) \qquad \begin{array}{ccc} +_0^{j,k}\, u_i(p) & : (\mathbb{C},0) \longrightarrow & (\mathbb{C},u_j(p)) \\ & h_j \downarrow & \downarrow h_j \\ +_0^{j,k}\, u_i'(h(p)) & : (\mathbb{C},0) \longrightarrow & (\mathbb{C},u_j'(h(p))) \end{array} ,$$

(we gave in the proof of Proposition I.4.3), we see that h_j $(\overline{h}_j)$, $j = 1,2,3$ are complex analytic at $0 \in \mathbb{C}$ hence $h = (u_1,u_2)^{-1}(h_1,h_2)(u_1,u_2)$ (or $\overline{h}$) is complex analytic at the origin $0 \in \mathbb{C}^2$.

This completes the proof of Theorem I.4.1.

Chapter II . Application to Projective geometry of Projective space curves.

In this chapter we study geometric structure of d-<u>webs</u> W_C <u>generated by</u> projective curves $C^d \subset \mathbb{P}_n$ of degree d. W_C is defined to be the collection of dual hyperplanes $y^\vee = \mathbb{P}_{n-1} \subset \mathbb{P}^\vee_n$, $y \in C$. Then W_C forms a complex analytic d-web beside the algebraic singular locus $\Sigma(W_C) \subset \mathbb{P}^\vee_n$. The structure of $\Sigma(W_C)$ is also studied in the descending sections.

Section 1. Proof of Theorem II .1.3.

We say two webs W_C , $W_{C'}$ generated by algebraic curves C , $C' \subset \mathbb{P}_n$ are <u>topologically equivalent</u> if there is a homeomorphism h of $\mathbb{P}_n$ such that for any leaf $x^\vee$, $x \in C$ of W_C , the image $h(x^\vee)$ is a leaf $x'^\vee$ for an x' C' . Then we denote $h: W_C \to W_{C'}$ or $h(W_C) = W_{C'}$. Our problem is to classify all webs W_C up to this equivalence relation, which is a classification of projective curves C.

In this chapter, we denote by $P(x_1,\dots,x_a)$ the subspace spanned by $x_1,\dots,x_a$ in $\mathbb{P}_n$ and denote by $W_C(x_1,\dots,x_a)$ the restriction of W_C to $P(x_1,\dots,x_a)$.

We define two singular sets $\mathrm{envl}(W_C)$ and $\mathrm{degn}(W_C)$ as follows:

$$\mathrm{envl}(\mathcal{W}_C) = \{x \in \mathbb{P}_n^\vee \mid x^\vee \text{ has a contact with } C \text{ at a smooth point or } x^\vee \cap \Sigma(C) \neq \emptyset\}$$

$$= \{x \in \mathbb{P}_n^\vee \mid m_{x'}(x^\vee, C) \geqq 2 \text{ for an } x' \in x^\vee\}$$

$$= \{x \in \mathbb{P}_n^\vee \mid \text{the geometric number of points of } x^\vee \cap C \text{ is less than } d\} \quad ,$$

$$\mathrm{degn}(\mathcal{W}_C) = \overline{\{x \in \mathbb{P}_n^\vee \mid x^\vee \cap C \text{ is degenerate in } x^\vee = \mathbb{P}_{n-1}\}},$$

where "degenerate" means that
some distinct n points $x_1, \ldots, x_n \in x^\vee \cap C$ are coplanner in $x^\vee$, i.e., $x_1, \ldots, x_n$ does not span $x^\vee$.
The variety $\mathrm{envl}(\mathcal{W}_C)$ is known as the <u>dual valiety</u> of C defined similarly to the dual plane curve (see [L,W]).

The detailed structure of $\Sigma(\mathcal{W}_C)$ are investigated in the next section. First we offer the following proposition.

Proposition II.1.1. $\Sigma(\mathcal{W}_C) = \mathrm{degn}(\mathcal{W}_C) \cup \mathrm{envl}(\mathcal{W}_C)$.

Proof. Let the multiplicity be $m_{x_i}(x^\vee, C) = m_i$ for $x_i \in x^\vee \cap C$. Then the geometric number of points of the intersection $x^\vee \cap C$ is $d' = d - \Sigma(m_i - 1)$. This shows that just d' leaves of $\mathcal{W}_C$ are passing through x . So we have $\mathrm{envl}(\mathcal{W}_C) \subset \Sigma(\mathcal{W}_C)$.

Let $x \notin \mathrm{envl}(\mathcal{W}_C)$. Then $m_i = 1$ for any $x_i \in x^\vee \cap C$ and $x^\vee$ meets transversally to C at distinct d points

$x_1,\dots,x_d$ and the germs (C,x_i) generate germs of nonsingular foliations $\mathcal{F}_i$ at x, which form a nondegenerate d-web of codimension 1 if and only if $x^\vee \subset C$ is nondegenerate in $x^\vee = \mathbb{P}_{n-1}$. This proves the proposition.

Proposition II.1.2. Let C, $C' \subset \mathbb{P}_n$ be projective curves and h a homeomorphism of the dual space $\mathbb{P}_n^\vee$ such that $h(\mathcal{W}_C) = \mathcal{W}_{C'}$. Then h induces the homeomorphism $h^\vee : C \to C'$ by $h(x^\vee) = h^\vee(x)^\vee$ for $x \in C$, which possesses the properties $h(P(x_1,\dots,x_{n-2})^\vee) = P(h(x_1),\dots,h(x_{n-2}))^\vee$ and $h(\mathcal{W}_C(x_1,\dots,x_{n-2})) = \mathcal{W}_{C'}(h^\vee(x_1),\dots,h^\vee(x_{n-2}))$.

Proof. Clearly $h^\vee$ is a continuous map of C into C', and $(h^{-1})^\vee \circ h^\vee = id$ holds by definition. So $h^\vee$ is a homeomorphism. Since $P(x_1,\dots,x_{n-2})^\vee \cap x^\vee = P(x_1,\dots,x_{n-2},x)$ for $x \in C$, and $h(P(x_1,\dots,x_{n-2},x)) = P(h^\vee(x_1),\dots,h^\vee(x_{n-2}),h^\vee(x))$, we have $h(\mathcal{W}_C(x_1,\dots,x_{n-2})) = \mathcal{W}_C(h^\vee(x_1),\dots,h^\vee(x_{n-2}))$.

Now we state our main theorem in this chapter.

