

Weighted inequalities for operators
on discrete-time martingales

東北大学 理 和泉沢正隆
(Masataka IZUMISAWA)

1. Introduction. Let \underline{M} be a family of martingales with discrete time-parameter on a probability system $(\Omega, \mathcal{F}, P; \underline{F} = (F_n)_{n \geq 1})$. It was showed in [7] that the weighted norm inequality for operators of matrix type on \underline{M} holds. The operators of matrix type which were introduced by Burkholder, Davis and Gundy[3,4] are a generalization of the square function $S(X)$ or of the maximal function $M(X)$ on martingales X . (see §2 for the precise definitions of these operators.) On the other hand, Chou[5] showed that the weighted norm inequalities for the operators $M(X) \vee S(X) = \text{Max}\{M(X), S(X)\}$ and $M(X) \wedge S(X) = \text{Min}\{M(X), S(X)\}$ on \underline{M} in the continuous time-parameter case.

Our purpose in this paper is to unify these two type inequalities. Our method is based on distribution function inequalities and on the Davis decomposition of martingales which were used by Bonami, Lépingle and Chou[1,5].

Throughout the paper, we fix a strictly positive random variable W with $E[W] = 1$ as a weight. We set the weighted probability measure $dP^W = W dP$ and denote by $E^W[\cdot]$ the expectation over Ω with respect to P^W . Put $W_n = E[W | F_n]$. The weight W is said to satisfy the condition (b^+)

[resp. (S^-)] if there exist constants $k > 1$ and $C > 0$ such that, for all n ,

$$(b^+) \quad E[W^k | \mathcal{F}_n]^{1/k} \leq C W_n$$

[resp. (S^-) $W_{n+1} \geq k W_n$].

Let us denote by $U(X)$ or $V(X)$ the operators of matrix type for $X \in \underline{M}$ and put $s(X) = \left\{ \sum_{n=1}^{\infty} E[(X_n - X_{n-1})^2 | \mathcal{F}_{n-1}] \right\}^{1/2}$ and $r(X) = \left\{ \sum_{n=1}^{\infty} E[|X_n - X_{n-1}| | \mathcal{F}_{n-1}]^2 \right\}^{1/2}$. Clearly $r(X) \leq s(X)$. For a predictable increasing process $D = (D_n)_{n \geq 1}$,

we denote by $\underline{M}(D)$ the family of $X \in \underline{M}$ such that

$|X_n - X_{n-1}| \leq D_n$ for all n . We say that a function Φ on R_+ is moderate if Φ is a nondecreasing continuous function satisfying $\Phi(0) = 0$ and the growth condition

$$\Phi(2\lambda) \leq c \Phi(\lambda) \text{ for all } \lambda > 0.$$

We are now in a position to state the results.

Theorem 1. Let Φ be a moderate function and $0 \leq q < p < \infty$. Suppose that W satisfies the condition (b^+) . Then there exist constants $c = c(\Phi, W, U, V)$ and $c' = c(p, q, W, U, V)$ such that

$$(1) \quad E^W[\Phi(J(X))] \leq c E^W[\Phi(K(X) + D_{\infty})]$$

and

$$(2) \quad E^W [J(X)^P / \{ K(X) + D_\infty \}^Q] \leq c' E^W [J(X)^{P-Q}]$$

for every $X \in \underline{M}(D)$ where $K(X) = U^*(X) \wedge V^*(X)$ or $K(X) = U^*(X) \wedge s(X)$ and $J(X) = U^{**}(X) \vee V^{**}(X) \vee s(X)$.

Theorem 2. Let Φ be a convex moderate function. Suppose that W satisfies the conditions (b^+) and (S^-) . Then there exists a constant $c = c(\Phi, W, U, V)$ such that

$$(3) \quad E^W [\Phi(U^{**}(X) \vee V^{**}(X) \vee r(X))] \leq c E^W [\Phi(U^*(X) \wedge V^*(X))]$$

for every P -martingale X .

