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Selected results on functions of uniformly bounded characteristic

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It is a great honor for me to be able to speak in the opening session of this assembly.*

Functions of uniformly bounded characteristic are functions meromorphic on a Riemann surface with "uniformly" bounded Shimizu-Ahlfors characteristic functions, so that we must begin with the definition of the characteristic function.

1. Shimizu-Ahlfors' characteristic function.

Let R be a Riemann surface, each point of which will be identified with its local-parametric image in the complex plane $\mathbb{C} = \{|z| < \infty\}$ if there is no risk of misunderstanding. By a pair, w, D, we always mean a point $w \in R$ and a domain (open and connected set) D, $w \in D \subset R$, such that the boundary ∂D consists of a finite number of mutually disjoint, analytic, simple, and closed curves. The radius $r = r(w,D)$ of D with respect to w is defined by

$$r = \exp\{\lim(g_D(z,w) + \log|z - w|)\},$$

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where $z \to w$ within the parametric disk of center w, and g_D is the Green function of D. Set

$$D_t = \{z \in D;\ g_D(z,w) \geqq \log(r/t)\}, \quad 0 < t < r.$$

Let $M(R)$ be the family of meromorphic functions on R, and set $f^{\#} = |f'|/(1 + |f|^2)$ for $f \in M(R)$, the spherical derivative of f (not a function in general). The Shimizu-Ahlfors characteristic function is then defined by

$$T(D,w,f) = \pi^{-1}\int_0^r t^{-1}\left[\iint_{D_t} f^{\#}(z)^2 dxdy\right]dt.$$

The terminology is justified because we obtain the familiar one in the specified case $R = \{z \in \mathbb{C};\ |z| < \rho\}$, $0 < \rho \leqq +\infty$, $w = 0$.

Since $R \in O_G$ cannot carry nonconstant, nonnegative, and superharmonic function, we shall hereafter assume that $R \notin O_G$, or, there exist the Green functions $g(z,w) \equiv g_R(z,w)$. Set

$$T(w,f) = T(R,w,f) = \lim T(D,w,f) \leqq \infty,$$

where $D \uparrow R$, the directed limit. Then $T(w,f)$ is the function on R. By definition, $f \in UBC(R)$ (of uniformly bounded characteristic on R) if the supremum of $T(w,f)$ for $w \in R$ is finite.

By elementary considerations we obtain

Theorem 1.1. *For* $f \in M(R)$ *and for* $w \in R$, *we have*

$$T(w,f) = \pi^{-1}\iint_R f^{\#}(z)^2 g(z,w)\,dxdy \qquad (z = x + iy).$$

Thus, $f \in UBC(R)$ if and only if the Green potential $T(w,f)$ of the measure $\pi^{-1}f^{\#}(z)^2 dxdy$ is bounded on R.

2. Harmonic majoration and F. Riesz' decomposition.

It follows from the celebrated Florack-Behnke-Stein theorem that for $f \in M(R)$ there exist holomorphic functions f_1 and f_2 with no common zero on R such that $f = f_1/f_2$. Then $\phi = (1/2)\log(|f_1|^2 + |f_2|^2) > -\infty$ is subharmonic because $\Delta\phi(z) = 2f^{\#}(z)^2 \geq 0$. With the aid of the Green formula we have

$$T(D,w,f) = \phi_D^{\wedge}(w) - \phi(w)$$

for each pair w, D, where $\phi_D^{\wedge}$ is the least harmonic majorant of ϕ in D, namely,

$$\phi_D^{\wedge}(w) = -\frac{1}{2\pi}\int_{\partial D}\phi(z)dg_D^{*}(z,w),$$

the Poisson integral of ϕ on ∂D being positively oriented. Let $BC(R)$ be the family of $f \in M(R)$ such that there exists $w = w(f) \in R$ with $T(w,f) < \infty$.

Theorem 2.1. (The F. Riesz decomposition of ϕ on R.) _For each_ $f \in BC(R)$ _there exists the least harmonic majorant_ $\phi_R^{\wedge}$ _of_ ϕ _on_ R, _the smallest among all the harmonic functions not less than_ ϕ _on_ R, _such that_

$$\phi(w) = \phi_R^{\wedge}(w) - T(w,f),\ w \in R.$$

Remark. The function $T(w,f)$ is of C^{∞} with respect to the real variables u and v with $w = u + iv$.

Corollary 2.1.1. _If_ $f \in BC(R)$, _then_ $T(w,f) < \infty$ _for each_ $w \in R$.

Corollary 2.1.2. *If* $f \in BC(R)$, *and if* f *has two expressions* $f = f_1/f_2 = F_1/F_2$, *described at the beginning of this section, then we set*

$$\phi = (1/2)\log(|f_1|^2 + |f_2|^2) \quad \text{and}$$

$$\Phi = (1/2)\log(|F_1|^2 + |F_2|^2).$$

Then the difference $\phi - \Phi$ *is harmonic in* R.