Theorem II.1.3. Let C , $C' \subset \mathbb{P}_n$ be algebraic curves in the projective n-space $(n \geqq 2)$ of degree d and W_C , $W_{C'}$ be the d-web generated by C , C' , respectively, and h be a homeomorphism of the dual space $\mathbb{P}_n^{\vee}$ such that $h(W_C) = W_{C'}$. If C , C' are irreducible and nondegenerate, i.e., are not contained in a hyperplane, and $d \geqq n + 2$, then h or the complex conjugate $\bar{h}$ is a projective linear transformation of $\mathbb{P}_n^{\vee}$ and in particular C' is isomorphic to C or the conjugate $\bar{C}$: the induced homeomorphism $h^{\vee}$ is $h^{\vee} = ({}^t h)^{-1} : C \to C'$ or $\overline{({}^t\bar{h})}^{-1}$: $C \to \bar{C}' \to C'$

Proof. Let $y \in \mathbb{P}_n^{\vee} - \Sigma(W_C)$ and $\{x_1,\dots,x_d\} = y^{\vee} \cap C$ and $\pi : \mathbb{P}_n \to \mathbb{P}_2$ be the projection with the center $\mathbb{P}_{n-3} = P(x_1,\dots,x_{n-2})$ to $P(x_1,\dots,x_{n-2})^{\vee *}$, where $*$ denotes the dual projective space of itself as $\mathbb{P}_2$ not in $\mathbb{P}_n$. The closure of the image $\pi(C - x_1,\dots,x_{n-2}) \subset \mathbb{P}_2$ is again an irreducible and nondegenerate algebraic curve of degree $d - (n-2)$, which we denote by $C(x_1,\dots,x_{n-2})$. Since $d \geqq n + 2$, we have $d - (n-2) \geqq 4$

Now we prove

Proposition II.1.4. The restriction $W_C(x_1,\dots,x_{n-2})$ is a web of $\mathbb{P}_2 = P(x_1,\dots,x_{n-2})^{\vee}$ generated by the algebraic curve $C(x_1,\dots,x_{n-2}) \cap \mathbb{P}_2 = P(x_1,\dots,x_{n-2})^{\vee *}$ (dual space).

Proof. The leaves $P(x_1,\dots,x_{n-2})^{\vee} \cap x^{\vee} = P(x_1,\dots,x_{n-2},x)^{\vee}$, $x \in C$ of $W_C(x_1,\dots,x_{n-2})$ are the intersections of the dual hyperplane of $\pi(x)$ in $\mathbb{P}_n$ with $P(x_1,\dots,x_{n-2})$. So we see that $\mathcal{W}_C(x_1,\dots,x_{n-2}) = \mathcal{W}_{C(x_1,\dots,x_{n-2})}$. This proves the proposition.

Therefore we can apply Graf-Sauer's theorem to the algebraic plane curve $C(x_1,\dots,x_{n-2})$ and the generated $d-(n-2)$ -web of the intersection $\mathbb{P}_2 = P(x_1,\dots,x_{n-2}) \subset \check{\mathbb{P}}_n$ of leaves $\check{x}_i$, and consequently we see that any 3-subweb of $\mathcal{W}_C(x_1,\dots,x_{n-2})$ is nowhere hexagonal beside the singular set $\Sigma(\mathcal{W}_C(x_1,\dots,x_{n-2}))$.

Since $\Sigma(\mathcal{W}_C) = \mathrm{degn}(\mathcal{W}_C)$ is defined pure topologically, $h(\Sigma(\mathcal{W}_C)) = \Sigma(\mathcal{W}_{C'})$ holds. Therefore we can apply Theorem I.4.1, and consequently we see that h or the conjugate $\bar{h}$ is complex analytic beside the singular set $\Sigma(\mathcal{W}_C)$, which is a proper subvariety of $\check{\mathbb{P}}_n$ for C is nondegenerate. By Hatog's extension theorem, h or $\bar{h}$ must be a complex analytic automorphism of $\check{\mathbb{P}}_n$ hence a projective linear transformation of $\check{\mathbb{P}}_n$.

The other statement is easy to see. This completes the proof of Theorem II .1.4.

Using the usual langage in algebraic geometry, the theorem can be rephrased as follows.

Corollary II.1.5. *Let* C, C' *be Reimann surfaces and* E, E' *be linear systems of effective divisors of degree* d *with no base points such that the associated morphisms* $\iota_E : C \to E^\vee$, $\iota_{E'} : C' \to E'^\vee$ *are birational and* $d-2 \geqq \dim E = \dim E' \geqq 2$. *Suppose that there is a homeomorphism* $h : C \to C'$ *such that* $h(E) = E'$, i.e., $h(\Sigma a_i x_i) = \Sigma a_i \cdot h(x_i) \in E'$ *for any* $\Sigma a_i x_i \in E$. *Then* $h : C \to C'$ *is holomorphic or anti-holomorphic diffeomorphism respectively whether* h *is orientation preserving or not*.

Proof. We identify the complete linear system $|D|$, $D \in E$ as $\mathbb{P}(H^0(C,\mathcal{O}(|D|))$ by $\Sigma a_i x_i \in |D| \leftrightarrow s \in H^0(C,\mathcal{O}(|D|))$ with $s^{-1}(0) = \Sigma a_i x_i$, and we suppose $E \subset |D| = \mathbb{P}_{\dim |D|}$.
The morphism $\iota_E : C \to E^\vee$ is defined by $x \in C \to H_x^\vee$, where $H_x = \{\Sigma a_i x_i \in E \mid x_i = x \text{ for an } i\} \subset E$. Then the image $\tilde{C} \subset E^\vee$ of C is a nondegenerate curve of degree d which generates the d-web $\mathcal{W}_{\tilde{C}}$ on E by the leaves H_x, $x \in C$.

The homeomorphism $h : C \to C'$ preserves the linear systems E, E' so h induces a homeomorphism $h^\vee : E \to E'$ which maps a leaf H_x, $x \in C$ to a leaf $H_{h(x)}$, $h(x) \in C'$, therefore $h^\vee(\mathcal{W}_{\tilde{C}}) = \mathcal{W}_{\tilde{C}'}$. Then applieing Theorem II.1.3, we see that $h^\vee$ or the complex conjugate $\bar{h}^\vee$ is a projective linear transformation and $({}^t h^\vee)^{-1}$ or $({}^t \bar{h}^\vee)^{-1}$ is a transformation of $\tilde{C}$ to $\tilde{C}'$, which lifts to an isomorphism of C to C' that is the original homeomorphism h.

This completes the corollary.

Riemann-Roch theorem says that $\dim|D| = d + 1 - g + \dim|K-D|$, where K is the canonical divisor and g is the genus of C. So, if $d \geqq g + 2$ then $\dim|D| \geqq 2$ and a linear system E of dimension 2 (net) exists. So roughly to say, a complex structure of a Riemann surface C of genus g is determined by a 2-dimensional family of linearly equivalent $g+2$ $(g \geqq 2)$ or 4 $(g = 1)$ point subsets of C.

Section 2. Structure of the envelope set $envl(\mathcal{W}_C)$: the dual space curves and the dual webs.

In this section, we turn into study of the envelope set $envl(\mathcal{W}_C) \subset \mathbb{P}_n^\vee$ of the web $\mathcal{W}_C$ generated by the projective curve $C \subset \mathbb{P}_n$.