Sekiguchi[11], Kazamaki[8], Izumisawa[7] and Muramoto[10] obtained partial forms of the above inequalities (1) and (3) by using a "BMO-martingale method" which is based on Fefferman's inequality, the square function and the relation between the condition (A_∞) and the class of BMO-martingales.

On the other hand, the same type inequality as (2) was treated by Fefferman, Gundy, Silverstein and Stein[6] for ratios of functionals of harmonic functions and by Yor[12] for continuous processes. Our method follows along the same line as in [6].

This paper contains the main results of the author's dissertation of Doctor of Science in Tohoku University, 1984.

2. Definitions. For an \underline{F} -adapted process $X = (X_n)_{n \geq 1}$,

we denote by its small letter x the difference sequence of X , so that $X_n = \sum_{k=1}^n x_k$. For convenience, we assume that $X_0 = x_0 = 0$, $\underline{F} = \bigvee_{n \geq 1} F_n$ and F_0 is generated by all P -null sets.

A matrix $(u_{jk}; j \geq 1, k \geq 1)$ will be called type B-G (B-G stands for Burkholder and Gundy) if it has the following properties:

(a) Each entry u_{jk} is an F_{k-1} -measurable random variable.

(b) There is a constant $d > 1$ such that for all $k \geq 1$,

$$1/d \leq \sum_{j=1}^{\infty} u_{jk}^2 \leq d.$$

We define $U(X)$, $U_n(X)$, $U_n^*(X)$ and $U_n^{**}(X)$ for a matrix (u_{jk}) of type B-G as follows:

$$U(X) = \left(\sum_{j=1}^{\infty} \limsup_{n \rightarrow \infty} \left| \sum_{k=1}^n u_{jk} x_k \right|^2 \right)^{1/2},$$

$$U_n(X) = \left(\sum_{j=1}^{\infty} \left| \sum_{k=1}^n u_{jk} x_k \right|^2 \right)^{1/2},$$

$$U_n^*(X) = \sup_{n \geq k} U_k(X)$$

and

$$U_n^{**}(X) = \left(\sum_{j=1}^{\infty} \sup_{n \geq i} \left| \sum_{k=1}^i u_{jk} x_k \right|^2 \right)^{1/2}.$$

We write simply $U^*(X)$ and $U^{**}(X)$ instead of $U_{\infty}^*(X)$ and $U_{\infty}^{**}(X)$. $U(X)$ is called an operator of matrix type and $U^{**}(X)$ is called an operator of maximal matrix type. In the same way, for a matrix of type B-G, we can define $V(X)$, $V^*(X)$ and $V^{**}(X)$ by using v_{jk} instead of u_{jk} . The square function $S(X)$ of X is an operator of matrix type and of maximal matrix type with (u_{jk}) the identity matrix, so that $S(X) = (\sum_{k=1}^{\infty} x_k^2)^{1/2}$. The maximal function $M(X) = \sup_n |X_n|$ of X is an operator of maximal matrix type with (u_{jk}) the single-row matrix. Martingale transforms and the "Littlewood-Paley" operator are examples of operators of matrix type.

For later use, we give here some properties of operators of matrix type or of maximal matrix type.

$$U^*(X) \leq U^{**}(X) .$$

[sub-linear] $U^*(X+Y) \leq U^*(X) + U^*(Y) .$

$$U^{**}(X+Y) \leq U^{**}(X) + U^{**}(Y) .$$

$$(4) \quad (1/\sqrt{d}) U^{**}(x_n) \leq |x_n| \leq 2\sqrt{d} U^*(X) .$$

Note that the conditioned square function $s(X)$ of X and the operator $r(X)$ are not of matrix type and that $r(X)$ is sub-linear but $s(X)$ is quasi-linear, i.e. $s(X+Y) \leq \sqrt{2}\{s(X) + s(Y)\}$.

