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Type II_1-factors with property T

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The property T for von Neumann algebras are defined by Connes-Jones in [5]. It is a reformulation of property T for groups due to Kazhdan [8]. The distinguished property of II_1-factors with property T is that the fundamental group of them are countable, and the group von Neumann algebra of a countable group G has property T if and only if G has property T [5]. Here I will discuss on type II_1-factors with property T.

In 2, I shall give some characterization of property T of type II_1-factors.

In 3, I shall investigate strongly ergodic outer actions of a group with property T on the hyperfinite II_1-factor R. In the case of an amenable discrete countable group K, for every outer action α of G on R, the crossed product $W^*(K,R,\alpha)$ is again a hyperfinite II_1-factor [4]. More precisely, in [9], Ocneanu proved that all outer actions of such a group on R are cocycle conjugate. In [7], Jones showed that the converse of this result is not true, that is, there are two non cocycle conjugate outer actions of a given non amenable group G on R. His secand action is given as the tensor product $\beta\otimes id$ of the so-called non-commutative Bernoulli shift action β and the trivial action id. Obviously, the action $\beta\otimes id$ is not ergodic. My first question is the following ; are there two ergodic non cocycle conjugate outer actions of a given non amenable group on R ? If two actions are cocycle conjugate, then the crossed products associated with them are isomorphic [10]. However, in general, the converse of this does not hold. So my second question is the following ; are there two ergodic outer actions of a group on R, which construct non isomorphic factors ? The result in §3 gives an answer for this second question.

2. Characterizations of property T for II_1-factors.

Let N be a type II_1-factor. A Hilbert space H is a correspondence from N to N if there are commuting both side actions of N on H which are continuous, that is, if (x_ι) in N converges to $x \in N$ in the *-strong topology, then $(x_\iota \xi)$ and (ξx_ι) converge to x weakly for all $\xi \in H$. The set of all correspondences from N to N is a topological space with some topology and N is said to have property T if the trivial correspondence $L^2(N,tr)$ is an isolated point in the set, which is equivalent to the following ; there are an $\varepsilon > 0$ and a finite subset $\{x_1,\dots,x_n\}$ of N such that if a correspondence H from N to N contains a norm 1 vector ξ with $\| x_i \xi - \xi x_i \| < \varepsilon$ $(i = 1,\dots,n)$, then there is a non zero vector $\eta \in H$ with $x\eta = \eta x$ for all $x \in N$ ([5]). Such a vector η is called a central vector of N in H.

In [1], Akemann-Walter give a characterization of property T for groups using positive definite functions. First, I shall give a characterization of property T for II_1-factors by complete positive maps.

Let N be a II_1-factor and N_* the predual of N. Then N_* is a partially ordered space with the order defined by positivity for linear functionals on N. For $f \in N_*$ and $x \in N$, let

$$x \circ f(y) = f(yx) \qquad f \circ x(y) = f(xy) \qquad (X \in N).$$

Let consider the *-strong topology for N and norm topology as linear functionals for N_*. Let N° be the opposite algebra of N

Theorem 1. <u>A type II_1-factor N has property T if and only if each net $\{\varphi_\iota\}$ of continuous completely positive linear maps from N to N_*°, such that $\varphi_\iota(1)$ is a state for each ι, satisfies the following ; if $\varphi_\iota(x) - \varphi_\iota(1)\circ x$ converges to 0 for all $x \in N$, then $\varphi_\iota(x) - \varphi_\iota(1)\circ x$ converges to 0 uniformly on the unit ball N_1 of N.</u>

<u>Proof</u>. Assume that N has property T. Let $\{\varphi_\iota\}$ be a net of continuous completely positive linear maps from N to N_*° such that $\varphi_\iota(1)$ is a state for all ι. A positive sesquilinear form on the algebraic tensor product $N \otimes N^{\circ}$ is defined by

$$(a \otimes b,\ c \otimes d) = (\ \varphi_\iota(c^*a))(bd^*) \qquad (a,c \in N,\ \ b,d \in N^{\circ})$$

because φ_ι is a completely positive linear map of N to N_*°. Let H be the Hilbert space obtained by this form from $N \otimes N^{\circ}$. Let N act on the left and right sides on H by the way ;

$$x(a \otimes b) = xa \otimes b, \qquad (a \otimes b)x = a \otimes bx \qquad (x,a,b \in N).$$