Let $\phi : \tilde{C} \to C$ be the normalization and $\tilde{\Phi} = (\phi_0, \ldots, \phi_n : \tilde{C} \to \mathbb{C}^{n+1} - 0$ be a local lift of ϕ, and suppose that $\tilde{\Phi}$ is nondegenerate i.e., the Wronskian $W(\phi_0, \ldots, \phi_n)$ is not identically zero on $\tilde{C}$, or in other words, C is not contained in a hyperplane. Let $C_{reg} = C - \text{sing } C$, $C_0 = C_{reg} - \phi(W^{-1}(0))$ and $\tilde{C}_0 = \tilde{C} - W^{-1}(0)$.

Let $\Sigma^{1,\overset{i}{\ldots},1}(\mathcal{W}_C) \subset envl(\mathcal{W}_C) \subset \mathbb{P}_n^\vee$ be the set of points y of which dual hyperplane $y^\vee$ has a contact with C_0 of order $\geq i+1$, and $envl^i(\mathcal{W}_C) = \overline{\Sigma}^{1,\overset{i}{\ldots},1}(\mathcal{W}_C)$ (closure). The <u>osculating</u> i-<u>plane</u> $Osc^i\, C_{\phi(t)}$ of C at $\phi(t)$ C_0 is the i-plane $\mathbb{P}_i$ which has a contact with C at $\phi(t)$ of order $\geq i+1$, which is i-space spaned by the points $\phi(x), \ldots, \phi^{(i-1)}(t)$ if $\text{rank}\begin{pmatrix}\phi_0, \ldots, \phi_n \\ \phi_0^{i-1}, \ldots, \phi_n^{i-1}\end{pmatrix}(t) = i$ or especially $W(\phi_0, \ldots, \phi_n)(t) \neq 0$. The osculating i-planes give the i-bundle: $Osc^i\, C \to \tilde{C}$ over $\tilde{C}$, which we call the <u>osculating</u> i-<u>bundle</u> of C, and we denote the restriction over $\tilde{C}_0$ by $Osc^i\, C_0$.

By the definition, we see that $\Sigma^{1,\overset{i}{\ldots},1}(\mathcal{W}_C)$, $envl^i(\mathcal{W}_C)$ are the unoin of the dual spaces of $Osc^i\, C_x$, $x \in C_0$, C, respectively, and we have

$$\mathbb{P}_n^{\vee} = \Sigma^0(\mathcal{W}_C) \supset \Sigma^1(\mathcal{W}_C) \supset \ldots \supset \Sigma^{\overbrace{1,\ldots,1}^{n-1}}(\mathcal{W}_C) \supset \ldots \quad ,$$

$$\mathbb{P}_n^{\vee} = \mathrm{envl}^0(\mathcal{W}_C) \supset \mathrm{envl}^1(\mathcal{W}_C) \supset \ldots \supset \mathrm{envl}^{n-1}(\mathcal{W}_C) \supset \ldots \quad .$$

In the later we will find that $\dim \Sigma^{\overbrace{1,\ldots,1}^{i}}(\mathcal{W}_C) = n - i$, $i = 1,\ldots,n$. The varieties $\mathrm{envl}^1(\mathcal{W}_C)$, $\mathrm{envl}^{n-1}(\mathcal{W}_C)$ are known as the dual varicty and the dual curve of C , respectively, so we denote $\mathrm{encl}^{n-1}(\mathcal{W}_C) = C^{\vee}$ which is given by the local mapping:

$$\phi^{\vee} = (W_0(\phi_0,\ldots,\phi_n):\ldots:W_n(\phi_0,\ldots,\phi_n)) \quad ,$$

where $W_i(\phi_C,\ldots,\phi_n) = (-1)^i \, W(\phi_0,\ldots,\phi_{i-1},\phi_{i+1},\ldots,\phi_n)$. By an easy calculation we have

$$\sum_{i=0}^{n} \phi_i^{(k)} \cdot W_i^{(\ell)} \equiv 0 \quad , \quad 0 \leqq k + \ell \leqq n - 1$$

$$\sum_{i=0}^{n} \phi_i^{(k)} \cdot W_i^{(\ell)} = (-1)^n \cdot W(\phi_0,\ldots,\phi_n) \quad , \quad k + \ell = n \, ,$$

from which we have

$$W(\phi_0,\ldots,\phi_n) \cdot W(W_0,\ldots,W_n) = (-1)^N \cdot W(\phi_0,\ldots,\phi_n)^{n+1} \quad ,$$

so we see that

$$W(\phi_0,\ldots,\phi_n)(t) = 0 \longleftrightarrow W(W_0,\ldots,W_n)(t) = 0 \quad ,$$

and by Cramer's rule, we have

$$\phi_i \cdot W(W_0,\ldots,W_n) = (-1)^{N'} \cdot W_i(W_0,\ldots,W_n) \cdot W(\phi_0,\ldots,\phi_n) ,$$

This shows the duality of the correspondence of space curves:

$$\phi(\tilde{C}) = C \longleftrightarrow C^{\vee} = \phi^{\vee}(\tilde{C}) ,$$

and bundles

$$(\mathrm{osc}^i\ C)^{\vee} = \mathrm{Osc}^{n-i-1}\ C^{\vee}$$

over $\tilde{C}$ and

$$\mathrm{envl}^i(\omega_C) = P^{\vee}(\mathrm{Osc}^{n-i-1}\ C^{\vee}) ,$$

where $P^{\vee}$ denotes the natural projection into the dual space $\mathbb{P}_n^{\vee}$.

Since the projection $P^{\vee} : (\mathrm{Osc}^0\ C_0)^{\vee} = \mathrm{Osc}^{n-1}\ C_0^{\vee} = \gamma \to \mathbb{P}_n^{\vee}$ has everywhere rank $\geq n - 1$, $P^{\vee}$ has only singularities of type A^k, $k = 0,1,\ldots$ (=Morin $\Sigma^{1,\overset{k}{\ldots},1}$ singularity). By the singular type, γ is filterated by subsundles as:

$$\mathbb{P}_n^{\vee} \times \tilde{C} \supset \gamma = \Sigma^0(P^{\vee}) \supset \Sigma^1(P^{\vee}) \supset \ldots \quad \Sigma^{1,\overset{n-1}{\ldots},1}(P^{\vee}) \quad \ldots ,$$

where $\Sigma^{1,\overset{i}{\ldots},1}(P^{\vee})$ is the set of points where $P^{\vee}$ is of type $\Sigma^{1,\overset{j}{\ldots},1}$-type, $j \geq i$.