For a process $H = (H_n)_{n \geq 1}$ with the difference sequence (h_n) , we define the predictable projection process $\hat{H} =$

$(\hat{H}_n)_{n \geq 1}$ of H as follows:

$$\hat{h}_k = E[h_k | \mathcal{F}_{k-1}] \quad \text{and} \quad \hat{H}_n = \sum_{k=1}^n \hat{h}_k .$$

Let us denote by $\underline{T}(\underline{F})$ the class of all \underline{F} -stopping times.

Many times, we denote by c or C a positive constant and by c_p a positive constant depending only on the parameter p . Both letters are not necessarily the same at each occurrence.

3. Preliminary Lemmas. In this section, we prepare four lemmas which are of use in subsequent sections. Let $(\Omega, \mathcal{F}, \nu)$ be a finite measure space and denote by $E^\nu[\cdot]$ the integral with respect to the measure ν , i.e., $E^\nu[\cdot] = \int_{\Omega} \cdot d\nu$.

Lemma 1. Suppose that f and g are nonnegative measurable functions and $\beta > 1$, $\delta > 0$, $\epsilon > 0$ are real numbers such that

$$\nu(f > \beta\lambda, g \leq \delta\lambda) \leq \epsilon \nu(f > \lambda) \quad \lambda > 0 .$$

Let Φ be a moderate function, $\gamma = \sup_{\lambda > 0} \frac{\Phi(\beta\lambda)}{\Phi(\lambda)}$ and $\eta = \sup_{\lambda > 0} \frac{\Phi(\lambda/\delta)}{\Phi(\lambda)}$. If $\gamma\epsilon < 1$, then

$$E^\nu[\Phi(f)] \leq \frac{\gamma\epsilon}{1-\gamma\eta} E^\nu[\Phi(g)] .$$

For the proof of Lemma 1, see Burkholder[2].

Lemma 2. Let $L = (L_n)_{n \geq 1}$ be a predictable increasing process and g be a nonnegative measurable function. Suppose that

$$E^V[(L_\infty - L_T) I_{\{T < \infty\}}] \leq E^V[g I_{\{T < \infty\}}]$$

for all $T \in \underline{T}(F)$.

If ϕ is a moderate convex function with the right derivative φ , then

$$E^V[\phi(L_\infty)] \leq E^V[g \varphi(L_\infty)]$$

and there exists a constant c' depending only on ϕ such that

$$E^V[\phi(L_\infty)] \leq c' E^V[\phi(g)] .$$

For the proof of Lemma 2, see Lenglart, Lépingle and Pratelli[9].

Lemma 3. Let f and g be nonnegative measurable functions. Suppose that $\{f=0\} = \{g=0\}$ and $c > 1$, $r > 0$ are real numbers such that

$$v(f > \beta\lambda, g \leq \lambda) \leq \{c/\beta^r\} v(f > \lambda)$$

for all $\beta > 4$ and $\lambda > 0$. If $0 \leq q < p < r$, then

$$E^v[f^p/g^q] \leq 4^{p+q} \{ c/(4^{(r-p)} - 1) + 1 \} E^v[f^{p-q}] .$$

Lemma 3 is proved in Fefferman, Gundy, Silverstein and Stein[6] for the case where f and g are functionals of harmonic functions. Their method is valid in our case.

Lemma 4. Let ϕ be a convex moderate function and \hat{H} be the predictable projection process of an \underline{F} -adapted process H . Suppose that W satisfies the condition (S^-) and that there exists a P^W -integrable random variable g satisfying $\sum_{k=1}^{\infty} |h_k| \leq g$. Then, there exists a constant c independent of H and g such that

$$E^W[\phi(g)] \geq (1/c) E^W[\phi(U^{**}(H) \vee V^{**}(H) \vee r(H) \vee U^{**}(\hat{H}) \vee V^{**}(\hat{H}))] .$$