Since φ_ι is continuous, this actions are normal, so that H is a correspondence from N to N. Put $\xi_\iota = 1 \otimes 1 \in H_\iota$. Then $\|\xi_\iota\|^2 = (\ \varphi_\iota(1))(1) = 1$. Assume that $\|\ \varphi_\iota(x) - \varphi_\iota(1)\circ x\ \| \longrightarrow 0$ for all $x \in N$. Then, for any unitary $u \in N$,

$$\|\ u\xi_\iota - \xi_\iota u\ \|^2 = 2 - 2\mathrm{Re}(\ \varphi_\iota(u))(u^*) \longrightarrow 0.$$

Since a element in N is decomposed into a linear combination of unitaries, by [5], we have a net $\{\eta_\iota\}$ such that

$$0 \neq \eta_\iota \in H\ , \quad \|\xi_\iota - \eta_\iota\| \to 0, \qquad x\eta_\iota = \eta_\iota x \qquad (x \in N).$$

We may assume that $\|\eta_\iota\| = 1$. Since the functional $(.\ \eta_\iota, \eta_\iota)$ is the canonical trace on N,

$$|(\varphi_\iota(x))(y) - (\varphi_\iota(1)\circ x)(y)|$$
$$= |(\xi_\iota y, x^*\xi_\iota) - (\xi_\iota y, \xi_\iota x^*)|$$
$$\leqq |(\xi_\iota y, x^*\xi_\iota - x^*\eta_\iota)| + |(\xi_\iota y, \eta_\iota x^* - \xi_\iota x^*)|$$
$$\leqq 2\|\xi_\iota - \eta_\iota\|$$

for all x and y in N_1. Hence

$$\sup \{ \|\varphi_\iota(x) - \varphi_\iota(1)\circ x\| \ ;\ x \in N_1 \} \longrightarrow 0.$$

Conversely, assume that N does not have property T. Then there exists a net $\{H_\iota, \xi_\iota\}$ of correspondences H_ι without non trivial central vectors and unit vectors $\xi_\iota \in H_\iota$ such that $\{x\xi_\iota - \xi_\iota x\}$ converges to 0 for all $x \in N$. Let take and fix an $x \in N$. Put

$$(\varphi_\iota(x))(y) = (x\xi_\iota y, \xi_\iota).$$

Since the action of N on correspondences is normal, $\varphi_\iota(x)$ is a σ-weakly continuous linear functional on N and φ_ι is a continuous linear map from N to $\overset{\circ}{N}_*$. Since ξ_ι is a unit vector, $\varphi_\iota(1)$ is a state. Identifing $M_n(N^*)$ with the dual $(M_n(N))^*$ by

$$f(a) = \sum_{i,j} f_{ij}(a_{ij}),$$

for $f = [f_{ij}] \in M_n(N^*)$, $a = (a_{ij}) \in M_n(N)$, in order to show the complete positivity of φ_ι, it is sufficient to prove

$$\sum \varphi_\iota(x_i^* x_j)(a_j a_i^*) = \|\sum_{i=1}^n x_i \xi_\iota a_i\|^2 \geqq 0.$$

Hence each φ_ι is completely positive. For an $x \in N$,

$$\|\varphi_\iota(x) - \varphi_\iota(1)\circ x\|$$
$$= \sup \{ |(x\xi_\iota y, \xi_\iota) - (\xi_\iota yx, \xi_\iota)| \ ;\ y \in N_1 \}$$

$$\leqq \| x^*\xi_\iota - \xi_\iota x^* \| \longrightarrow 0.$$

Suppose that $\{ \varphi_\iota(x) - \varphi_\iota(1)\circ x \}$ converges to 0 uniformly on N_1. Then for any $\varepsilon > 0$, there are ι such that

$$\| \varphi_\iota(x) - \varphi_\iota(1)\circ x \| < \varepsilon \quad \text{for all} \quad x \in N_1.$$

