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Approximate innerness of positive linear maps of factors of type II

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We in this paper shall discuss the properties of positive linear maps which continue from the former work by the author [7].

Let M be a σ-finite, semi-finite von Neumann algebra, then there exists a faithful, normal semi-finite trace Tr and we can define a norm $\|\cdot\|_2$ on the ideal $S = \{x \in M; Tr(x^*x) < +\infty\}$. In particular, if M is a finite von Neumann algebra, then $S = M$.

Let A and B be C^*-algebras. A linear map ρ of A to B is said to be n-positive if the multiplicity map ρ_n from the matrix algebra $M_n(A)$ over A to the algebra $M_n(B)$ over B defined by $\rho_n([a_{ij}]) = [\rho(a_{ij})]$ is a positive map. If ρ is n-positive for every positive integer n, we call ρ completely positive. Many autors (for example, [1], [5], [7], [8] and [9]) studied the completely positive linear maps of C^*-algebras. In particular, we have the following Stinespring's theorem [5]: Let A be a C^*-algebra and ρ a completely positive linear map of A to $B(H)$ where $B(H)$ is the von Neumann algebra of all bounded operators on a Hilbert space H. Then, there exists a representation π of A to a Hilbert space K and a bounded operator v of H to K such that $\rho(x) = v^*\pi(x)v$ for every

$x \in A$. In particular, if ρ is unital (ie. $\rho(1) = 1$), v is an isometry. Furthermore, A is a von Neumann algebra and ρ is normal, then π is a normal representation. We in general can not take the above operator v in A. For this problem, we have the following result by Haagerup [3; Proposition 2.1]: Let N be a properly infinite von Neumann algebra and let F be a finite dimensional subfactor. Let ρ be a completely positive map from F to N. Then there exists an element $a \in N$ such that $\rho(x) = a^*xa$ for every $x \in F$. In this report, we shall consider the above problem for finite von Neumann algebras by using the approximate innerness and extend the obtained results to the semi-finite von Neumann algebras. Thus, we here introduce the notation of the approximate innerness.

Definition 1. Let M be a σ-finite, finite von Neumann algebra with a fixed faithful, normalized normal trace Tr and A a C*-subalgebra of M. A positive linear map ρ of A into M is approximate inner if there exists a net $\{a_\lambda\}$ (not necessarily bounded) in M satisfying $\lim\|\rho(x) - a_\lambda{}^*xa_\lambda\|_2 = 0$ for every $x \in A$.

If we consider the approximate innerness for positive linear maps, we can show that those positive linear maps are closely related to the *-homomorphisms. Before we denote the theorems, we shall mention the following lemma by Choi [1] (and also see [9]).

Lemma 2. Let A and B be unital C*-algebra and ρ a unital completely positive map of A to B. If ρ is a C*-homomorphism (ie., $\rho(a^2) = \rho(a)^2$ for every self-adjoint element a of A), then ρ is a *-homomorphism of A to B.

Consider Lemma 2, we have the following theorem that a positive linear map with the approximate innerness is closely related to *-homomorphism. The following theorem is in a sense a generalization of Theorem 3 in [7].

Theorem 3. Let M be a σ-finite,finite von Neumann algebra and A a C*-subalgebra with the unit in M. Let ρ be a positive linear map of A to M and approximate inner with respect to a net $\{a_\lambda\}$ such that $||a_\lambda{}^*a_\lambda - e||_2 \to 0$ and $||a_\lambda a_\lambda{}^* - f||_2 \to 0$ for a projection e of M and a projection f in A. Then, ρ is a *-homomorphism of fAf to eMe.

Proof. By the assumption for the net $\{a_\lambda\}$ and the approximate innerness of ρ with respect to $\{a_\lambda\}$, $\rho(1) = e$ and $\rho(1 - f) = 0$. Thus, we can assume that $fa_\lambda e = a_\lambda$ for every $\lambda \varepsilon \Lambda$. By the remark before Definition 1, ρ is completely positive map. So ρ is a unital completely positive map of C*-algebra fAf to von Neumann algebra eMe. To show that ρ is a *-homomorphism of fAf to eMe, we must show by Lemma 2 that

$\rho(x^2) = \rho(x)^2$ for every self-adjoint element $x \varepsilon fAf$. Given an arbitrary self-adjoint element $x \varepsilon fAf$. Then,

$$||\rho(x) - a_\lambda^* x a_\lambda||_2^2 = Tr(\rho(x)^2) - 2Tr(\rho(x)a_\lambda^* x a_\lambda) + Tr(a_\lambda^* x a_\lambda a_\lambda^* x a_\lambda).$$