The projection $P^{\vee} : \mathrm{Osc}^{n-i-1}\ C_C^{\vee} \to \mathbb{P}_n^{\vee}$ is locally given by the mapping:

$$(u,t) \in \mathbb{P}_n \times \tilde{C} \mapsto {}^t\begin{vmatrix} W_0 & \cdots & W_n \\ & & \\ W_0^{(n-i-1)} & \cdots & W_n^{(n-i-1)} \end{vmatrix}(t)\, u \quad ,$$

so we see by an easy induction, that

$$\Sigma^{\overbrace{1,\ldots,1}^{i}}(P^{\vee}) = P^{\vee}(\mathrm{Osc}^{n-i-1}\, C_0^{\vee}) \quad ,$$

$$\Sigma(P^{\vee}:\Sigma^{\overbrace{1,\ldots,1}^{i}}(P^{\vee}) \to \mathbb{P}_n^{\cdot}) = \Sigma^{\overbrace{1,\ldots,1}^{i+1}}(P^{\vee}) \quad ,$$

$$P^{\vee}(\Sigma^{\overbrace{1,\ldots,1}^{i}}(P^{\vee}) = P^{\vee}(\mathrm{Osc}^{n-i-1}\, C_0^{\vee}) = \Sigma^{\overbrace{1,\ldots,1}^{i}}(\omega_C)$$

and $P^{\vee}(\Sigma^{\overbrace{1,\ldots,1}^{i+1}}(P^{\vee})) = \Sigma^{\overbrace{1,\ldots,1}^{i+1}}(\omega_C)$ is the envelope set of $\Sigma^{\overbrace{1,\ldots,1}^{i}}(\omega_C)$ foliated by fibres cf $\mathrm{Osc}^{n-i-1}\, C_0^{\vee}$.

Conversely we have also

$$P^{\vee}(\mathrm{Osc}^{n-i-1}\, C^{\vee}) = \mathrm{Tan}\ \mathrm{envl}^{i+1}(\omega_C)$$

$$= \mathrm{Tan}\ P^{\vee}(\overline{\Sigma}^{\overbrace{1,\ldots,1}^{i+1}}(P^{\vee})) = P^{\vee}(\overline{\Sigma}^{\overbrace{1,\ldots,1}^{i}}(P^{\vee}))$$

$$= \mathrm{envl}^{i}(\omega_C) \quad ,$$

where Tan X denotes the tangent varicty of X which is the closure of the union of tangent spaces of X at nonsingular points.

Summarizing the fact above, we can see the following proposition.

Proposition II.2.1. <u>We have sequences:</u>

$$\mathbb{P}_n^{\vee} = \mathrm{envl}^0(\omega_C) \supset \mathrm{envl}^1(\omega_C) \supset \ldots \supset \mathrm{envl}^{n-1}(\omega_C) = C^{\vee} \quad ,$$

$$\mathbb{P}_n = \mathrm{envl}^0(\omega_C\vee) \supset \mathrm{envl}^1(\omega_C\vee) \supset \ldots \supset \mathrm{envl}^{n-1}(\omega_C\vee) = C$$

<u>and</u>

$$\mathrm{envl}^i(\omega_C) = \mathrm{Tan}\ \mathrm{envl}^{i+1}(\omega_C) = (\mathrm{Tan})^{n-i-1}\ C^{\vee}$$

$$= P^{\vee}(\mathrm{Osc}^{n-1}\ C^{\vee})$$

<u>and</u>

$$\mathrm{envl}^i(\omega_C)^{\vee} = \mathrm{envl}^{n-i-1}(\omega_C\cdot) \quad ,$$

<u>for</u> $i = 1,\ldots,n-1$.

Next we prove

Proposition II.2.2.

$$\mathrm{envl}(\omega_C) = \mathrm{envl}^1(\omega_C) \cup (\mathrm{sing}\ C)^{\vee}$$

$$= (\mathrm{Tan})^{n-2}\ C^{\vee} \cup (\mathrm{sing}\ C)^{\vee}$$

$$= \mathrm{Tan}(\mathrm{Tan}(\ \ldots(\mathrm{Tan}\ C^{\vee}))\ldots) \cup (\mathrm{sing}\ C)^{\vee}$$

Proof. The inclusion $\mathrm{envl}(\mathcal{W}_C) \supset \mathrm{envl}^1(\mathcal{W}_C) \cup (\mathrm{sing}\ C)^\vee$ is clear. Suppose $x \in \mathrm{envl}(\mathcal{W}_C) - (\mathrm{sing}\ C)^\vee$. Then the dual hyperplane has a contact with C_{reg} of crder $\geqq 2$ at $\phi(t)$. Since the matrix $\begin{vmatrix} \phi_0 & ,\ldots, & \phi_n \\ \phi_0^{(1)} & ,\ldots, & \phi_n^{(1)} \end{vmatrix}(t)$ has rank 2, x is in the dual space of the line spanned by $\phi(t)$, $\phi^{(1)}(t)$. The closure of the union of those dual spaces is presicely the set $\mathrm{envl}^1(\mathcal{W}_C)$. So the converse cf the inclusion holds. The other part of the statement follows from Proposition II.2.1.

The structure of $\Sigma(\mathcal{W}_C) \cap (C - C_0)^\vee$ is more complecated depending on the degeneracy of $W(\phi_0,\ldots,\phi_n)$. But here we will not discuss furthermore.

We remark that all singular subsets abcve are defined only by the topological properties of the web $\mathcal{W}_C$, $\mathcal{W}_{C^\vee}$, because order of contact of subspace with C, $C^\vee$ is a topological quantity which can be recovered by the topological structure of $\mathcal{W}_C$, $\mathcal{W}_{C^\vee}$.

Finally to analyze the whole singular set $\Sigma(\mathcal{W}_C) = \mathrm{degn}(\mathcal{W}_C)$ in a similar way to abcve, we define the secant variety:

$$\mathrm{Sec}^n(C) = \overline{\{P(x_1,\ldots,x_n) \mid x_i \in C \text{ are all distinct and } \dim P(x_1,\ldots,x_n) \leqq n-2\}}$$

Then by the definition we have

$$\mathrm{degn}(\mathcal{W}_C) = \mathrm{envl}(\mathcal{W}_C) \cup (\mathrm{Sec}^n\, C)^{\vee} \quad .$$

Of course we can define a filteration of $\mathrm{Sec}^n\, C$ by the degree of degeneracy in the same manner as $\mathrm{envl}^i(\mathcal{W}_C)$. However, the author does not know whether there exists any duality like Proposition II .2.1, 2.2, between subsets of $\mathrm{degn}(\mathcal{W}_C)$ and $\mathrm{degn}(\mathcal{W}_{C^\vee})$ nor what the set $(\mathrm{Sec}^n\, C)^{\vee}$ is .

For a point $p \in \mathbb{P}_n - C$, the <u>normal bundle of the projection</u> π_p of C from p is

$$\mathcal{N}_p = \pi_p{}^* \, T\mathbb{P}_{n-1} \, / \, TC \quad .$$

The structure of $\mathcal{N}_p$ is completely translated into the geometry of the hyperplane section $p^{\vee} \cdot \mathcal{W}_C$. A purely geometric approach might helpful to understand $\mathcal{N}_C$. In the papers $[E_1, E_2, H]$, very interesting problems are discussed on $\mathcal{N}_p$.

Section 3. Graf-Sauer's theorem, Quasi product.

In the following sections, we consider the case that the curve $C \subset \mathbb{P}_n$ is of degree $\leqq n+1$. This is exceptional in Theorem II .1.3.