So there exists a ξ_ι such that

$$| (u \xi_\iota u^*, \xi_\iota) - 1 | < \varepsilon < 1 \quad \text{for all unitary} \quad u \in N.$$

Hence there exists a non zero $\eta \in H_\iota$ such that $u\eta = \eta u$ for all unitary $u \in N$ (cf,[6]). This contradicts that H_ι has no non zero central vectors. Hence $\{\varphi_\iota(x) - \varphi_\iota(1)\circ x\}$ does not converge to 0 uniformly on N_1.

Corollary 2. <u>Assume that a II_1-factor has property</u> T. <u>Let</u> $\{\varphi_\iota\}$ <u>be a net of continuous completely positive linear maps on</u> N <u>such that</u> $\varphi_\iota(1) = 1$. <u>If</u> $\| \varphi_\iota(x) - x \|_2 \longrightarrow 0$ <u>for all</u> $x \in N$, <u>then</u>

$$\sup \{ \| \varphi_\iota(x) - x \|_2 \ ; \ x \in N, \|x\| \leqq 1 \} \longrightarrow 0.$$

Let M be a von Neumann algebra and H a correspondence from M to M. A linear map δ from M to H is called a <u>derivation</u> if $\delta(ab) = a\delta(b) + \delta(a)b$ for all a and b in the domain $D(\delta)$ of δ. A linear map δ from M to H is called <u>bounded</u> if $\sup \{ \| \delta(a) \| \ ; a \in M, \|a\| \leqq 1 \} < +\infty$.

Theorem 3. <u>If a type II_1-factor</u> N <u>does not have property</u> T, <u>there exists a unbounded derivation from</u> N <u>to a correspondence, the domain of which is a weakly dence *-subalgebra of</u> N <u>containing the identity</u>.

<u>Proof.</u> Let S_o be a countable subset of N_1 such that S_o is dense in N_1 with respect to $\| \cdot \|_2$ defined by the canonical trace tr of N. We may assume that S_o contains the identity of N. Let S be the set of products of finite elements in $S_o \cup S_o^*$. Then S is a countable $\| \cdot \|_2$-dense subset of N_1. Put $S = \{ x_1, x_2, \dots, x_n, \dots \}$.

If N does not have property T, then for each integer n, there exist a correspondence H_n from N to N without non zero central vector for N and a $\xi_n \in H_n$ such that $\| \xi_n \| = 1$ and that $\| x_j \xi_n - \xi_n x_j \| < 1/4^n$. Assume that for each $\varepsilon > 0$ there exists an integer m such that $\| x_i \xi_m - \xi_m x_i \| < \varepsilon$ for all i. Since S is a $\| \cdot \|_2$-dense in N_1, for any $\varepsilon > 0$, there exists an m such that $\| u \xi_m u^* - \xi_m \| < \varepsilon$ for all unitary $u \in N$. Let ζ be $\| \zeta \| = \inf \{ \| \xi \| ;\ \xi$ is in the norm closure of the convex hull of $u \xi_m u^*$, u is a unitary in $N \}$. Then ζ is a non zero central vector in H_m. This is a contradiction. Hence we have a sequence $\{ H_n,\ \xi_n \in H_n,\ \alpha > 0,\ a_n \in S \}$ such that

$$\| \xi_n \| = 1,\ \| x_j \xi_n - \xi_n x_j \| < 1/4^n \ (1 \leqq j \leqq n),\ \| a_n \xi_n - \xi_n a_n \| \geqq \alpha .$$

