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On the Schwarzian derivatives of univalent functions and finite dimensional Teichmüller spaces

Hiroshige Shiga
(志賀啓成)
Kyoto University

1. Introduction and main results.

In the celebrated paper[1], Bers introduced Teichmüller spaces of Fuchsian groups and investigated them and their boundaries. Sice his investigation, several authors have studied the Teichmüller spaces as well as their boundaries. In this article, we shall be concerned in the relation between Teichmüller spaces and the spaces of Schwarzian derivatives of univalent functions.

Let Γ be a finitely generated Fuchsian group of the first kind acting on the upper half plane U and let $T(\Gamma)$ be the Teichmüller space of Γ. It is well known that $T(\Gamma)$ is identified with a bounded domain in $B_2(L,\Gamma)$ by Bers' embedding, where $B_2(L,\Gamma)$ is the Banach space of all holomorphic functions ϕ on the lower half plane L with

$$\phi(\gamma(z))\cdot\gamma'(z)^2 = \phi(z) \qquad \gamma\in\Gamma,\ z\in L \quad \text{and}$$

$$\|\phi\| = \sup_{z\in L} (\mathrm{Im}\, z)^2 |\phi(z)| < +\infty .$$

Namely, ϕ is in $T(\Gamma)$ if there is a meromorphic function W_ϕ on L such that the Schwarzian derivative $\{W_\phi, z\}$ of W_ϕ on L is equal to $\phi(z)$ and W_ϕ has a quasiconfomal extension (compatible with Γ) to the sphere $\hat{\mathbb{C}}$.

We denote by $S(\Gamma)$ the set of all ϕ in $B_2(L,\Gamma)$ such that the above W_ϕ is univalent. It is known that $S(\Gamma)$ is closed and contains $\overline{T(\Gamma)} = T(\Gamma) \cup \partial T(\Gamma)$ (cf. Bers[1]). In the sequel, we denote by W_ϕ for ϕ in $B_2(L,\Gamma)$ a locally univalent meromorphic function on L satisfying

$$\{W_\phi, z\} = \phi(z) \quad \text{and}$$

$$W_\phi(z) = (z+i)^{-1} + O(|z+i|) \quad \text{as} \quad z \to -i.$$

For every ϕ in $B_2(L,\Gamma)$, W_ϕ yields an isomorphism χ_ϕ of Γ with $W_\phi \circ \gamma = \chi_\phi(\gamma) \circ W_\phi$ $(\gamma \in \Gamma)$, and if ϕ is in $S(\Gamma)$, then the group $\Gamma^\phi = \chi_\phi(\Gamma) = W_\phi \Gamma (W_\phi)^{-1}$ is a Kleinian group. Furthermore, if ϕ is in $T(\Gamma)$, then Γ^ϕ is a quasi-Fuchsian group, i. e., a Kleinian group with two simply connected invariant components, and if ϕ is in $\partial T(\Gamma)$, then Γ^ϕ is a b-group, i. e., a Kleinian group with only one simply connected invariant component.

First, we shall show

Theorem 1. Int $S(\Gamma)$, the interior of $S(\Gamma)$ on $B_2(L,\Gamma)$, is connected and is equal to $T(\Gamma)$.

In the proof of the theorem, the " λ-lemma " (cf. Mane, Sad and Sullivan[4]) will play an important role. From this theorem, we shall show thw following;

Corollary. Let D be a simply connected invariant component of a finitely generated non-elementary Kleinian group G. Then D is a quasi-disk (i. e., D is the image of the unit disk by a quasi-conformal automorphism of $\hat{\mathbb{C}}$ and G is a quasi-Fuchsian group) if and only if there exists a constant $C > 0$ such that every meromorphic function f on D is univalent whenever f satisfies the conditions

$$|\{f, z\}| \leqq C\rho_D(z)^2 \quad \text{and}$$

$$\{f, g(z)\} \cdot g'(z)^2 = \{f, z\} \quad (\forall g \in G).$$

When $G = \{id.\}$. Ahlfors-Gehring showed the similar property of quasi-disks called the Schwarzian derivative property in Gehrins[2].

Next, we shall show a geometric property of $T(\Gamma)$.

Theorem 2. Let Γ, $T(\Gamma)$ and $B_2(L, \Gamma)$ be as above, and let H be a hyperplane in $B_2(L, \Gamma)$ with $T(\Gamma) \cap H \neq \emptyset$. Then $H - H \cap \overline{T(\Gamma)}$ is connected and $\hat{\partial}(H - H \cap \overline{T(\Gamma)}) = H \cap \partial T(\Gamma)$, where $\hat{\partial}$ is the boundary operator considered in H. In particular, Ext $T(\Gamma)$, the exterior of $T(\Gamma)$ in $B_2(L, \Gamma)$, is connected and $\partial(\text{Ext } T(\Gamma)) = T(\Gamma)$.

Theorem 2 is an extension of the following author's result.

Theorem 2' ([5] Theorem 2). Let Γ, $T(\Gamma)$, $B_2(L, \Gamma)$ and H be as above, and let V_∞ be the unique component of $H - H \cap \overline{T(\Gamma)}$ which is not relatively compact in H. Then every ϕ in $\hat{\partial}(H \cap T(\Gamma))$ is contained in $\hat{\partial} V_\infty$.

Finally, we shall touch upon some results related to the above considerations.