4. Proof of Theorem 1. We prove the case of $K(X) = U^*(X) \wedge V^*(X)$. In the rest of this paper, we fix a constant $d > 1$ such that

$$1/d \leq \sum_{j=1}^{\infty} u_{jk}^2, \quad \sum_{j=1}^{\infty} v_{jk}^2 \leq d .$$

First, we prove the following inequality:

$$(5) \quad P^W(U^{**}(X) > \beta\lambda , V^*(X) \vee D_\infty \leq \delta\lambda)$$

$$\leq c_{d,\beta,\delta} P^W(U^{**}(X) > \lambda)$$

for all $X \in \underline{M}(D)$ where $\beta > 3$, $0 < \delta < \{(\beta^2/3d) - (1/d)\}^{1/2}$
and $\lambda > 0$. Let

$$\tau = \inf \{ n ; U_n^{**}(X) > \lambda \},$$

$$\mu = \inf \{ n ; U_n^*(X) > \beta\lambda \},$$

$$\sigma = \inf \{ n ; V_n^*(X) \vee D_{n+1} > \delta\lambda \},$$

$\bar{x}_k = I_{\{\tau < k \leq \sigma\}} x_k$ and $\bar{X}_n = \sum_{k=1}^n \bar{x}_k$. Since D is a
predictable process and $\{\tau < k \leq \sigma\} \in F_{k-1}$, these τ , μ
and σ are stopping times and $\bar{X} = (\bar{X}_n) \in \underline{M}(D)$. Using an
elementary inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$, we
obtain on a set $\{\tau < n\}$

$$\begin{aligned} & \left(\sum_{k=1}^n u_{jk} x_k \right)^2 \\ & \leq 3 \left\{ \left(\sum_{k=1}^{\tau-1} u_{jk} x_k \right)^2 + (u_{j\tau} x_\tau)^2 + \left(\sum_{k=\tau+1}^n u_{jk} x_k \right)^2 \right\}. \end{aligned}$$

Put a set $A = \{ U^{**}(X) > \beta\lambda , V^*(X) \vee D_\infty \leq \delta\lambda \}$. Then $A =$
 $\{\tau < \infty\} \cap \{\mu < \infty\} \cap \{\sigma = \infty\}$. Hence we have, on A ,

$$\begin{aligned}
& \sup_n \left(\sum_{k=1}^n u_{jk} x_k I_{\{\tau < k\}} \right)^2 \\
& \geq \frac{1}{3} \sup_n \left(\sum_{k=1}^n u_{jk} x_k \right)^2 - \sup_n \left(\sum_{k=1}^{n \wedge (\tau-1)} u_{jk} x_k \right)^2 \\
& \quad - |u_{j\tau} x_\tau|^2
\end{aligned}$$

Taking a sum with respect to j , we get, on the set A ,

$$\begin{aligned}
U^{**}(\bar{X})^2 & \geq \frac{1}{3} U^{**}(X)^2 - U_{\tau-1}^{**}(X)^2 - d |x_\tau|^2 \\
& \geq (\beta^2/3) \lambda^2 - \lambda^2 - d \delta^2 \lambda^2 \\
& = \{(\beta^2/3) - 1 - d \delta^2\} \lambda^2 .
\end{aligned}$$

So we obtain

$$\begin{aligned}
(6) \quad E[I_A |F_\tau] & = E[I_A I_{\{\tau < \sigma\}} |F_\tau] \\
& \leq \{(\beta^2/3 - 1 - d\delta^2)\lambda^2\}^{-1} E[U^{**}(\bar{X})^2 I_{\{\tau < \sigma\}} |F_\tau] \\
& = (\lambda^2 c_{d,\beta,\delta})^{-1} E[\sum_{j=1}^{\infty} \sup_n \left| \sum_{k=1}^n u_{jk} \bar{x}_k \right|^2 |F_\tau] I_{\{\tau < \sigma\}} \\
& = (\lambda^2 c_{d,\beta,\delta})^{-1} \sum_{j=1}^{\infty} E[\sup_n \left| \sum_{k=1}^n u_{jk} \bar{x}_k \right|^2 |F_\tau] I_{\{\tau < \sigma\}} .
\end{aligned}$$