Put

$$H = \sum_{n=1}^{+\infty} \oplus H_n .$$

We shall define a new action of N on H by

$$x \zeta = (x \zeta_n), \quad \zeta x = (\zeta_n x) \quad (x \in N,\ \zeta_n \in H_n,\ \zeta = (\zeta_n) \in H).$$

Then H is an again correspondence from N to N. For each n, let $\tilde{\xi}_n$ be $\zeta = (\zeta_i) \in H$ such that $\zeta_n = \xi_n$ and $\zeta_i = 0$ if $i \neq n$. Put

$$\delta(x) = \sum_{n=1}^{+\infty} 2^n (x \tilde{\xi}_n - \tilde{\xi}_n x), \qquad (x \in S).$$

Let take an $x \in S$, then $x = x_i$ for some i. Since $\| x_i \xi_n - \xi_n x_i \| < 1/4^n$ for all $n > i$, the series converges. Hence δ is well defined on S and we shall extend δ to the set $D(\delta)$ of finite combinations of elements in S by the natural method. Then δ is a derivation of N to the correspondence H. By the following inequality, δ is unbounded ;

$$\| \delta(a_m) \|^2 = \sum_{n=1}^{+\infty} 2^{2n} \| a_m \xi_n - \xi_n a_m \|^2 \geqslant 2^{2m} \alpha^2 .$$

A type II_1-factor N with property T does not have property Γ of Murray and von Neumann. A. Connes gives some characterization of property Γ ([4]). Next, we shall give a characterization of a negation of property T.

Theorem 4. <u>A II_1-factor N does not have property T if and only if, for any finite subset $\{ x_1, \dots, x_n \}$ of N, there are a correspondence H from N to N with a non trivial central vector and a sequence (ξ_k) of unit vectors in H such that $\| x_j \xi_k - \xi_k x_j \| \xrightarrow[k\to\infty]{} 0$ for all $j = 1, \dots, n$ but that (ξ_k) does not tend to any central vector when $k \to \infty$.</u>

<u>Proof</u>. Assume that N does not have property T. Let take any finite subset $\{ x_1, \dots, x_n \}$ of N. Then for any integer k, there is a correspondence H_k from N to N with a unit vector η_k such that $\| \eta_k x_j - x_j \eta_k \| < 1/k$ for all $j = 1, \dots, n$ but H_k does not contain any non zero central vector. Let tr be the canonical trace. Let

$$H = L^2(N, tr) \oplus \sum_{k=1}^{+\infty} \oplus H_k .$$

For an $x \in N$ and $\zeta = (\zeta_j)$ ($\zeta_0 \in L^2(N,tr)$, $\zeta_k \in H_k$ for $k \geqslant 1$), let $x\zeta = (x\zeta_j)$, and $\zeta x = (\zeta_j x)$. Then H is a correspondence from N to N. Since all H_k's have no non zero central vectors, central vectors in H are contained in $\mathbb{C}1 \subset L^2(N, tr)$. Let ξ_k be the vector in H such that $\xi_k = (0,\ldots,0,\eta_k,0,\ldots)$, where η_k is in the k-component. Then the sequence (ξ_k) satisfies the properties of the statement.

Convesely,assume that N has property T. Then by [5], there exist $\varepsilon > 0$, $y_1,\ldots,y_m \in N$, $K > 0$ such that for any correspondence H from N to N and a vector $\xi \in H$, $\|\xi\| = 1$ with $\|y_j\xi - \xi y_j\| < \varepsilon' < \varepsilon$, there is a central vector η with $\|\eta - \xi\| < K\varepsilon'$. Let take $\{y_1,\ldots,y_m\}$ as a finite subset $\{x_1,\ldots,x_n\}$. If there are such a correspondence H from N to N and a sequence (ξ_k) as in the statement, (ξ_k) must converge to some central vector. This is a contradiction.

In the case where we can take the trivial correspondence as the correspondence H in Theorem, it is a caracterization of property Γ.

Let N be a II_1-factor and H a correspondence from N to N. Let u be the unitary representaion of the unitary group $U(N)$ of N defined by $\rho(u)\xi = u\xi u^*$ for all $u \in U(N)$ and $\xi \in H$.