Here we consider the Nevanlinna-Parreau-Sario characteristic function $T_S(D,w,f)$ of $f \in M(R)$. Set

$$m_S(D,w,f) = -\frac{1}{2\pi}\int_{\partial D} \log^+|f(z)|dg_D^*(z,w).$$

Let $n(t,f)$ be the number of the roots of the equation $f = \infty$ in D_t and let $n(0,f)$ be the limit of $n(t,f)$ as $t \to 0$. Set

$$N_S(D,w,f) = \int_0^r t^{-1}[n(t,f) - n(0,f)]dt + n(0,f)\log r,$$

$$T_S(D,w,f) = m_S(D,w,f) + N_S(D,w,f),$$

$$T(w,f) = \lim_{w \in D \uparrow R} T_S(D,w,f).$$

We compare T with T_S in

Theorem 2.2. *For* $f \in M(R)$,

$$|T(w,f) - T_S(w,f)| \leq k(w,f), \qquad w \in R,$$

where k *is a constant; read* $T = \infty$ *if and only if* $T_S = \infty$.

Corollary 2.2.1. *Let* $f \in M(R)$. *Then* $f \in BC(R)$ *if and*

<u>only if there exists</u> $w \in R$ <u>such that</u> $T_S(w,f) < \infty$.

3. <u>Removable singularity; classification of Riemann surfaces.</u>

A closed set E on R is said to be of capacity zero if the intersection of E with each parametric disk, considered to be a subset of $\mathbb{C}$, is of logarithmic capacity zero. We claim that a compact set on R of capacity zero is always UBC-removable, namely,

Theorem 3.1. <u>Let</u> E <u>be a compact set of capacity zero on</u> R. <u>Then, for each</u> $f \in UBC(R \setminus E)$ <u>there exists</u> $F \in UBC(R)$ <u>such that the restriction of</u> F <u>to</u> $R \setminus E$ <u>coincides with</u> f.

Let $BMOA(R)$ be the family of functions f holomorphic in R with

$$\sup_{w \in R} T^*(w,f) < \infty,$$

where

$$T^*(w,f) = T^*(R,w,f) = \lim_{w \in D \uparrow R} T^*(D,w,f)$$

with

$$T^*(D,w,f) = \pi^{-1}\int_0^r t^{-1}\Big[\iint_{D_t} |f'(z)|^2 dxdy\Big]dt.$$

An easy calculation yields the Green potential expression:

$$T^*(w,f) = \pi^{-1}\iint_R |f'(z)|^2 g(z,w)\,dxdy, \qquad w \in R.$$

If $|f|^2$ admits a harmonic majorant on R, then

$$|f(w)|^2 = (|f|^2)^{\wedge}_R(w) - 2T^*(w,f), \qquad w \in R,$$

where $(|f|^2)^{\wedge}_R$ is the least harmonic majorant of $|f|^2$. This is the F. Riesz decomposition of the subharmonic function $|f|^2$. Let $UBCA(R)$ be the family of all the pole-free members of $UBC(R)$.

Theorem 3.2. $BMOA(R) \subset UBCA(R)$ <u>and the inclusion relation is proper in case</u> R <u>is the open unit disk</u> $\Delta = \{|z| < 1\}$.

Let O_X be the family of Riemann surfaces R such that $R \in O_G$ or $R \notin O_G$ with $X(R) \equiv \mathbb{C}$.

Theorem 3.3. $O_{UBCA} \underset{\neq}{\subset} O_{BMOA}$.

4. Counting function.

For $f \in M(R)$ and for w, D we set

$$N(D,w,f) = \sum_{\substack{f(b)=\infty \\ b\in D}} g_D(w,b).$$

Then $N(D,w,f) = N_S(D,w,f)$ if $f(w) \neq \infty$. Actually,

$$N(D,w,f) = \int_0^r t^{-1} n(t,f)\,dt.$$

For $z \in \mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, we set

$$N(D,w,z,f) = N(D,w,\tfrac{1}{f-z}) = \sum_{\substack{f(\zeta)=z \\ \zeta\in D}} g_D(w,\zeta),$$

with $N(D,w,\infty,f) = N(D,w,f)$. We further set

$$N(w,z,f) = \lim_{w \in D \uparrow R} N(D,w,z,f).$$

We call $f \in M(R)$ Lindelöfian if $N(w,z,f) < \infty$ for each pair of points $w \in R$, $z \in \mathbb{C}^* \setminus \{f(w)\}$. It is known that $f \in BC(R)$ if and only if f is Lindelöfian. Thus, $N(w,z,f)$ serves for deciding whether $f \in BC(R)$ or not.