Now, since

$$|Tr(a_\lambda^* x a_\lambda a_\lambda^* x a_\lambda - a_\lambda^* x^2 a_\lambda)|$$

$$= |Tr(a_\lambda^* x(a_\lambda a_\lambda^* - f)x a_\lambda)| = |Tr((a_\lambda a_\lambda^* - f)x a_\lambda a_\lambda^* x)|$$

$$\leqq Tr((a_\lambda a_\lambda^* - f)^2)^{1/2} Tr(x a_\lambda a_\lambda^* x^2 a_\lambda a_\lambda^* x)^{1/2}$$

$$\leqq ||a_\lambda a_\lambda^* - f||_2 \cdot ||x|| \cdot Tr(x a_\lambda a_\lambda^* a_\lambda a_\lambda^* x)^{1/2}$$

$$\leqq ||x||^2 \cdot ||a_\lambda a_\lambda^* - f||_2 \cdot Tr(a_\lambda a_\lambda^* a_\lambda a_\lambda^*)^{1/2}$$

$$= ||x||^2 \cdot || a_\lambda a_\lambda^*||_2 \cdot ||a_\lambda a_\lambda^* - f||_2,$$

$\{||a_\lambda a_\lambda^*||_2\}$ is bounded and $\ell im||a_\lambda a_\lambda^* - f||_2 = 0$, we have the relation

$$\ell im\{Tr(a_\lambda^* x a_\lambda a_\lambda^* x a_\lambda) - Tr(a_\lambda^* x^2 a_\lambda)\} = 0.$$

Thus, since $\ell im\ Tr(a_\lambda^* x^2 a_\lambda) = Tr(\rho(x^2))$ by the assumption,

$\ell im\ Tr(a_\lambda{}^*xa_\lambda a_\lambda{}^*xa_\lambda) = Tr(\rho(x^2))$. Furthermore, since

$$|Tr(\rho(x)a_\lambda{}^*xa_\lambda) - Tr(\rho(x)^2)| = |Tr(\rho(x)(a_\lambda{}^*xa_\lambda - \rho(x))|$$

$$\leqq ||\rho(x)||_2 \cdot ||\rho(x) - a_\lambda{}^*xa_\lambda||_2,$$

$\ell im\ Tr(\rho(x)a_\lambda{}^*xa_\lambda) = Tr(\rho(x)^2)$. By the above considerations and the relation $\ell im\ ||\rho(x) - a_\lambda{}^*xa_\lambda||_2 = 0$,

$$Tr(\rho(x)^2) - 2Tr(\rho(x)^2) + Tr(\rho(x^2)) = 0.$$

So, $Tr(\rho(x^2) - \rho(x)^2) = 0$. Now, since ρ is a completely positive map, $\rho(x)^2 \leqq \rho(x^2)$. Therefore, $\rho(x^2) = \rho(x)^2$ and so, by Lemma 2, ρ is a *-homomorphism of fAf to eMe. q.e.d.

Under the definition of approximate innerness, if ρ is approximate inner, then ρ is completely positive like as [7]. Furthermore, we can replace the conditions in Theorem 3 as the following by the remark in [7]. That is, if ρ is approximately inner with respect to $\{a_\lambda\}$ and $\rho(1) = e$ is a projection, then the conditions in Theorem 3 is equivalent that A has a projection f satisfying $\rho(1 - f) = 0$ and $Tr(e) = Tr(f)$.

By considering Theorem 3 and a Sakai's result [4], we have the following theorem.

Theorem 4. Let M be an approximately finite dimensional factor of type II_1. Let ρ be a positive linear map of M into M such that $\rho(1) = e$ is a projection, $\rho(1 - f) = 0$ and $Tr(e) = Tr(f)$ for a projection f of M. Then ρ is approximately inner with respect to a net $\{a_\lambda\}$ if and only if ρ is a *-isomorphism of fMf to eMe.

Proof. Necessity: By Theorem 3, ρ is a *-homomorphism of fMf to eMe, and so the kernel of ρ in fMf is a closed two-sided ideal. Since M is a finite factor, the kernel of $\rho = \{0\}$ and so ρ is a *-isomorphism of fMf to eMe.