In [1], Akemann-Walter gave a characterization of property T for locally compact groups using some distinguished projection. Next we shall give a characterization of property T for factors by some projection.

Let H be a correspondence from N to N with a non zero central vector. Let P_c be the projection of H onto the subspace spaned by all central vectors.

Theorem 5. <u>A II_1-factor</u> N <u>has property</u> T <u>if and only if, for every correspondence with non zero central vectors, the projection</u> P_c <u>is contained in the</u> C*-<u>algebra</u> $C^*(\rho(U(N))$ <u>generated by</u> $\rho(U(N))$.

<u>Proof</u>. Assume that N has property T. Then by Theorem 5, there exists a finite subset $\{u_1,\dots,u_n\}$ of N such that, for any correspondence H from N to N and a sequence (ξ_k) of unit vectors in H, if $\| x_j\xi_k - \xi_k x_j\| \to 0$ for all j, then (ξ_k) must tend to some central vector when $k \to \infty$. We may assume the set is a set of unitaries in N which contains the identity 1 and *-closed. Let take a correspondence H from N to N with non zero central vectors. Put

$$a = \sum_{j=1}^{n} \rho(u_j),$$

then a is an self adjoint operator in $C^*(\rho(U(N))$ such that $a\zeta = n\zeta$ for all cental vectors $\zeta \in H$. Let $K = H \ominus P_c(H)$. If the restriction of $a - n1$ to K is not invertible, then there exists a sequence (ξ_k) of unit vectors in K such that $\| a\xi_k - n\xi_k\| \to 0$ when $k \to \infty$. Since

$$| \; \|a\xi_k\| - n \; | \leqq \|a\xi_k - n\xi_k\| \underset{k\to\infty}{\longrightarrow} 0,$$

$$\|\rho(u_j)\xi_k\| = 1 \quad \text{and} \quad \|a\xi_k\| \leqq n \quad \text{for all } k,j,$$

we have that

$$\| u_j\xi_k - \xi_k u_j\| = \|\rho(u_j)\xi_k - \rho(1)\xi_k\| \underset{k\to\infty}{\longrightarrow} 0 \quad (1 \leq j \leq n).$$

Hence the sequence (ξ_k) of unit vectors in K must converge to a vector in $P_c(H)$. This is a contradiction. Therefore the proper value n of a is an isolated point of spectrum of a. Hence the projection P_c is contained in the C*-subalgebra of $C^*(\rho(U(N)))$ generated by a.

Conversely, assume that N does not have property T. Then there is a net $\{H_\iota, \xi_\iota\}$ of correspondences H_ι without non zero central vectors and an $\xi_\iota \in H_\iota$ such that $\|\xi_\iota\| = 1$ and $\|x_j \xi_\iota - \xi_\iota x_j\| \to 0$ for every finite subset $\{x_1, \ldots, x_n\}$ of N. Let H be the direct sum of $L^2(N, tr)$ and the direct sum of all H_ι's. Let ζ_ι be the vector in H such that the ι- component is only one non zero vector ξ_ι. Let N act on H from left and right sides by the diagonal form. Then H is a correspondence from N to N with the central vectors $\mathbb{C}1$ in $L^2(N,tr)$. The projection P_c is the 1-dimensional projection on 1. If P_c is contained in $C^*(\rho(U(N)))$, then for any $\varepsilon > 0$, there are a finite set of unitaries $\{u_1, \ldots, u_n\}$ and complex numbers $\{c_1, \ldots, c_n\}$ such that

$$\| P_c - \sum_j c_j \rho(u_j) \| < \varepsilon .$$

Then

$$\left| \, \left| \sum_{j=1}^{n} c_j \right| - 1 \, \right| \leqq \| (\sum_j c_j \rho(u_j))(1) - P_c(1) \| < \varepsilon$$

and

$$\begin{aligned} \left| \sum_{j=1}^{n} c_j \right| &= \| \sum_j c_j \xi_\iota \| \\ &\leqq \| \sum_j c_j \xi_\iota - \sum_j c_j \rho(u_j) \zeta_\iota \| + \| \sum_j c_j \rho(u_j) \zeta_\iota - P_c \zeta_\iota \| \\ &\leqq 2 \varepsilon \end{aligned}$$

for sufficiently large ι. This is a contradiction.

