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ALMOST P-V-VERSAL DEFORMATIONS

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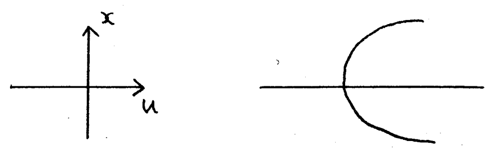
1. Introduction

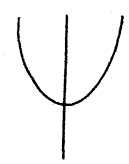
Let $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be a smooth map germ. For each $u \in (R^r, 0)$, we have a germ of "varieties" $f_u^{-1}(0)$ defined by $f_u = f|R^n \times u$. In this note we shall study bifurcations of these varieties.

EXAMPLE 1. Let $f, g : (R\times R, 0) \longrightarrow (R, 0)$ be map germs defined by $f(x,u) = x^3 - ux$ and $g(x,u) = u^3 - ux$. Then $f^{-1}(0)$ and $g^{-1}(0)$ are diffeomorphic by a rotation. But bifurcations of varieties are as follows.

	$u < 0$	$u = 0$	$u > 0$
$\#(f_u^{-1}(0))$	1	1	3

	$u < 0$	$u = 0$	$u > 0$
$\#(g_u^{-1}(0))$	1	$2^{\aleph_0}$	1





We now introduce two equivalence relations which distinguish bifurcations of varieties.

Definition (1.1). Let f, g : $(R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be smooth map germs.

(1) We say that f and g are P-K-equivalent if there exists a diffeomorphism germ $\Phi : (R^n \times R^r, 0) \longrightarrow (R^n \times R^r, 0)$ of the form $\Phi(x,u) = (\Phi^1(x,u), \phi(u))$ such that $\Phi^*(I(f)) = I(g)$. Here, I(f) is the ideal in the ring of smooth function germ which is generated by coordinate functions of f.

(2) We say that f and g are P-V-equivalent if there exists a homeomorphism germ Φ: $(R^n \times R^r, 0) \longrightarrow (R^n \times R^r, 0)$ of the form $\Phi(x,u) = (\Phi^1(x,u), \phi(u))$ such that $\Phi^{-1}(f^{-1}(0)) = g^{-1}(0)$.

For each smooth map germ f : $(R^n \times R^r, 0) \longrightarrow (R^p, 0)$, the bifurcation map germ $\pi_f : (f^{-1}(0), 0) \longrightarrow (R^r, 0)$ is defined by $\pi_f(x,u) = u$. The above equivalence relations describe the C^∞-type and the topological type of π_f. The P-K-equivalence theory has been studied in ([2], [3], [5], [10]).

There are some motivations to study bifurcations of varieties. One of the most convenient motivation is the following : Let f : $(R^n \times R^r, 0) \longrightarrow (R^n, 0)$ be a smooth map germ. Consider a system of a parametrized ordinary differential equation,

$$(*) \qquad \frac{dx}{dt} = f(x,u)$$

then the equilibrium solutions of (*) are $f^{-1}(0)$. Hence, the theory of bifurcations of varieties contains the theory of bifurcations of equilibrium solutions of parametrized ordinary differential equations.

EXAMPLE 2. P-K-modality (P-K-cod(f) = 5).

Let $f_\lambda : (R\times R,0) \longrightarrow (R,0)$ be a smooth map germ defined by $f_\lambda(x,u) = x(x+u)(x-\lambda u)$. The picture of $f_\lambda^{-1}(0)$ is the following.

If $\lambda \neq \lambda'$ $(0 < \lambda, \lambda' < \infty)$, then f_λ and $f_{\lambda'}$ are not P-K-equivalent. Hence, in this case λ is a moduli parameter.

But, numbers of bifurcations of varieties are the following.

	$u < 0$	$u = 0$	$u > 0$
$\#(f_\lambda)_u^{-1}(0)$	3	1	3

for any λ $(0 < \lambda < \infty)$.

This is one of the motivation to study the P-V-equivalence theory.

2. Almost P-V-versal deformations

One of the most usefull concept in the singularity theory is the versal deformation (unfolding). Hence, it is an important problem to seek for P-V-versal deformations.
In relation to this problem, we shall introduce a new concept about deformations which is called almost P-V-versal deformations.