The Riemann sphere $\mathbb{C}^*$ has the chordal distance $\chi(a,b)$, the Euclidean distance for the subspace $\mathbb{C}^*$ of $\mathbb{R}^3$. Let

$$\Gamma(a,\rho) = \{z \in \mathbb{C}^*;\ \chi(z,a) = \rho\}, \quad a \in \mathbb{C}^*,\ 0 < \rho < 1.$$

For $w \in R$, $0 < \rho < 1$, and $f \in M(R)$ we set

$$C(w,\rho,f) = \sup_{z \in \Gamma(f(w),\rho)} N(w,z,f).$$

Theorem 4.1. <u>The following are mutually equivalent for</u> $f \in M(R)$.

(1) $f \in UBC(R)$.

(2) <u>There exists</u> ρ, $0 < \rho < 1$, <u>such that</u> $\sup_{w \in R} C(w,\rho,f) < \infty$.

(3) <u>For each</u> ρ, $0 < \rho < 1$, $\sup_{w \in R} C(w,\rho,f) < \infty$.

5. <u>The case</u> $R = \Delta$.

By the uniformization theory there exists an analytic projection map π from Δ onto $R \notin O_G$. In many cases we can reduce the problems on R to Δ via π. The following result is fundamental.

Theorem 5.1. <u>For</u> $f \in M(R)$ <u>and for</u> $\delta \in \Delta$, <u>we have</u>

$$T(R,\pi(\delta),f) = T(\Delta,\delta,f\circ\pi).$$

Corollary 5.1.1. <u>For</u> $f \in M(R)$ <u>we have</u>

$$f \in UBC(R) \Leftrightarrow f\circ\pi \in UBC(\Delta).$$

Set

$$N(\Delta) = \{f \in M(\Delta);\ \sup_{z\in\Delta}(1 - |z|^2)f^{\#}(z) < \infty\}$$

and

$$N(R) = \{f \in M(R);\ f\circ\pi \in N(\Delta)\};$$

each member of $N(R)$ is called a normal meromorphic function on R.

It immediately follows from $UBC(\Delta) \subset N(\Delta)$ that

Theorem 5.2. $UBC(R) \subset N(R)$.

In case $R = \Delta$, the inclusion is sharp; there exists a holomorphic function f in Δ which is

(i) normal in Δ;

(ii) of Hardy class $H^p(\Delta)$ for each $0 < p < \infty$;

(iii) not a member of $UBC(\Delta)$.

Each $f \in UBC(\Delta)$ has, as a member of $BC(\Delta)$, the decomposition $f = (b_1/b_2)F$, where b_1 and b_2 are Blaschke products without common zeros, and $F \in BC(\Delta)$ is pole- and zero-free.

Theorem 5.3. $f \in UBC(\Delta) \Rightarrow F \in UBC(\Delta)$.

The converse is false. There exists a Blaschke quotient

$b_1/b_2 \notin N(\Delta)$ so that we have only to let $F = 1$.

Algebraically $UBC(\Delta)$ is not good:

Theorem 5.4. $UBC(\Delta)$ <u>is closed neither for summation nor for multiplication.</u>

For $f \in M(\Delta)$ and $w \in \mathbb{C}^*$ we let $n(w,f)$ be the number of the roots of the equation $f = w$ in Δ. Our next result is concerned with exceptional sets.

Theorem 5.5. <u>Let</u> $f \in M(\Delta)$ <u>and let</u> $k \geq 0$ <u>be an integer. Then,</u>

$$\mathrm{cap}\{w \in \mathbb{C}^*;\ n(w,f) \leqq k\} > 0 \Rightarrow f \in UBC(\Delta),$$

<u>where</u> cap <u>means the logarithmic capacity.</u>

Another theorem on the value distribution is

Theorem 5.6. <u>Suppose for</u> $f \in M(\Delta)$ <u>that</u>

$$\iint_\Delta f^{\#}(z)^2 dxdy < \infty.$$

<u>Then</u>

$$\lim_{|w|\to 1} T(\Delta, w, f) = 0.$$

A sequence $\{z_n\}_{n=1}^{\infty}$ of points in Δ is called interpolating if

$$\inf_{n\geq 1} \prod_{\substack{k=1\\k\neq n}}^{\infty} \left| \frac{z_k - z_n}{1 - \overline{z}_k z_n} \right| > 0.$$

If $\{z_n\}$ is interpolating, then $\Sigma(1 - |z_n|) < \infty$.

Theorem 5.7. _Let_ $\{a_n^{(k)}\}_{n=1}^{\infty}$ $(k = 1, 2)$ _be disjoint interpolating sequence of points in_ Δ. _Set for_ $k = 1, 2,$

$$B_k(z) = \prod_{n=1}^{\infty} \frac{|a_n^{(k)}|}{a_n^{(k)}}(a_n^{(k)} - z)/(1 - \overline{a_n^{(k)}}\, z)$$

$(|a_n^{(k)}|/a_n^{(k)} = 1$ _if_ $a_n^{(k)} = 0)$. _Then the following are mutually equivalent._

(I) $B_1/B_2 \in N(\Delta)$.

(II) $B_1/B_2 \in UBC(\Delta)$.

(III) $\{a_n^{(1)}\} \cup \{a_n^{(2)}\}$ _is interpolating._

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