3. Two strongly ergodic actions of $Sp(n,Z)$ on R.

Here I shall give an answer for second question in §1. Non compact groups with property T are non amenable. Special linear groups with the orders larger than 3 are typical groups with property T. In[3], we showed the existence of two strongly ergodic outer actions of $SL(n,Z)$ on the hyperfinite II_1-factor R, one of which construct a II_1-factor with property T and the other gives a II_1-factor without property T. Another examples of groups with property T is symplectic groups. In this place, I shall give two strongly ergodic actions of $Sp(n,Z)$ on R. The grøoup $SL(n,Z)$ can be naturally imbedded into the group $Sp(n,Z)$. Then under the inclusion, the restricted actions of $Sp(n,Z)$ to $SL(n,Z)$ coinsides with the actions in [3].

Let take and fix an integer n. Let $B(a,b)$ be the symplectic form of a ans b in the vector space Z^{2n} (Z is the set of all integers) :

$$B(a,b) = \sum_{i=1}^{n} a_i b_{n+i} - \sum_{i=1}^{n} a_{n+i} b_i,$$

where $a = (a_1,\dots,a_n, a_{n+1},\dots,a_{2n}) \in Z^{2n}$.

Let consider two groups

$$K = Sp(n,R) \times_s R^{2n} \qquad \text{and} \qquad G = Sp(n,Z) \times_s Z^{2n} \quad ;$$

the semidirect product,the product of whose elements are defined by

$$(S,a)(T,b) = (ST,\ T^{-1}(a) + b), \quad (S,T \in Sp(n,R),\ a,b \in R^{2n}),$$

where the action of $Sp(n,R)$ on R^{2n} is defined by the obvious manner.

Lemma 6. The groups K and G have property T .

Proof. In [12], it is proved that the semidirect product $SL(n,R) \times_s R^n$ has property T, where the product is defined by the same manner as K. The proof is due to the following :

The semidirect product $\Gamma \times_s A$, of a locally compact group A by a locally compact group Γ with property T under an action α, has property T if

(i) Γ has a subgroup L such that for every continuous unitary representation u of Γ, there is a non zero fixed point under u when the restriction u to L has one and that A is generated by the set $\{a \in A;$ there is a $g \in \Gamma$ with $\alpha_g(a) = \alpha_{hg}(a)$ for all $h \in L\}$,

(ii) the group generated by $\{\alpha_g(a)a^{-1}\ ;\ a \in A,\ g \in \Gamma\}$ is dense in A.

In the case of $\Gamma = Sp(n,R)$, let L be the subgroup of matrices $\begin{pmatrix} E & B \\ 0 & E \end{pmatrix}$, where E is the identity n by n matrix and B is an n by n matrix with $B = {}^tB$(the transposed matrix of B). Then L satisfies the conditions in (i)([6]). The group $Sp(n,R)$ contains the matrix $\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ and $\begin{pmatrix} D & 0 \\ 0 & {}^tD^{-1} \end{pmatrix}$.

Hence the natural action of $Sp(n,R)$ on R^{2n} satisfies the conditions in (i) and (ii). Therefore K has property T. The group G has a finite covolume in K, so that G has property T.