Note that the following inequalities hold for every locally bounded martingale $Y = (Y_n)_{n \geq 1}$ with $\sum_{k=1}^{n \wedge \tau} y_k = 0$;

$$(7) \quad E[Y_\infty^2 | F_\tau] \leq E[S(Y)^2 | F_\tau] = E[\sum_{k=\tau+1}^{\infty} Y_k^2 | F_\tau]$$

and

$$(8) \quad E[S(Y)^2 | F_\tau] \leq c' E[Y_\infty^2 | F_\tau] .$$

As $\{ \sum_{k=1}^n u_{jk} \bar{x}_k \}$ is a locally bounded martingale with $\sum_{k=1}^{n \wedge \tau} u_{jk} \bar{x}_k = 0$, by Doob's maximal inequality and (7), we have

$$\begin{aligned} & E[\sup_n | \sum_{k=1}^n u_{jk} \bar{x}_k |^2 | F_\tau] I_{\{ \tau < \sigma \}} \\ & \leq 4 \sup_n E[| \sum_{k=1}^n u_{jk} \bar{x}_k |^2 | F_\tau] I_{\{ \tau < \sigma \}} \\ & = 4 E[| \sum_{k=1}^{\infty} u_{jk} \bar{x}_k |^2 | F_\tau] I_{\{ \tau < \sigma \}} \\ & \leq 4 E[\sum_{k=1}^{\infty} | u_{jk} \bar{x}_k |^2 | F_\tau] I_{\{ \tau < \sigma \}} \\ & = 4 E[\sum_{k=\tau+1}^{\sigma} u_{jk}^2 x_k^2 | F_\tau] I_{\{ \tau < \sigma \}} . \end{aligned}$$

Hence, using inequalities $(\sum_j u_{jk}^2) < d < d^2 (\sum_j v_{jk}^2)$ and

(8), we get

$$\begin{aligned} & \sum_{j=1}^{\infty} E[\sum_{k=\tau+1}^{\sigma} | u_{jk} x_k |^2 | F_\tau] I_{\{ \tau < \sigma \}} \\ & = E[\{ \sum_{k=\tau+1}^{\sigma-1} x_k^2 (\sum_{j=1}^{\infty} u_{jk}^2) \} + x_\sigma^2 (\sum_{j=1}^{\infty} u_{jk}^2) | F_\tau] I_{\{ \tau < \sigma \}} \end{aligned}$$

$$\begin{aligned}
&\leq E[d^2 \sum_{k=\tau+1}^{\sigma-1} \sum_{j=1}^{\infty} |v_{jk} x_k|^2 + d D_{\sigma}^2 |F_{\tau}] I_{\{\tau < \sigma\}} \\
&\leq c'd^2 E[\sum_{j=1}^{\infty} | \sum_{k=\tau+1}^{\sigma-1} v_{jk} x_k |^2 |F_{\tau}] I_{\{\tau < \sigma\}} + d\delta^2 \lambda^2 \\
&\leq 2c'd^2 E[v_{\sigma-1}(X)^2 + v_{\tau}(X)^2 |F_{\tau}] I_{\{\tau < \sigma\}} + d\delta^2 \lambda^2 \\
&\leq 4c'd^2 E[v_{\sigma-1}^*(X)^2 |F_{\tau}] + d\delta^2 \lambda^2 \leq (4c'd+1)d\delta^2 \lambda^2 .
\end{aligned}$$