Definition (2.1). Let $F, G : (R^n\times R^r\times R^s,0) \longrightarrow (R^p,0)$ be s-parameter deformations of $f : (R^n\times R^r,0) \longrightarrow (R^p,0)$. (i.e. $F|R^n\times R^r\times 0 = G|R^n\times R^r\times 0 = f$). We say that F, G are almost P-V-equivalent as deformations, if there exists a map germ

$\Phi : (R^n \times R^r \times R^s, 0) \longrightarrow (R^n \times R^r \times R^s, 0)$ of the form $\Phi(x,u,v) = (\Phi^1(x,u,v), \phi(u,v), \psi(v))$ which is not necessary continuous such that $\psi : (R^s, 0) \longrightarrow (R^s, 0)$ and $\Phi_v : (R^n \times R^r \times v, (0,v)) \longrightarrow (R^n \times R^r \times \psi(v), \Phi(0,v))$ are homeomorphism germs for any $v \in (R^s, 0)$ and $\Phi^{-1}(F^{-1}(0)) = G^{-1}(0)$.

Remark. We say that F, G are P-V-equivalent as deformations, if Φ is a homeomorphism germ in the above definition.

Definition (2.2). Let $F : (R^n \times R^r \times R^s, 0) \longrightarrow (R^p, 0)$ and $G : (R^n \times R^r \times R^t, 0) \longrightarrow (R^p, 0)$ be deformations of $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$. An A.P-V-morphism $\Phi : G \longrightarrow F$ is a map germ $\Phi : (R^n \times R^r \times R^t, 0) \longrightarrow (R^n \times R^r \times R^s, 0)$ of the form $\Phi(x,u,v) = (\Phi^1(x,u,v), \phi(u,v), \psi(v))$ which is not necessary continuous such that $\psi : (R^t, 0) \longrightarrow (R^s, 0)$ is continuous, $\Phi_v : (R^n \times R^r \times v, (0,v)) \longrightarrow (R^n \times R^r \times v, \Phi(0,v))$ is a homeomorphism germ for any $v \in (R^t, 0)$ and $\Phi^{-1}(F^{-1}(0)) = G^{-1}(0)$.

Remark. We say that $\Phi : G \longrightarrow F$ is a P-V-morphism, if Φ is continuous in the above definition.

Definition (2.3). Let $F : (R^n \times R^r \times R^s, 0) \longrightarrow (R^p, 0)$ be a deformation of $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$. We say that F is an almost P-V-versal deformation of f if for any deformation $G : (R^n \times R^r \times R^t, 0) \longrightarrow (R^p, 0)$ of f, there exists an A.P-V-morphism $\Phi : G \longrightarrow F$.

Remarks. 1) The definition of the P-V-versal deformation is the same as the above. It is clear that F is a P-V-versal deformation of f then it is almost P-V-versal.

2) Let $F : (R^n \times R^r \times R^s, 0) \longrightarrow (R^p, 0)$ be the almost P-V-versal deformation of f, then it has any informations about bifurcations of f with respect to P-V-equivalence.

3. Results

In this section we will state some results about the P-V-equivalence theory. Detailed proof will appear elsewhere.

The first statement is the following generic classification theorem with respect to the P-V-equivalence.

Theorem (3.1). (Thom-Varčenko type theorem with respect to P-V). For each $\ell \geq 1$, there exists a partition of the space $J^\ell(n+r,p)$ into disjoint semialgebraic subsets $\Sigma_0^\ell(n+r,p), V_1^\ell, \ldots, V_{n(\ell)}^\ell, V_{n(\ell)+1}^\ell, \ldots, V_{s(\ell)}^\ell$ with the following properties :

1) V_i^ℓ are C^∞-submanifold of $J^\ell(n+r,p)$ and codim $V_{n(\ell)+j}^\ell = 0$ in $J^\ell(n+r,p)$.

2) If $f, g : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ are map germs such that $j_0^\ell f, j_0^\ell g \in V_i^\ell$. Then germs f and g are P-V-equivalent.

3) If $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ is a map germ such that $j_0^\ell f \in V_{n(\ell)+j}^\ell$, then $\pi_f : (f^{-1}(0), 0) \longrightarrow (R^r, 0)$ is the MT-stable germ. Call a map germ MT-stable if it satisfies the sufficient condition of the topological stability in [1].

4) Let k and ℓ be positive integers such that $k > \ell$. For any i $(n(\ell)+1 \leq i \leq s(\ell))$, there exists j $(n(k)+1 \leq j \leq s(k))$

such that $(\pi^{k,\ell})^{-1}(V_i^\ell) \subset V_j^k$ *and for any* i $(1 \le i \le n(\ell))$, *there exists* j $(1 \le j \le n(k))$ *such that* $(\pi^{k,\ell})^{-1}(V_i^\ell) = V_j^k$.