Here we refer the following theorem.

Theorem II .3.1. (Graf-Saure [BB,GS]). <u>Let</u> (C_i, x_i), $i = 1,2,3$ <u>be germs of nonsingular projective curves in</u> $\mathbb{P}_n$ <u>and</u> x_i $X^{\cdot}$ <u>all distinct and</u> W <u>be the</u> 3-<u>web generated by</u> (C_i, x_i), $i = 1,2,3$. <u>Then</u> W <u>is hexagonal if and only if</u> (C_i, x_i) <u>are germs of the same cubic curve</u> C .

The proof and the beautiful picture of the hexagonal 3-webs of the cubic curves are found in [BB] or [GS].

This theorem alludes that the web structure of W_C of cubic curves is everywhere homogeneous off the singular set $\Sigma(W_C)$, so may admit many topological symmetry other from their projective symmetries.

In another point of view, its known that cubic curves, possiblly singular, admit group structure on their smooth parts. This structure is, as well known, intrinsically implied by Abel's theorem, which implies also the hexagonality of the webs. These relations are summulized and generalized in [CG]. Now we recall some results on these subjects, which is a preliminary for the forthcoming sections.

A symmetric quasigroup is a set E with a binary composition law $E \times E \to E : (x,y) \to x \circ y$ with the condition:

$$x \circ y = y \circ x \quad , \quad x \circ (x \circ y) = y \quad .$$

In other words, the law is defined by a subset of relation L_E $E \times E \times E$ invariant under the permutations of three entrie, such that the projections $P_i : L_E \to E \times E$ forgetting i-th factors are bijective, by $x \circ y = z$ if $(x,y,z) \in L_E$. If E is an analytic manifold and L_E is an analytic hyper surface then we say E an analytic quasigroup.

We introduce a new composition law $\bullet$ defined by

$$x \bullet y = u \circ (x \circ y) \quad ,$$

for a base point (unit) $u \in E$. Then

$$(x,y,z) \in L_E \Leftrightarrow x \bullet (y \bullet z) = (x \bullet y) \bullet z = u \circ u \ .$$

We call E an Abelian symmetric quasigroup if the new composition $\bullet$ makes E an Abelian group. Then, for any $u' \subset E$, the corresponding composition is again abelian (see [M]).

Let $C \subset \mathbb{P}_2$ be a irreducible cubic curve in the projective plane and C_{reg} be its smooth part. The relation $L_C \subset {C_{reg}}^3$ is defined as

$$(x,y,z) \in L_C \Leftrightarrow x,y,z \in C_{reg} \text{ are collinear} .$$

We will see that L_C is nonsingular surface if C is nonsingular curve, i.e., an elliptic curve. The group structure of the cubic curves C_{reg} is defined as above with this symmetric quasi group.

If C is redusible, i.e. contains a line L as a irreducble component, then $x \circ y$ is not defined for any $x , y \in L$. However we can define the group structure on it. (For a space curve $C \subset \mathbb{P}_n$ of degree $n+1$, we can define n-nary symmetric quasi product.)

The web structure of cubic curve is equivalent to symmetric quasigroup structure of the curve, which is the geometry of the surface $L_C \subset C_{reg}{}^3$, folliated by the coordinate lines in $C_{reg}{}^3$ which forms a 3-web of codimension 1 on L_C.

Section 4. Nonsingular curves of degree n , n + 1 .

In this section we study the structure of the web W_C for the case that C is nonsingular curve of degree n , n+1 in projective n space.

It is known that such a curve is a rational or an elliptic curve. In general, any curve in $\mathbb{P}_n$ is obtained by projecting a normal curve in projective space of dimension $\geqq n$. By a generalization of Proposition II . 1.1 , 1.2, the web generated by the projection is given by the plane section of the web of the normal curve. So, in this section, we restrict ourselves to the case of rational or elliptic normal curves in projective n space.

A. The rational normal curve.

This curve is projectively transformed to the twisted curve which is imbedded by the Veronese imbedding:

$$v(t) = (1:t:t^2:,\dots:t^n):\mathbb{P}_1 \hookrightarrow \mathbb{P}_n \quad .$$

Let t_i , $i = 1,\dots,n$ be distinct points. Then the intersection of the dual hyperplanes $W(t) = \bigcap_{i=1}^{n} v(t)^\vee$ $\mathbb{P}_n^*$ is the dual of the image of the matrix

$$\begin{pmatrix} 1 , & t_1 , & t_1^2 , & \dots , & t_1^n \\ \vdots & & & & \vdots \\ 1 , & t_n , & t_n^2 , & \dots , & t_n^n \end{pmatrix} \quad ,$$

of which $n \times n$ minors gives the plücker coordinates of the image. We can caluclate this as

$$v^\vee = (\sigma_n \Pi : \sigma_{n-1} \Pi : \dots : \Pi) \quad ,$$

where $\Pi(t) = \prod_{i<j} (t_i - t_j)$ and σ_i is the basic

symmetric polynomial $\sigma_i(t) = \Sigma\, t_{s_1} \cdot \ldots \cdot t_{s_i}$, where $\{s_1,\ldots,s_i\}$ runs over all i point subsets of $\{1,\ldots,n\}$. By this form we see that $v^\vee\colon \mathbb{P}_1^n - \Delta \to \mathbb{P}_n$ extends analytically to a mapping $v^\vee\colon \mathbb{P}_1^n \to \mathbb{P}_n$ as $v^\vee = (\sigma_n : \sigma_{n-1} : \ldots : \sigma_1 : 1)$ (the quotient map) ,so that

$$v^\vee(\{t_1,\ldots,t_n) \in \mathbb{P}_1^n \mid t_i = t_i'\}) = v(t_i')^\vee \subset \mathbb{P}_n^\vee \;.$$

So $\mathcal{W}_C$ is octahedral.

A homeomorphism of $\mathcal{W}_C$ lifts to an invariant nomeomorphism of $\mathbb{P}_1^n$ preserving the octahedral n-web by leaves $L_{t_i'} = \{(t_1,\ldots,t_n) \in \mathbb{P}_1^n \mid t_i = t_i'\}$, $t_i' \in \mathbb{P}_1$, $i = 1,\ldots,n$, so of the form $h \times \ldots \times h$ with a homeomorphism h of $\mathbb{P}_1$.

Conversely, any homeomorphism h of $\mathbb{P}_1$ induces a homeomorphism h^n of $\mathbb{P}_1^n$ hance a homeomorphism of the quatient space $(\mathbb{P}_n, \mathcal{W}_C)$. By this correspondence, we have

Proposition II.4.1. *Let* $C \subset \mathbb{P}_n$ *be a rational normal curve in projective n-space. Then*

$$\mathrm{Homeo}(\mathcal{W}_C) = \mathrm{Homeo}(\mathbb{P}_1)$$

and

$$\mathrm{Homeo}(\mathcal{W}_C) \cap PGL(n+1,\mathbb{C}) = \mathrm{Aut}(\mathbb{P}_1) \quad ,$$

where $\mathrm{Homeo}(\mathcal{W}_C)$ *denotes the group of homeomorphism of* $\mathcal{W}_C$.