Let take an irrational s in $[0, \pi/2]$ mod. 2π. Let

$$\mu_s((S,a),(T,b)) = \exp(isB(T^{-1}a,b)),$$

for $S,T \in Sp(n,Z)$ and $a,b \in Z^{2n}$. Then μ_s is a normalized 2-cocycle of $G \times G$ to the torus T :

(1) $\mu_s(g,h) = 1$

(2) $\mu_s(1,g) = \mu_s(h,1) = \mu_s(g,g) = 1$

(3) $\mu_s(f,h)\,\mu_s(fh,k) = \mu_s(h,k)\,\mu_s(f,hk)$

for all f,g,h,k in G. In fact, it is clear that the condition (2) is satisfied because $B(.,.)$ is the symplectic form. The relation (3) is proved by the following :

$$\mu_s((S,a),(Tb))\,\mu_s((S,a)(T,b),(S',c))$$

$$= \exp(is(\,B(T^{-1}(a),b) + B(T^{-1}(a),S'(c)) + B(b,S'(c))\,))$$

$$= \mu_s((T,b),(S',c)) \quad \mu_s((S,a),(T,b)(S',c)).$$

The left μ_s-regular representation of G on $l^2(G)$ is defined by the following :

$$(\,\lambda^s(g)\,\xi\,)(h) = \mu_s(h^{-1},g)\,\xi(g^{-1}h), \qquad (g,h\in G,\ \xi\in l^2(G)).$$

Then λ^s is a unitary representation of G with the cocycle μ_s, that is,

$$\lambda^s(g)\,\lambda^s(h) = \mu_s(g,h)\,\lambda^s(gh), \qquad (g,h\in G).$$

Lemma 7. <u>The von Neumann algebra</u> R_s <u>generated by</u> $\lambda^s(1,Z^{2n})$ <u>is a hyperfinite</u> II_1<u>-factor</u>.

<u>Proof.</u> Let $\{\,e_j\ ;\ 1\leqq j\leqq 2n\,\}$ be the natural basis in Z^{2n}. Then the unitaries $\{\,\lambda^s(e_j)\ ;\ 1\leqq j\leqq 2n\,\}$ satisfies

$$\{\,\lambda^s(e_j)\ ;\ 1\leqq j\leqq n\,\}\ \text{are commutative,}$$

$$\{\,\lambda^s(e_{n+j})\ ;1\leqq j\leqq n\,\}\ \text{are commutative}$$

$$\lambda^s(e_j)\,\lambda^s(e_{n+k}) = \exp(\,\delta_{j,k}\,2is)\,\lambda^s(e_{n+k})\,\lambda^s(e_j),$$

for $j,k = 1,\dots,n$. Let A_j be the von Neumann algebra generated by $\{ \lambda^s(e_j), \lambda^s(e_{n+j}) \}$. Then A_j is a hyperfinite II_1-factor because s is irrational. Since A_j and A_k commute for $j \neq k$, the algebra R_s is a hyperfinite II_1-factor.

Next we shall define an action α^s of $Sp(n,Z)$ on R_s by

$$\alpha^s(T)(a) = \lambda^s(T,0)\, a\, \lambda^s(T,0)^* \qquad (a \in R_s).$$

Lemma 8. The action α^s is an ergodic outer action of $Sp(n,Z)$ on the hyperfinite II_1-factor R_s.

Proof. BY the definition,

$$\alpha^s(T)(\lambda^s(1,a)) = \lambda^s(1, T(a)) \qquad (T \in Sp(n,Z),\ a \in Z^{2n}).$$

Hence α^s is an action of $Sp(n,Z)$ on R_s. The ergodicity and outerness of the action α^s is proved by a similar method as in [3].

Lemma 9. The von Neumann algebra $N(s,G)$ generated by $\lambda^s(G)$ is isomorphic to the crossed product $W^*(Sp(n,Z), R_s, \alpha^s)$.

Proof. The algebra is generated by R_s and the unitary group $\lambda^s(Sp(n,Z),0)$. Since $N(s,G)$ is finite and the action $Ad\ \lambda^s(T,0)$ is outer on R_s for all $T \in SP(n,Z)$, $N(s,G)$ is isomorphic to $W^*(Sp(n,Z), R_s, \alpha^s)$ by [13].

Lemma 10. The crossed product $W^*(Sp(n,Z),R_s, \alpha^s)$ is a type II_1-factor with property T.