Combining the above inequality with (6), we have

$$E[I_A |F_{\tau}] \leq c_{d,\beta,\delta}$$

and, by the condition (b⁺) and by Hölder's inequality,

$$\begin{aligned}
E^W[I_A |F_{\tau}] &= E[I_A (W/W_{\tau}) |F_{\tau}] \\
&\leq \{E[I_A |F_{\tau}]\}^{(k-1)/k} \{E[(W/W_{\tau})^k |F_{\tau}]\}^{1/k} \leq c_{d,\beta,\delta,k} .
\end{aligned}$$

Since $\{\tau < \infty\} \in F_{\tau}$ and $A \subset \{\tau < \infty\}$, we obtain

$$E^W[I_A |F_{\tau}] \leq c I_{\{\tau < \infty\}}$$

and $P^W(A) \leq c P^W(\tau < \infty)$. Thus the inequality (5) is established.

Secondly we prove the inequality (1) in the case of $\Phi(\alpha) = \alpha^2$. By virtue of Lemma 1 and the inequality (5), the

following inequality is obtained:

$$(9) \quad E^W[\Phi(U^{**}(X))] \leq c E^W[\Phi(V^*(X) + D_\infty)]$$

for all $X \in \underline{M}(D)$. Set $\bar{x}_k = x_k I_{\{T < k\}}$, $\bar{X}_n = \sum_{k=1}^n \bar{x}_k$, $\bar{D}_k = D_k I_{\{T < k\}}$, $\bar{X} = (\bar{X}_n)_{n \geq 1}$ and $\bar{D} = (\bar{D}_n)_{n \geq 1}$ for $T \in \underline{T}(F)$.

Using the inequality (9) for $\bar{X} \in \underline{M}(\bar{D})$ and $\Phi(\alpha) = \alpha$, we get

$$E^W[U^{**}(\bar{X})] \leq c E^W[V^*(\bar{X}) + \bar{D}_\infty]$$

and

$$(10) \quad E^W[\{ U^*(X) - U_{T-1}^* \} I_{\{T < \infty\}}] \leq E^W[(U^*(\bar{X}) + c\bar{D}_\infty) I_{\{T < \infty\}}] \\ \leq c E^W[V(\bar{X}) + \bar{D}_\infty] \leq 2c E^W[(V^*(X) + D_\infty) I_{\{T < \infty\}}]$$

by the sub-linearity of operators and by the fact that $\bar{X} = 0$ on $\{T = \infty\}$, $\bar{X}_\infty = (X_\infty - X_{T-1}) - x_T$, $\bar{D}_\infty = D_\infty I_{\{T < \infty\}}$ and $|x_T| \leq \bar{D}_\infty$ on $\{T = \infty\}$. Applying Lemma 2 to $\Phi(\alpha) = \alpha^2$ and to the predictable increasing process $(U_{n-1}^*(X))_{n \geq 1}$, we get

$$E^W[U^*(X)^2] \leq E^W[c(V^*(X) + D_\infty)(2U^*(X))] \\ = c E^W[(V^*(X) + D_\infty) U^*(X)]$$

by (10). Combining this inequality with (9), we have

$$\begin{aligned}
E^W[U^{**}(X)^2] &\leq c E^W[U^*(X)^2 + D_\infty^2] \\
&\leq c E^W[(V^*(X) + D_\infty) U^*(X) + D_\infty^2] \\
&\leq c E^W[(V^*(X) + D_\infty)(U^*(X) + D_\infty)] .
\end{aligned}$$

In the same way, we obtain $E^W[V^{**}(X)^2] + E^W[s(X)^2] \leq c E^W[(V^*(X) + D_\infty)(U^*(X) + D_\infty)]$. Therefore we have

$$\begin{aligned}
&E^W[(J(X) + D_\infty)^2] \\
&\leq c E^W[U^{**}(X)^2 + V^{**}(X)^2 + s(X)^2 + c'D_\infty^2] \\
&\leq c E^W[(U^*(X) + D_\infty)(V^*(X) + D_\infty) + c'D_\infty^2] \\
&\leq c E^W[(U^*(X) + D_\infty)(V^*(X) + D_\infty)] \\
&= c E^W[(U^*(X) \vee V^*(X) + D_\infty)(U^*(X) \wedge V^*(X) + D_\infty)] \\
&\leq c E^W[(J(X) + D_\infty)(K(X) + D_\infty)] \\
&\leq c \{ E^W[(J(X) + D_\infty)^2] \}^{1/2} \times \{ E^W[(K(X) + D_\infty)^2] \}^{1/2} .
\end{aligned}$$