Let

$$\Delta^i = \{(t_1,\dots,t_n) \in \mathbb{P}^n_1 \mid i+1 \text{ of } t_i\text{'s are the same}\} \subset \mathbb{P}^n .$$

Then we have

$$v^\vee(\Delta^i) = \mathrm{envl}^i(\omega_C) \quad , \text{ irreducible for } i = 0,\dots,n-1$$

$$v^\vee(\Delta^1) = \mathrm{envl}^1(\omega_C) = \Sigma(\omega_C) \quad ,$$

$$v^\vee(\Delta^{n-1}) = C^\vee \text{ : dual curve} \quad .$$

Since $v^\vee = (1:\sigma_1:\dots:\sigma_n)$, the restriction $v^\vee|\Delta^{n-1}$ is Voernese imbedding. So the dual curve $C^\vee$ is again the rational normal curve.

In the following we refer the result by Piene [P] :

Proposition II .4.2. <u>Let</u> $C \subset \mathbb{P}_n$ <u>be the rational normal curve</u>. <u>Then</u> $\mathrm{envl}^i(\omega_C)$ <u>is an irreducible variety of dimension</u> $n - i$ <u>and</u>

$$\text{degree } \mathrm{envl}^i(\omega_C) = (i+1)(n-i) \quad .$$

<u>for</u> $i = 0,\dots,n-1$.

Proof. The degree is presented in [P]. The other statements are seen in above.

B. The elliptic normal curve of degree $n+1$ (nonsingular curve of degree $n+1$ with genus 1).

This curve is projectively transformed to the elliptic normal turve cannonically imbedded in $\mathbb{P}_n$. First we recall a classical result on the elliptic curve.

The elliptic curve C is a complex manifold given by the quationt of $\mathbb{C}$ by a nondegenerate lattice $\Lambda = (\omega_1, \omega_2)$ The complex structure of C is determined by the well known j invariant:

$$j(\lambda) = \frac{4(\lambda^2 - \lambda + 1)^3}{27\,\lambda^2(\lambda - 1)^2} \quad , \quad \lambda = \frac{\omega_2}{\omega_1} \quad .$$

and an imbedding of C to $\mathbb{P}_n$ is given by doublly periodic functions. Here we refer from the paper by Hulek [H] , an explicit construction of imbedding of C with degree $n+1$:

The Weierstrass σ function is defined by

$$\sigma(z) = z \cdot \prod_{\omega \in \Gamma - 0} \left(1 - \frac{z}{\omega}\right) e^{\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right)} \quad .$$

Withrespect translation by ω_1 , ω_2 , the following fundamental formulas hold:

$$\sigma(z + \omega_i) = - e^{\eta_i \left(z + \frac{\omega_i}{2}\right)} \cdot \sigma(z) \quad ,$$

where η_i is the priod constant of the Weierstrass ζ - function.

Case $n \geqq 2$, even. For $p , q \in \mathbb{Z}$, define

$$\sigma_{p,q}(z) = \sigma(z - \frac{p\omega_1 + q\omega_2}{n+1})$$

and

$$\omega = - e^{-\frac{\eta_1}{2}\frac{\eta_2\omega_1}{2}} , \quad \theta = e^{-\frac{\eta_1\omega_1}{2n+2}}$$

and

$$x_m(z) = \omega^m \cdot \theta^{m^2} \cdot e^{m\eta_1 z} \cdot \sigma_{m,0}(z) \cdot \ldots \cdot \sigma_{m,n}(z) .$$

Case $n \geqq 3$, odd.

$$\tilde{\sigma}_{p,q}(z) = \sigma(z - \frac{p\omega_1 + q\omega_2}{n+1} - \frac{1}{2}(\omega_1 + \frac{\omega_2}{n+1})) ,$$

$$\tilde{\omega} = e^{-\frac{1}{2}(\eta_1\omega_1 + \eta_2\omega_1)}$$

and

$$x_m(z) = \tilde{\omega}^m \cdot \theta^{m^2} \cdot e^{m\eta_1 z} \tilde{\sigma}_{m,0}(z) \cdot \ldots \cdot \tilde{\sigma}_{m,n}(z) ,$$

Then we see $x_{n+1+m}(z) = x_m(z)$, for any integer m and $x_0 , \ldots , x_n$ are basis of $\Gamma(O_C((n+1)0))$ and the map $v = (x_0 : \ldots : x_n) : C \hookrightarrow \mathbb{P}_n = \mathbb{P}\Gamma(O_C((n+1)0))^\vee$ is a nondegenerate normal imbedding of degree $n + 1$.

Let $\varepsilon = e^{\frac{2\pi\sqrt{-1}}{n+1}}$. Then the followings hold:

$$(1) \qquad x_i(-z) \sim (-1)^{n+1}\, x_{-i}(z)$$

$$(2) \qquad x_i\left(z - \frac{\omega_1}{n+1}\right) \sim x_{i+1}(z)$$

$$(3) \qquad x_i\left(z + \frac{\omega_2}{n+1}\right) \sim e^i \cdot x_i(z) \qquad ,$$

where $\sim$ means that equality holds up to a common nowhere vanishing function independent of i. Then, by (2),(3), the action of the group $\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}$ by translation of $\frac{\omega_1}{n+1}, \frac{\omega_2}{n+1}$ is compatible with the action on $C = \mathbb{C} / \Lambda$ generated by

$$(x_0 : \ldots : x_n) \longmapsto (x_n : x_0 : \ldots : x_{n-1})$$

$$(x_0 : \ldots : x_n) \longmapsto (x_0 : \varepsilon x_1 : \ldots : \varepsilon^n x_n) \qquad ,$$

and by (1) the involution

$$(x_0 : \ldots : x_n) \longmapsto (x_0 : x_{-1} : \ldots : x_{-n})$$

induces the involution $z \to -z$ on $C = \mathbb{C} / \Lambda$.