5) $\operatorname{codim} \Sigma_0^\ell(n+r,p) \to \infty$ *as* $\ell \to \infty$.

6) $\{V_1^\ell,\dots,V_{n(\ell)}^\ell,V_{n(\ell)+1}^\ell,\dots,V_{s(\ell)}^\ell\}$ *is a Whitney stratification of* $J^\ell(n+r,p) - \Sigma_0^\ell(n+r,p)$.

For any $\ell \ge 1$, we denote the stratification of $J^\ell(n+r,p) - \Sigma_0^\ell(n+r,p)$ which is constructed in the above theorem by $S^\ell(n+r,p)$. Let $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be a map germ such taht $j_0^\ell f \notin \Sigma_0^\ell(n+r,p)$, then let $S^\ell(f)$ be the strata of $S^\ell(n+r,p)$ which contains $j_0^\ell f$.

Definition (3.2). Let $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be a map germ.

(1) We say that f *has a finite almost P-V-codimension* if there exists an integer ℓ such that $j_0^\ell f \notin \Sigma_0^\ell(n+r,p)$.

(2) If f has a finite almost P-V-codimension, we define

$\text{A.P-V-cod}(f) = \operatorname{codim} S^\ell(f) + p$.

We call it an *almost P-V-codimension*.

By Theorem (3.1), A.P-V-cod(f) is well-defined.

The following notions are due to [8]. If $A_k \subset J^k(n+r,p)$ are semialgebraic sets, with $A_k \subset (\pi^{k+1,k})^{-1}(A_k)$, the set A of map germ f with $j_0^k f \in A_k$ for all k is said to be *prosemialgebraic*. Clearly, $\operatorname{codim} A_k \le \operatorname{codim} A_{k+1}$: we write codim A for the limit as $k \to \infty$. A property of map germ which holds for all except those in prosemialgebraic set of infinite codimension is said to <u>hold in general</u>.

By Theorem (3.1), we have the following theorem.

<u>Theorem (3.3)</u>. (1) <u>Map germs have finite almost P-V-codimensions in general</u>.

(2) <u>Let</u> $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ <u>be a map germ such that</u> $A.P\text{-cod}(f) < +\infty$. <u>Then</u> f <u>is finitely determined relative to P-V</u>.

Throughout remainder of this section we consider the almost P-V-versality theorem.

We now define a stratification of

$R^n \times R^r \times (R^p - \{0\}) \times J^\ell(n+r,p) \cup R^n \times R^r \times \{0\} \times (J^\ell(n+r,p) - \Sigma_0^\ell(n+r,p))$ by

$S^\ell(R^n \times R^r, R^p) = \{R^n \times R^r \times (R^p - \{0\}) \times J^\ell(n+r,p)\} \cup (R^n \times R^r \times \{0\} \times S^\ell(n+r,p))$.

We let $\tilde{S}^\ell(f) = R^n \times R^r \times \{0\} \times S^\ell(f)$.

<u>Definition (3.4)</u>. Let $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be a map germ such that $j_0^\ell f \notin \Sigma_0^\ell(n+r,p)$. Let $F : (R^n \times R^r \times R^s, 0) \longrightarrow (R^p, 0)$ be a deformation of f. We define a set germ $D_\ell(F;P\text{-}V)$ by $\{v \in (R^s,0) \mid j_0^\ell F_v \in R^n \times R^r \times \{0\} \times (J^\ell(n+r,p) - TS^\ell(n+r,p))\}$. We call it an <u>$\ell$-P-V-discriminant of</u> F. Here, $F_v(x,u) = F(x,u,v)$, $TS^\ell(n+r,p) = V^\ell_{n(\ell)+1} \cup \cdots \cup V^\ell_{s(\ell)}$.

If $A.P\text{-}V\text{-cod}(f) = s$ and $F : (R^n \times R^r \times R^s, 0) \longrightarrow (R^p, 0)$ is a deformation of f. We define $J_F^\ell : (R^s, 0) \longrightarrow J^\ell(R^n \times R^r, R^p)$ by $J_F^\ell(v) = j_0^\ell F_v$. It is clear that codim $\tilde{S}^\ell(f) = A.P\text{-cod}(f) = s$ and codim $\Sigma_0^\ell(n+r,p) > s$ for sufficiently large ℓ. In this situation if J_F^ℓ is transverse to $\tilde{S}^\ell(f)$, then J_F^ℓ avoids $\Sigma_0^\ell(n+r,p)$.