Proof. It is clear that $W^*(Sp(n,Z), R_s, \alpha^s)$ is a II_1-factor. By Lemma 6, the group G has property T. Hence the group von Neumann algebra $N(s,G)$ defined by the normalized 2-cocycle $\mu_s(.,.)$ has property T. Therefore by Lemma 9, we have Lemma 10.

Let j be the imbedding of $GL(n,R)$ into $Sp(n,R)$ defined by

$$j(T) = \begin{pmatrix} T & 0 \\ 0 & {}^tT^{-1} \end{pmatrix} \qquad (T \in GL(n,R))$$

Put $\gamma^s = \alpha^s \circ j$, then γ^s is an action of $GL(n,Z)$ on the hyperfinite II_1-factor R_s. This action γ^s is the action α^s of $GL(n,Z)$ on the hyperfinite II_1-factor R, which is discussed in the previous paper [3]. In [3], we proved that the crossed product $W^*(SL(n,Z), R, \gamma^s)$ is a II_1-factor with property T. Now we have another proof of it as follows.

Lemma 11. <u>The semidirect products</u> $j(SL(n,R)) \times_s R^{2n}$ <u>and</u> $j(SL(n,Z)) \times_s Z^{2n}$ <u>have property</u> T <u>for</u> $n \geqq 3$.

<u>Proof</u>. Let consider the subgroup $\left\{ j(T) \; ; \; T = \begin{pmatrix} 1 & 0 \\ x & I_{n-1} \end{pmatrix}, x \in R^{n-1} \right\}$. Then by a similar way as the proof of Lemma 6, it is prooved that these two groups have ptoperty T.

Lemma 12. <u>The crossed product</u> $W^*(SL(n,Z), R, \gamma^s)$ <u>has property</u> T <u>for</u> $n \geqslant 3$.

<u>Proof</u>. The algebra $W^*(SL(n,Z),R, \gamma^s)$ is isomorphic to the algebra $(\lambda^s(j(SL(n,Z) \times_s Z^{2n}))''$, which has property T by Lemma 11.

Let G be a countable group. Let A be a finite von Neumann algebra with a faithful normal normalized trace tr. Let M be the infinite tensor product $\bigotimes_{g \in G}(A_g, tr_g)$, where $A_g = A$ and $tr_g = tr$ for all $g \in G$. Let j_g be the canonical injection of A to $\cdots \otimes 1 \otimes A_g \otimes 1 \otimes \cdots \subset M$. The Bernoulli shift action of G for

(A,tr) is the action β of G on M such that $\beta_g(j_h(x)) = j_{gh}(x)$ for all $x \in A$. Then in [3], we proved that for every gtoup G the Bernoulli shift action β constructs by the crossed product a von Neumann algebra which does not have property T. Hence we have the following :

Theorem 13. <u>The symplectic group</u> Sp(n,Z)($n \geq 2$) <u>and the special linear group</u> SL(n,Z) ($n \geq 3$) <u>have two kind of ergodic outer actions on the hyperfinite</u> II_1-<u>factor</u> R, <u>one of which constructs a</u> II_1-<u>factor with property</u> T <u>and the other gives a full</u> II_1-<u>factor without property</u> T.

In the last I denote some remarks. A II_1-factor N is full if N does not have property Γ of Murray and von Neumann. Then the crossed products $W^*(GL(n,Z), R, \gamma^s)$, $W^*(GL(n,Z), R, \beta)$ $W^*(F_2, R, \gamma^s)$ and $W^*(F_2, R, \beta)$ are all full II_1-factors.

Let α (resp. β) be an action of a group G on a von Neumann algebra A(resp. B). Then α and β are <u>conjugate</u> if there exists an isomorphism θ of A onto B such that $\beta_g \theta = \theta \alpha_g$ for all $g \in G$. The actions $\{\alpha^s\}$ are non conjugate actions of Sp(n,Z) on the hyperfinite II_1-factor R.

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