Thus we obtain $E^W[(J(X) + D_\infty)^2] \leq c E^W[(K(X) + D_\infty)^2]$ and

$$(11) \quad E^W[J(X)^2] \leq c E^W[(K(X) + D_\infty)^2]$$

which is the inequality (1) in the case of $\Phi(\alpha) = \alpha^2$. Remark that the constant c in (11) is independent of martingales X .

Thirdly, we show the following inequality;

$$(12) \quad P^W(J(X) > \beta\lambda, K(X) + D_\infty \leq \lambda) \leq (c/\beta^2) P^W(J(X) > \lambda).$$

Set $A = \{ J(X) > \beta\lambda, K(X) + D_\infty \leq \lambda \}$, $X_n^{(T)} = \sum_{k=1}^{T \wedge n} x_k$, $J_n(X) = J(X^{(n)})$ and $K_n(X^{(n)})$ for $T \in \underline{T}(F)$. Put $\tau = \inf \{ n ; J_n(X) > \lambda \}$, $\mu = \inf \{ n ; J_n(X) > \beta\lambda \}$, $\eta = \inf \{ n ; J_n(X) > \beta\lambda/3 \}$, $\sigma = \inf \{ n ; K_n(X) + D_\infty > \lambda \}$ and $\bar{X} = X^{(\sigma)} - X^{(\eta-1)}$. Then we have, on the set A , $J(X)^2 \leq 2\{ J(\bar{X})^2 + J(X^{(\eta-1)})^2 \} \leq 4\{ J(\bar{X})^2 + J_{\eta-1}(X)^2 \} \leq 4\{ J(\bar{X})^2 + (\beta\lambda/3)^2 \}$,

$$J(\bar{X})^2 \geq (1/4)J(X)^2 - (\beta\lambda/3)^2 \geq (1/4 - 1/9) \beta^2 \lambda^2$$

and

$$\begin{aligned} K(\bar{X}) + D_\infty &= K_\sigma(\bar{X}) + D_\sigma \leq K_{\sigma-1}(X) + |x_\sigma| + D_\sigma \\ &\leq 2 \{ K_{\sigma-1}(X) + D_\sigma \} \leq 2\lambda. \end{aligned}$$

Applying the inequality (11) to the $(F_{n+\eta})$ -martingale \bar{X} , we obtain

$$\begin{aligned}
P^W(A) &= P^W(\mu < \infty, \sigma = \infty) \\
&\leq (c/\beta^2 \lambda^2) E^W[J(\bar{X})^2 ; \tau \leq \mu < \infty, \sigma = \infty] \\
&\leq (c/\beta^2 \lambda^2) E^W[\{ K(\bar{X}) + D_\infty \}^2 ; \tau < \sigma] \\
&\leq (c/\beta^2) P^W(\tau < \infty) = (c/\beta^2) P^W(J(X) > \lambda) .
\end{aligned}$$

Therefore the inequality (12) holds.

Immediately, by Lemma 1, we get the inequality (1) and

$$(13) \quad E^W[J(X)^r] \leq c E^W[\{ K(X) + D_\infty \}^r] \quad \text{for } 0 < r < \infty .$$

Using the inequality (13) and the same argument as in third step, we obtain the inequality (12) with the constant (c/β^r) instead of (c/β^2) . Finally, by Lemma 3, we have inequality (2) which completes the proof.