Sufficiency: Since M is an approximately finite dimensional factor of type II_1, both fMf and eMe are so. Let $fMf = \widetilde{\bigcup A_n}$ ($\widetilde{\cdot}$ means the weak closure of $\cdot$) where A_n is a subfactor of type I_{2^n} of fMf satisfying $A_n \subset A_{n+1}$ ($n = 1, 2, \ldots$). Let $\{f_{ij}^{(n)}\}_{i,j=1}^{2^n}$ be the matrix units of A_n. Put $B_n = \rho(A_n)$, then $\rho(fMf) = N = \widetilde{\bigcup B_n}$ eMe and B_n is a factor of type I_{2^n}. Furthermore, put $e_{ij}^{(n)} = \rho(f_{ij}^{(n)})$, then $\{e_{ij}^{(n)}\}$ is the matrix units for B_n. It is sufficient for us to show that, for an arbitrary finite set $\{a_1, \ldots , a_k\}$ in fMf and each $\varepsilon > 0$, there exists an element $u \in M$ such that $||\rho(a_j) - u^*a_ju||_2 < \varepsilon$ ($j = 1, 2, \ldots , k$). Given any finite set $\{a_1, \ldots, a_k\}$ in fMf and $\varepsilon > 0$, then there exist a positive integer m and $\{b_1, \ldots, b_k\}$ in A_m such that $||a_j - b_j||_2 < \varepsilon/2$ $(j = 1, 2, \ldots, k)$. Since $Tr(e) = Tr(f)$,

$$\sum_{i=1}^{2^m} e_{ii}^{(m)} = e \quad \text{and} \quad \sum_{i=1}^{2^m} f_{ii}^{(m)} = f,$$

$Tr(f_{11}^{(m)}) = Tr(e_{11}^{(m)})$. And so, there exists a partial isometry v in M such that $vv^* = f_{11}^{(m)}$ and $v^*v = e_{11}^{(m)}$. Put $u = \sum_{i=1}^{2^m} f_{i1}^{(m)} v e_{1i}^{(m)}$, then u is an element of M and $u^*u = e$. Furthermore, we have the following;

$$u^*f_{ij}^{(m)}u = \left(\sum_{s=1}^{2^m} e_{s1}^{(m)} v^* f_{1s}^{(m)}\right) f_{ij}^{(m)} \left(\sum_{t=1}^{2^m} f_{t1}^{(m)} v e_{1t}^{(m)}\right)$$

$$= \sum_{s,t=1}^{2^m} e_{s1}^{(m)} v^* f_{1s}^{(m)} f_{ij}^{(m)} f_{t1}^{(m)} v e_{1t}^{(m)} = \sum_{s,t=1}^{2^m} e_{s1}^{(m)} v^* (\delta_{si}\delta_{tj} f_{11}^{(m)}) v e_{1t}^{(m)}$$

$$= e_{i1}^{(m)} v^* f_{11}^{(m)} v e_{1j}^{(m)} = e_{i1}^{(m)} v^* v e_{1j}^{(m)} = e_{i1}^{(m)} e e_{1j}^{(m)} = e_{ij}^{(m)}.$$

Thus, $u^*f_{ij}^{(m)}u = e_{ij}^{(m)}$ for $i,j = 1, 2, \ldots, 2^m$. And so, $u^*xu = \rho(x)$ for every $x \in A_m$. In particular, $\rho(b_j) = u^*b_ju$ $(j = 1, 2, \ldots, k)$. Furthermore, we have the following relations;

$$||\rho(a_j) - \rho(b_j)||_2 = Tr((\rho(a_j - b_j)^*\rho(a_j - b_j))^{1/2}$$

$$= Tr(f)^{1/2}\, Tr((a_j - b_j)^*(a_j - b_j))^{1/2} = Tr(f)^{1/2}||a_j - b_j||_2$$

$\leqq ||a_j - b_j||_2 < \varepsilon/2$ and

$$||u^*a_ju - u^*b_ju||_2 = Tr(u^*(a_j - b_j)^*(a_j - b_j)u)^{1/2}$$

$$= Tr(uu^*(a_j - b_j)^*(a_j - b_j))^{1/2} = Tr((a_j - b_j)^*(a_j - b_j))^{1/2}$$

$$= ||a_j - b_j||_2 < \varepsilon/2.$$

Thus, we have

$$||\rho(a_j) - u^*a_ju||_2$$

$$\leqq ||\rho(a_j) - \rho(b_j)||_2 + ||\rho(b_j) - u^*b_ju||_2 + ||u^*b_ju - u^*a_ju||_2$$

$$< \varepsilon/2 + \varepsilon/2 < \varepsilon \quad \text{for} \quad j = 1, 2, \ldots, k.$$

Therefore, we have the complete proof of Theorem 4. <u>q.e.d.</u>

<u>Remark</u>. In the former work [6] by the author, we have the error in the proof of Theorem 1 in [6] and so we must replace that. Because the results in this report are closely related to the results in [6]. Consider the results in this report and [2] and [3] in the references we have the following considerations for [6]. We replace Theorem 1 in [6] as Theorem 4 in this report and Proposition 2 in [6] as Theorem 3 in this report. Further-

more, if we consider a Haagerup's result [3], the C*-subalgebra A appeared in Theorem 2 in [6] was an MAF-C*-subalgebra but we must replace the algebra A as an AF-C*-subalgebra. The last result (Corollary 4) in [6] is right by [2].

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