Then $(J_F^\ell)^{-1}(S_0^\ell(R^n\times R^r,R^p))$ is a Whitney stratification of $D_\ell(F;P\text{-}V)$, where $S_0^\ell(R^n\times R^r,R^p) = R^n\times R^r\times\{0\}\times S_0^\ell(n+r,p)$ and $S_0^\ell(n+r,p) = S^\ell(n+r,p) - S^\ell|TS^\ell(n+r,p)$. We denote this stratification by $\mathcal{D}_\ell(F;P\text{-}V)$.

Our almost P-V-versality theorem is the following.

Theorem (3.5). *Let $f : (R^n\times R^r,0) \longrightarrow (R^p,0)$ be a map germ such that A.P-V-cod(f) = s. Suppose that ℓ is sufficiently large such that codim $\Sigma_0^\ell(n+r,p) > n + r + p + s + 1$. Let $F : (R^n\times R^r\times R^s,0) \longrightarrow (R^p,0)$ be a deformation of f such that J_F^ℓ is transverse to $\tilde{S}^\ell(f)$ at 0. Then F is an almost P-V-versal deformation of f.*

For the proof of the above theorem, it is enough to prove the following uniqueness theorem.

Theorem (3.6). *With the same hypothesis of Theorem 3.5, if F, G : $(R^n\times R^r\times R^s,0) \longrightarrow (R^p,0)$ are deformations of f such that J_F^ℓ and J_G^ℓ are transverse to $\tilde{S}^\ell(f)$ at 0, then these are A.P-V-equivalent as deformations.*

Theorem (3.5) is a corollary of Theorem (3.6). It is shown by the usual technique. The proof of Theorem (3.6) is given by using Thom's local second isotopy lemma. By the proof of Theorem (3.6), deformations F, G which appeared in the theorem have homeomorphic ℓ-P-V-discriminants.

4. P-V-invariants

In this section we introduce some P-V-invariants of analytic map germs : $(K^n \times K^r, 0) \longrightarrow (K^p, 0)$, where K = C or R.

For any integer m, we write $[m]_2$ for the congruence class of the integers congruent to m modulo 2.

In general, we have the following lemma.

Lemma (4.1). *Let* $f : (K^n \times K^r, 0) \longrightarrow (K^p, 0)$ *be a map germ such that* $\mathcal{K}\text{-cod}(f, \pi_r) < +\infty$. *Then*

$\Sigma^*(f) \cap \pi_r^{-1}(0) = \Sigma(f, \pi_r) \cap (f, \pi_r)^{-1}(0) \subset \{0\}$.

Here, $\Sigma^*(f) = \{(x,u) \in (R^n \times R^r, 0) \mid \text{rank}(\frac{\partial f_i}{\partial x_j}(x,u)) < p$ *and* $f(x,u) = 0\}$.

Definition (4.2). We write

$$Q(f, \pi_r) = \begin{cases} {}_{n+r}\mathcal{O}/\langle f_1, \ldots, f_p, u_1, \ldots, u_r \rangle & \text{if } K = C \\ C_0^\infty(R^n \times R^r)/\langle f_1, \ldots, f_p, u_1, \ldots, u_r \rangle & \text{if } K = R. \end{cases}$$

Here, ${}_{n+r}\mathcal{O}$ is the local ring of holomorphic function germs at $0 \in C^n \times C^r$.

We remark that if f has a complexification f_C, then $Q(f_C, \pi_r) = Q(f, \pi_r) \otimes_R C$. In this case, we have $\dim_R Q(f, \pi_r) = \dim_C Q(f_C, \pi_r)$.

We distinguishe three cases.

A) n = p : We need the following lemma.

Lemma (4.3). a) Let $f : (C^n \times C^r, 0) \longrightarrow (C^p, 0)$ be a map germ such that $K\text{-cod}(f,\pi_r) < +\infty$, and suppose that the power series representing f converge on $D_n \times D_r \subset C^n \times C^r$ has been chosen small enough so that $f^{-1}(0) \cap D_n \times \{0\} = \{0\}$. Then there is a neighbourhood U of 0 in C^r such that for almost all $u \in U$

$$\dim_C Q(f,\pi_r) = \#(\pi_f^{-1}(u) \cap (D_n \times D_r)).$$

b) Let $f : (R^n \times R^r, 0) \longrightarrow (R^p, 0)$ be an analytic map germ such that $K\text{-cod}(f,\pi_r) < +\infty$, and suppose that the power series representing f converge on $D_n \times D_r \subset C^n \times C^r$ about 0. Suppose that $D_n \times D_r$ has been chosen small enough so that $f_C^{-1}(0) \cap D_n \times \{0\} = \{0\}$. Then there is a neighbourhood U of 0 in R^n such that for almost all $u \in U$

$$\dim_R Q(f,\pi_r) \equiv \#(\pi_f^{-1}(u) \cap (D_n \times D_r)) \bmod 2.$$

The following statement is an obvious statement.

