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On Morin Singularities

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Introduction

In this paper we study the problem of finding a smooth map between smooth manifolds with nice Morin singularities in a given homotopy class. A geometric interpretation of Morin singularities of a smooth map $f:N \to P$ is as follows. Let $S^i(f)$ denote the set of all points x of N such that the kernel rank of df_x is i. For a certain map f, $S^i(f)$ becomes a submanifold of N and we may define $S^{i,j}(f)$ as the set $S^j(f|S^i(f))$ for $f|S^i(f):S^i(f) \to P$ similarly. Let $n = \dim N$, $p = \dim P$ and $i = \max(1, n-p+1)$. Let I_r be the r-sequence $(i,1,\ldots,1)$. Then we may continue to define $S^{I_r}(f)$ as $S^1(f|S^{I_{r-1}})$ inductively. A point of $S^{i,0}(f)$ or $S^{I_r}(f)$ is called a Morin singularity of symbol $(i,0)$ or I_r respectively. However this approach does not make it clear for what part of smooth maps f, $S^{I_r}(f)$ can be defined. For this we review the following important observation due to Boardman[2].

There exist a submanifold $\Sigma^{i,0}(N,P)$ and a series of submanifolds; $\Sigma^{I_1}(N,P) \supset \Sigma^{I_2}(N,P) \supset \ldots \supset \Sigma^{I_r}(N,P) \supset \ldots$ in the infinite jet space $J^\infty(N,P)$. The codimension of $\Sigma^{i,0}(N,P)$ is $i(p-n+i)$ and that of $\Sigma^{I_r}(N,P)$ is $n-p+r$ for $n \geq p$ and

$r(p-n+1)$ for $n < p$. He has shown that if a jet map $j^{\infty}f:N \to J^{\infty}(N,P)$ of f is transverse to all submanifolds $\Sigma^{i,0}(N,P)$ and $\Sigma^{I_r}(N,P)$, then $S^{i,0}(f)$ and $S^{I_r}(f)$ coincide with $(j^{\infty}f)^{-1}(\Sigma^{i,0}(N,P))$ and $(j^{\infty}f)^{-1}(\Sigma^{I_r}(N,P))$ respectively. Therefore for generic maps f we may consider $S^{i,0}(f)$ and $S^{I_r}(f)$.

For any integer $r \geqq 1$ we define a subset $\Omega_r(N,P)$ of $J^{\infty}(N,P)$ as the set of all jets z such that either z is of maximal rank or a point of $\Sigma^{i,0}(N,P)$ or $\Sigma^{I_2}(N,P)\setminus\Sigma^{I_{r+1}}(N,P)$. Then $\Omega_r(N,P)$ becomes an open subbundle of the fibre bundle $J^{\infty}(N,P)$ over N. The first result of this paper is the following

Theorem 1. Let $p \geqq 2$. Then for any section s of N into $\Omega_r(N,P)$, there exists a smooth map $g:N \to P$ such that $j^{\infty}g$ becomes a section of N into $\Omega_r(N,P)$ homotopic to s in $\Omega_r(N,P)$.

Next we will study the problem of eliminating the Morin singularities $S^{I_r}(f)$ with codim $S^{I_r}(f) = n$ from f admitting only Morin singularities. Theorem 1 reduces it to a problem of finding a section of N into $\Omega_{r-1}(N,P)$ homotopic to $j^{\infty}f$. We will show that if $j^{\infty}f$ is transverse to $\Sigma^{I_r}(N,P)$ for a connected and closed manifold N, then the number of points of $S^{I_r}(f)$ modulo 2 is the unique obstruction of finding the above section. We should note that this number is just the Thom polynomial of the topological closure $\overline{\Sigma^{I_r}}(N,P)$ for f (see the definition of [9]).

Theorem 2. Let $r \geqq 2$, $p \geqq 2$ and codim $\Sigma^{I_r}(N,P) = n$. Let N and P be orientable manifolds. Then

(1) A smooth map f with $j^{\infty}f(N) \subset \Omega_r(N,P)$ is homotopic to

a smooth map g such that $j^{\infty}g(N) \subset \Omega_{r-1}(N,P)$ and $j^{\infty}f$ and $j^{\infty}g$ are homotopic as sections of N into $\Omega_r(N,P)$ if and only if the Thom polynomial of $\overline{\Sigma^{I_r}(N,P)}$ for f vanishes.

(2) In particular f is homotopic to such a smooth map g in the following cases;

i) $n > p$ and $r \equiv 1 \pmod 4$

ii) $n > p$, $r \equiv 2,3$ or $4 \pmod 4$ and $n-p \equiv 1 \pmod 2$ and

iii) $n \leqq p$ and $n+p+r+\frac{1}{2}r(r+1) \equiv 0 \pmod 2$.

It will be shown by the Morse inequalities that the similar statement of Theorem 1 for $p = 1$ is not true. If N is an open manifold, then Theorem 1 is a direct consequence of Gromov[7, Theorem 4.1.1] and if $n < p$, it is also a special case of [4, Theorem B]. So the rest cases will be treated in this paper. The case $r = 2$ of Theorem 1 should be compared with [6, Theorem 1.3] which will play an important role in a proof of Theorem 1 (Sections 2 and 3).

The case $n \geqq p$ and $p = 2$ of Theorem 2 has been proved by Levine[11, Theorems 1 and 2] for $n > 2$ and by Eliasberg[5, Corollary of Theorem 4.9] for $n = 2$.

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