5. Proof of Theorem 2. We use the Davis decomposition of $X \in M$. Let us define

$$D_n = \sup_{k \leq n-1} |x_k|, \quad A = \{ |x_k| \geq 2D_k \}, \quad h_k = x_k I_A,$$

$$\tilde{h}_k = E[h_k | F_{k-1}], \quad H_n = \sum_{k=1}^n h_k, \quad \tilde{H}_n = \sum_{k=1}^n \tilde{h}_k,$$

$$Z_n = H_n - \tilde{H}_n \quad \text{and} \quad Y_n = X_n - Z_n .$$

Note that $D = (D_n)_{n \geq 1}$ is a predictable increasing process and $Y = (Y_n)_{n \geq 1} \in \underline{M}(4D)$. Indeed, $|Y_k| = |x_k - z_k| = |x_k - h_k + \hat{h}_k| = |x_k I_{A^c} + E[x_k I_A | \mathcal{F}_{k-1}]| \leq |x_k| I_{A^c} + |E[x_k - x_k I_A | \mathcal{F}_{k-1}]| \leq 4D_k$ because $E[x_k | \mathcal{F}_{k-1}] = 0$. Let us set $L(\cdot) = U^{**}(\cdot) \vee V^{**}(\cdot) \vee r(\cdot)$ and $K(\cdot) = U^*(\cdot) \wedge V^*(\cdot)$, for convenience. Since $2D_{k+1} \geq 2|x_k| = |x_k| + |x_k| \geq |x_k| + 2D_k$ on the set A , we have

$$\sum_{k=1}^{\infty} |h_k| = \sum_{k=1}^{\infty} |x_k| I_A \leq 2D_{\infty}.$$

Furthermore, the inequality (4) implies that $D_{\infty} \leq 2\sqrt{d} K(X)$.

By virtue of Lemma 4, we obtain

$$(14) \quad E^W[\phi(L(H)) + \phi(L(\hat{H}))] \leq c E^W[\phi(2D_{\infty})] \\ \leq c E^W[\phi(4\sqrt{d} K(X))] \leq c E^W[\phi(K(X))].$$

From the sub-linearity of operators, we get

$$L(X) \leq \{ L(Y) + L(H) + L(\hat{H}) \}$$

and

$$K(Y) \leq \{ U^*(X) + U^*(H) + U^*(\hat{H}) \} \wedge \{ V^*(X) + V^*(H) + V^*(\hat{H}) \} \\ \leq K(X) + 2 \{ U^{**}(H) \vee U^{**}(\hat{H}) \vee V^{**}(H) \vee V^{**}(\hat{H}) \}.$$

Thus, by (14), we obtain

$$E^W[\phi(L(X))]$$

$$\begin{aligned} &\leq c \{ E^W [\Phi(L(Y))] + E^W [\Phi(L(H)) + \Phi(L(\hat{H}))] \} \\ &\leq c E^W [\Phi(U^{**}(Y) \vee V^{**}(Y) \vee s(Y))] + c' E^W [\Phi(K(X))] \end{aligned}$$

and

$$\begin{aligned} &E^W [\Phi(K(Y))] \\ &\leq c E^W [\Phi(K(X))] + c' E^W [\Phi(U^{**}(H) \vee U^{**}(\hat{H}) \vee V^{**}(H) \vee V^{**}(\hat{H}))] \\ &\leq c E^W [\Phi(K(X))] . \end{aligned}$$

Finally, using Theorem 1, we have

$$\begin{aligned} &E^W [\Phi(L(X))] \\ &\leq c E^W [\Phi(U^{**}(Y) \vee V^{**}(Y) \vee s(Y))] + c' E^W [\Phi(K(X))] \\ &\leq c E^W [\Phi(K(Y) + 4D_\infty)] + c' E^W [\Phi(K(X))] \\ &\leq c E^W [\Phi(K(Y))] + c' E^W [\Phi(D_\infty)] + c E^W [\Phi(K(X))] \\ &\leq c E^W [\Phi(K(X))] \end{aligned}$$

which completes the proof.

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