Assertion (4.4). Let $f : (K^n \times K^r, 0) \longrightarrow (K^p, 0)$ be an analytic map germ such that $K\text{-cod}(f,\pi_r) < +\infty$.

a) If $K = C$, then $\dim_C Q(f,\pi_r)$ is a P-V-invariant.

b) If $K = R$, then $[\dim_R Q(f,\pi_r)]_2$ is a P-V-invariant.

B) $n < p$: In this case $\dim_C Q(f,\pi_r)$ does not equal to the branching order of (f,π_r). Let ${}_nM$ be the field of germs of meromorphic functions on C^n at 0 and $C(z_1,\dots,z_n)$ be the field of rational functions on C^n. Then we need the following well-known theorem.

Theorem (4.5). Let $f : (C^n,0) \longrightarrow (C^p,0)$ be a finite analytic map germ of branching order r. Then

$[{}_nM : f^*({}_pM)] = r.$

Moreover if f is a polynomial map germ, then

$[C(z_1,\ldots,z_n) : f^*(C(y_1,\ldots,y_p))] = r.$

Here $[K:k]$ is the degree of field extension K/k.

For the proof of the above theorem, see Theorem 10 in ([4], P35) and Theorem 7 in ([7], P117).

Definition (4.6). Let $f : (C^n,0) \longrightarrow (C^p,0)$ be a finite analytic map germ. We write

$$d(f) = \begin{cases} [C(z_1,\ldots,z_n) : f^*(C(y_1,\ldots,y_p))] & \text{if } f \text{ is a polynomial} \\ [{}_nM : f^*({}_pM)] & \text{otherwise.} \end{cases}$$

By the same reason as the case A), we have the following statement.

Assertion (4.6). Let $f : (K^n \times K^r,0) \longrightarrow (K^p,0)$ be an analytic map germ such that $K\text{-cod}(f,\pi_r) < +\infty$. We put $F = (f,\pi_r)$.

a) If $K = C$, then $d(F)$ is a P-V-invariant.

b) If $K = R$, then $[d(F_C)]_2$ is a P-V-invariant.

C) $n > p$: Let $f : (C^n \times C^r,0) \longrightarrow (C^p,0)$ be an analytic map germ. For sufficiently small positive numbers ε_1, ε_2, δ, we put $X = \{(x,u) \in C^n \times C^r \mid \|x\| < \varepsilon_1, \|u\| < \varepsilon_2, \|f(x,u)\| < \delta\}$ and

$T = \{u \in C^r \mid \|u\| < \varepsilon_2\}$.

Proposition (4.7). *If* K-cod$(f,\pi_r) < +\infty$, *then* $\pi_f| : f^{-1}(0) \cap X - \pi_f^{-1}(C^*(f)) \cap X \longrightarrow T - C^*(f)$ *is a locally trivial smooth fibration and its fibre has a homotopy type of* $\mu(F)$*-bouquet of* $(n-p)$*-sphere. Here* $\mu(F)$ *is the Milnor-Hamm number of* $F = (f,\pi_r)$. *Here* $C^*(f) = \pi_f(\Sigma^*(f))$.

By the theorem of Wall [9], we have the following statement.

Assertion (4.8). *Let* $f : (K^n \times K^r, 0) \longrightarrow (K^p, 0)$ *be an analytic map germ such that* K-cod$(f,\pi_r) < +\infty$.

a) *If* $K = \mathbf{C}$, *then* $\mu(F)$ *is a P-V-invariant*.

b) *If* $K = \mathbf{R}$, *then* $[\mu(F_{\mathbf{C}})]_2$ *is a P-V-invariant*.

Remark. By the formulae of Lê [6], we have the following calculation :

$$\mu(F) = \sum_{j=1}^{p} (-1)^{j+1} \dim_{\mathbf{C}} ({}_{n+r}\mathcal{O} / \langle u_1, \ldots, u_r, f_1, \ldots, f_{p-j}, J_x(f_1, \ldots, f_{p-j+1}) \rangle).$$

Here $J_x(f_1, \ldots, f_k) = \langle \text{minors} \frac{\partial(f_1, \ldots, f_k)}{\partial(z_{i_1}, \ldots, z_{i_k})} \mid 1 \le i_1 < \cdots < i_k \le n \rangle$ for $f_i \in {}_{n+r}\mathcal{O}$.

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