The inversion of the imbedding v is given by the abelian integral

$$v(z) \longmapsto \int_0^z dz \equiv z \pmod{\Lambda} \qquad .$$

Then Abel's theorem says that two divisors

$$\sum_{i=0}^{n} v(a_i)\ ,\ \sum_{i=0}^{n} v(b_i)$$ are linearly equivalent

if and only if

$$\sum_{i=0}^{n} a_i \equiv \sum_{i=0}^{n} b_i \quad (\mathrm{mod}\ \Lambda).$$

Let $\mathbb{P}_{n-1} = \{x_0=0\} \subset \mathbb{P}_n$ and $v(C) \cdot \mathbb{P}_{n-1} = \sum_{i=0}^{n} v(p_i)\ ,\ p_i \in C = \mathbb{C} / \Lambda$. Then this theorem is rephrased as:

$v(z_i)$, $i = 0,\ldots,n$ are coplanner

if and only if

$$\sum_{i=0}^{n} z_i \equiv \sum_{i=0}^{n} p_i \quad (\mathrm{mod}\ \Lambda).$$

By the explicit form of $x_m(z)$, we see

$$p_i = \begin{cases} \dfrac{i\ \omega_2}{n+1} \in C & n : \text{even} \\ \dfrac{i\ \omega_2}{n+1} + \dfrac{1}{2}(\omega_1 + \dfrac{\omega_2}{n+1}) & n : \text{odd} \end{cases} ,$$

so, in any case, we have

$$\sum_{i=0}^{n} p_i \equiv 0 \quad (\mathrm{mod}\ \Lambda) .$$

Therefore we have proven that:

$v(z_i)$, z_i $C = \mathbb{C}/\Lambda$, $i = 0,\dots,n$ are coplanner if and only if

$$\sum_{i=0}^{n} a_i \equiv 0 \pmod{\Lambda} \quad .$$

Then we denote the hyperplane $\mathbb{P}_{n-1}$ with $\mathbb{P}_{n-1} \cdot v(C) = v(z_0) + \dots + v(z_n)$ by $P(z_0,\dots,z_n)$ and $v^{\vee}(z_0,\dots,z_n) = \bigcap_{i=0}^{n} v(z_i)^{\vee} = P(z_0,\dots,z_n)^{\vee}$.

Let

$$L = \{(z_0,\dots,z_n)\ C^{n+1} \mid \Sigma\, a_i \equiv 0 \bmod \Lambda\} \subset C^{n+1}$$

and

$$\Delta^i = \{(z_0,\dots,z_n) \in C^{n+1} \mid i+1 \text{ of } z_i\text{'s are the same}\}$$

(Then $\Delta^n \cap L$ consists of $(n+1)^2$ points). Clearly we see that:

$P(z_0,\dots,z_n)$ has a contact with $C_{n+1} = v(C)$ of order $\geq i+1$ if and only if $(z_0,\dots,z_n) \in \Delta^i$,

and the map $v^{\vee}: L \to \mathbb{P}_n^{\vee}$ possesses the following properties:

(1) $v^{\vee}$ is an n+1-sheated covering with branch locus Δ^1 (the quotient map by the symmetry group).

(2) $v^{\vee}(\ z_i = a\ ) = v(a)^{\vee}$

(3) $\omega_{C_{n+1}}$ <u>is octahedral off the singular set</u> $\Sigma(\omega_{C_{n+1}})$

(4) $v^{\vee}(\Delta^i) = \mathrm{envl}^i(\omega_{C_{n+1}})$, irreducible

(5) $v^{\vee}(\Delta^{n-1}) =$ <u>the dual curve of</u> C_{n+1}

(6) $v^{\vee}(\Delta^n) = (n+1)^2$ singular points of $v^{\vee}(\Delta^{n-1})$

= duals of hyperosculating hyperplane of C_{n+1} at n+1 torsion points

(7) $\mathrm{envl}^1(\omega_{C_{n+1}}) = \Sigma(\omega_{C_{n+1}})$ (for the n+1 web of L is nowhere degenerate).

Here we refer again a result from the paper [P].

Proposition II.4.5. <u>Let</u> $C_{n+1} \subset \mathbb{P}_n$ <u>be the elliptic normal curve of degree</u> n+1. <u>Then</u> $\mathrm{envl}^i(\omega_C)$ <u>is an irreducible vriety of dimension</u> n - i <u>and</u>

$$\text{degree } \mathrm{envl}^1(\omega_C) = (i+1)(n+1) .$$

Proof. The degree is presented in the paper [P]. The other statements are seen in above.

Now we consider the topological symmetry of the web $W_{C_{n+1}}$ of $\mathbb{P}^{\vee}_{n}$.

Let $\Lambda = (\omega_1, \omega_2)$, $\Lambda' = (\omega_1', \omega_2')$ be nondegenerate rattices of $\mathbb{C}$ and C_{n+1} , C'_{n+1} be the correspondeng normal elliptic curves cannonically imbedded in $\mathbb{P}_n$ as previlusely, and suppose that generated n+1 webs $W_{C_{n+1}}$, $W_{C'_{n+1}}$ are topologically equivalent by a homeomorphism h of $\mathbb{P}^{\vee}_{n}$. Then h induces a homeomorphism $h^{\vee}: C_{n+1} \to C'_{n+1}$ such that $h^{\vee n+1}: L_{\Lambda} \to L_{\Lambda'}$ is a homeomorphism and $(n+1)\cdot h^{\vee}(0) \equiv 0$ mod Λ' (Proposition II .1.2). Compositing the translation $T:(\mathbb{C}/\Lambda, 0) \to (\mathbb{C}/\Lambda', 0)$, $T \circ h^{\vee}$ becomes an isomorphism of $\mathbb{C}/\Lambda$ to $\mathbb{C}/\Lambda'$ as topological groups. Then $T \circ h^{\vee}$ have to be a real linear isomorphism of the torus group T^2, which we identify naturally with an element of $GL(2,\mathbb{Z})$ acting on the lattice $\mathbb{Z} \times \mathbb{Z}$ $\mathbb{R}^2 = \mathbb{C}$. So $h^{\vee}$ is a composition of $T \circ h^{\vee} \in GL(2,\mathbb{Z})$ with T^{-1} , $T^{-1}(0) \in \mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1} \subset T^2$. These compositions form a lattice $\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}$ preserving subgroup G of the affine transformation group of T^2. Note that the group G is a extension of $GL(2,\mathbb{Z})$ by $\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}$:

$$0 \to \mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1} \to G \to GL(2,\mathbb{Z}) \to 0 .$$

From now on we denote G by a semi direct product $GL(2,\mathbb{Z}) \ltimes (\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1})$.

Conversely any linear isomorphism $h^{\vee}: \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ preserving the lattice $\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}$ induces a homeomorphism h of $\check{\mathbb{P}}_n$ such that $h(\mathcal{W}_{C_{n+1}}) = \mathcal{W}_{C'_{n+1}}$.

Summarizing the result above, we have

Proposition II.4.3. For any elliptic normal curves C, $C' \subset \mathbb{P}_n$ of degree $n+1$, the generated $n+1$ webs $\mathcal{W}_C$, $\mathcal{W}_{C'}$ are topologically equivalent.

Let $C = C'$, i.e., $\Lambda = \Lambda'$. Then above correspondence of the group G with homeomorphisms of $\mathcal{W}_C$ to $\mathcal{W}_{C'}$ gives a representation of G to the group $\mathrm{Homeo}(\mathbb{P}_n)$. It is easy to see that $G \cap PGL(n+1,\mathbb{C}) = \mathbb{Z}_i \ltimes (\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1})$, where $\mathbb{Z}_i$ is the cyclic subgroup of $GL(2,\mathbb{Z})$ generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ ($i=4$) if Λ is square, $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ ($i=6$) if Λ is triangular, and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ ($i=2$) for otherwise.

Finally we summarize as follows:

Proposition II.4.4. Let $C \subset \mathbb{P}_n$ be an elliptic normal of degree $n+1$. Then the generated $n+1$ web $\mathcal{W}_C$ is hexagonal off the singular set $\Sigma(\mathcal{W}_C) = \mathrm{envl}^1(\mathcal{W}_C)$. Then

$$\mathrm{Homeo}(\mathcal{W}_C) = GL(2,\mathbb{Z}) \ltimes (\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1})$$

and

$$\mathrm{Homeo}(\mathcal{W}_C) \cap PGL(n+1,\mathbb{C}) = \mathbb{Z}_i \ltimes (\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}) ,$$

where $i = 4$ if C is square, $i = 6$ if C is triangular and $i = 2$ for otherwise, and $\ltimes$ denotes the semi direct product.

Section 5. Singular cubics.

By projective transformations, a cubic plane curve is isomorphic to one on the normal form in the table (Table 1). We will see the web structure of the curves there by using their gqoup structure. The case (1) is the simplest case of the elliptic normal curves in Section 4, and the case (2), (3) are almost the same as (1).

Case (4): Conic and line, $x(x^2+y^2-z^2)=0$.

This curve is a union of the conic $C_1 = P_1 = \{x^2+y^2-z^2=0\}$ and the line $C_2 = \mathbb{P}_1 = \{y=0\}$. We take their parametrizations:

$$F_1(t) = \left(\frac{1-t^2}{1+t^2} : \frac{-2t}{1+t^2} : t\right) \quad , \quad F_2(t) = \left(\frac{t-1}{t+1} : 0 : 1\right) \quad ,$$

$t \in \mathbb{P}_1$, and we put a group structure $C^* \times \mathbb{Z}_2$ on $C_{reg} = C_1 \cup C_2 - (\pm 1:0:1)$ as follows:

We can easily see that

$F_1(a)$, $F_1(b)$, $F_3(c)$ are collinear
if and only if $a\ b\ c = 1$.

Then we define the symmetric quasi product $\circ$ (see Section 3) by

$$F_1(a) \circ F_1(b) = F_2(c)$$

and the multiplication $\cdot$ on C_{reg} by

$$F_1(a) \cdot F_1(b) = F_1(1) \circ (F_1(a) \circ F_1(b)) \in C_1$$

$$F_1(a) \cdot F_2(b) = F_1(1) \circ (F_1(a) \circ F_2(b)) \in C_2$$

$$F_2(a) \cdot F_2(b) = (F_2(a) \circ F_1(1)) \circ (F_2(b) \circ F_1(1)) \in C_1 .$$

By computation, we see the fomulas

$$F_1(a) \cdot F_1(b) = F_1(ab) ,$$

$$F_1(a) \cdot F_2(b) = F_2(ab) ,$$

$$F_2(a) \cdot F_2(b) = F_1(ab) ,$$

which makes C_{reg} the group $C^* \times \mathbb{Z}_2$, and we see that

$p, q, r \in C_{reg}$ <u>are collinear</u>
<u>if and only if</u> $p \cdot q \cdot r = F_1(1)$.

Let W_C be the 3-web generated by C and h be a homeomorphism of W_C. Since the group structure of C is recovered by the symmetric quasiproduct, h induces a homeomorphism $h^{\vee}$ of $C = C_1 \cup C_2$ such that

$$h^{\vee 3}(L) = L , \ L = \{(a,b,c) \in (\mathbb{C}^* \times \mathbb{Z}_2)^3 \mid abc = 1\} .$$

So $h^{\vee}: \mathbb{C}^* \times \mathbb{Z}_2 \to \mathbb{C}^* \times \mathbb{Z}_2$ is a two copy of a homeomorphism $h^{\vee\prime}$ of $\mathbb{C}^*$ such that

$$h^{\vee\prime 3}(L') = L' \ , \ L' = \{(a,b,c) \in \mathbb{C}^{*3} \mid abc=1\} \quad .$$

Then $h^{\vee\prime}(1)^3 = 1$ and $h^{\vee\prime}(1)^{-1}\, h^{\vee\prime} : \mathbb{C}^* \to \mathbb{C}^*$ is a group homeomorphism.

Conversely a composition $T \circ h : \mathbb{C}^* \to \mathbb{C}^*$, $T^3 = 1$, $h \in \mathrm{Aut}(\mathbb{C}^*)$ extends to a homeomorphism $h^{\vee}$ of $C = C_1 \cup C_2$ preserving the collinear relation. So $h^{\vee}$ induces a homeomorphism h of $\mathcal{W}_C$. This shows that

$$\mathrm{Homeo}(\mathcal{W}_C) = \mathrm{Aut}_{top}(C^*) \times \mathbb{Z}_3 = \mathbb{R}^* \times \mathbb{Z}_3$$

For the other cases (5) - (7), we can analyze similary with the group structures.

In the following, we refer the table of the group $\mathrm{Homeo}(\mathcal{W}_C) \subset \mathrm{Homeo}(\mathbb{P}_2)$.

Table 1. Singular cubics.

		group structure	$Homeo(W_C)$	$Homeo(W_C) \cap PGL(2,\mathbb{C})$
(1)	nonsingular cubic (elliptic curve)	$\mathbb{C}/\Lambda$	$GL(2,\mathbb{Z}) \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)$	$\mathbb{Z}_i \times (\mathbb{Z}_3 \times \mathbb{Z}_3)$,

where $i = 4$ if Λ is square, $i = 6$ if Λ is triangular, and $i = 2$ for otherwise.

(2)	cuspidal cubic	$\mathbb{C}$	$GL(2,\mathbb{R})$	$\mathbb{C}^* = \{\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} \mid \lambda \neq 0\}$
(3)	nodal cubic	$\mathbb{C}^* = \mathbb{C}/2\pi i$	$\mathbb{R}^* \times \mathbb{Z}_2$	$\mathbb{Z}_3$
(4)	conic and line	$\mathbb{C}^* \times \mathbb{Z}_2$	$\mathbb{R}^* \times \mathbb{Z}_2$	$\mathbb{Z}_3$
(5)	conic and tangent	$\mathbb{C} \times \mathbb{Z}_2$	$GL(2,\mathbb{R})$	$\mathbb{C}^*$
(6)	triangle	$\mathbb{C}^* \times \mathbb{Z}_3$	$\mathbb{R}^* \times \mathbb{Z}_3 \times \sigma_3$	$\mathbb{Z}_3 \times \sigma_3$
(7)	three concurrent line	$\mathbb{C} \times \mathbb{Z}_3$	$GL(2,\mathbb{R}) \times \sigma_3$	$\mathbb{C}^* \times \sigma_3$
(8)	two lines one repeated	–	$Homeo(\mathbb{P}_1)^2$	$PGL(2,\mathbb{C})^2$
(9)	triple lines	–	–	–

where σ_3 is the symmetric group of oeder 3 and $\ltimes$ denotes the semi direct product (see also Proposition II.4.4).

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