2. Proof of Theorem 1 and Corollary.

Proof of Theorem 1. Žuravlev[7] has already shown that $T(\Gamma)$ is equal to a component of $\operatorname{Int} S(\Gamma)$ containing the origin. Hence it suffices to show that $\operatorname{Int} S(\Gamma)$ has no other component other than $T(\Gamma)$. Let S be such a component of $\operatorname{Int} S(\Gamma)$. Then for each ϕ in S, $\Gamma^{\phi} = \chi_{\phi}(\Gamma) = W_{\phi}\Gamma(W_{\phi})^{-1}$ is a Kleinian group with a simply connected invariant component $W_{\phi}(L)$. Indeed, let Ω_{ϕ} be a component of Γ^{ϕ} containing $W_{\phi}(L)$. Suppose that there exists a point p in $\Omega_{\phi} - W_{\phi}(L)$. Then, for any $\varepsilon > 0$, $N_{\varepsilon}(p) = \{z \in \mathbb{C} : |z - p| < \varepsilon\}$ is not contained in $W_{\phi}(L) \cup \{p\}$ because $W_{\phi}(L)$ is simply connected. This implies that $N_{\varepsilon}(p)$ contains infintely many points of $\Omega_{\phi} - W_{\phi}(L)$ for any $\varepsilon > 0$ and the Riemann surface $\Omega_{\phi}/\Gamma^{\phi}$ contains infintely many points that are not contained in $W_{\phi}(L)/\Gamma^{\phi}$. However, L/Γ is a Riemann surface of conformally finite type and, by Ahlfors' finiteness theorem so is $\Omega_{\phi}/\Gamma^{\phi}$. This is absurd because the Riemann surface $W_{\phi}(L)/\Gamma^{\phi}$ is conformally equivalent L/Γ. Thus, $\Omega_{\phi} = W_{\phi}(L)$. Clearly, $W_{\phi}(L)$ is invariant under Γ^{ϕ}. Hence $W_{\phi}(L)$ is a simply connected invariant component of Γ^{ϕ}.

Therefore, Γ^{ϕ} has one or two simply connected invariant components by a theorem of Accola (cf. [1]). Namely, Γ^{ϕ} is a quasi-Fuchsian group or b-group.

If Γ^{ϕ} is a quasi-Fuchsian group, then the limit set $\Lambda(\Gamma^{\phi})$ of Γ^{ϕ} is a quasi-circle. Therefore, W_{ϕ} has a quasiconformal extention by Ahlfors' theorem, and ϕ belongs to $T \cap B_2(L, \Gamma)$, where T is the universal Teichmüller space. On the other hand, Kra[3] showed that $T(\Gamma) = T \cap B_2(L, \Gamma)$ if Γ is a finitely generated Fuchsian group of the first kind. Thus, ϕ is in $T(\Gamma)$. But this is a contradiction. Hence, Γ^{ϕ} is a b-group.

Since a function $(\text{trace } \chi_{\phi}(\gamma))^2$ for a fixed $\gamma \in \Gamma$ is analyti on $B_2(L, \Gamma)$ and Γ consists of countable elements, there exists a ϕ in S such that $(\text{trace } \chi_{\phi}(\gamma))^2 \neq 4$ for every non-parabolic element γ in Γ, namely, a b-group Γ^{ϕ} is not a cusp. Therefore, Γ^{ϕ} is a totally degenerate group with $\Omega(\Gamma^{\phi}) = W_{\phi}(L)$, where $\Omega(\Gamma^{\phi})$ is the region of discontinuity of Γ^{ϕ}. From now on, we shall consider such ϕ and Γ^{ϕ}.

Here, we note the following fact called the " λ-lemma ".

Proposition (Mane, Sad and Sullivan[4]). Let A be a subset of $\mathbb{C}$ and $\{i_\lambda\}$ be a family of injections of A into $\hat{\mathbb{C}}$, where λ is in the unit disk D. Furthermore, let $i_\lambda(z)$ be analytic with respect to $\lambda \in D$ for each z in A and $i_0(z) \equiv z$. Then, i_λ for each $\lambda \in D$ is automatically a quasiconformal mapping on $\bar{A}$, that is, i_λ is a homeomorphism of $\bar{A}$ into $\hat{\mathbb{C}}$ with

$$\sup_{z \in \bar{A}} \lim_{r \to 0} \frac{\inf \{d(i_\lambda(z), i_\lambda(z')) : d(z, z') = r, z' \in \bar{A}\}}{\sup \{d(i_\lambda(z), i_\lambda(z')) : d(z, z') = r, z' \in \bar{A}\}}$$

$$< +\infty,$$

where $d(\cdot,\cdot)$ is the spherical distance in $\hat{\mathbb{C}}$.

We proceed to prove Theorem 1. Since ϕ is in S, there exists a constant $r > 0$ such that $\{\psi \in B_2(L,\Gamma) : \|\psi - \phi\| < r\}$ is contained in Int $S(\Gamma)$. For each $\lambda \in D$ we set $\phi_\lambda = \phi + \lambda(\psi_0 - \phi)$ and $i_\lambda = W_{\phi_\lambda} \circ (W_\phi)^{-1}$ on $W_\phi(L)$, where ψ_0 is in $B_2(L,\Gamma)$ with $0 < \|\psi_0 - \phi\| < r$. Then i_λ is conformal on $W_\phi(L) = \Omega(\Gamma^\phi)$ and satisfies the condition of the above proposition for $A = \Omega(\Gamma^\phi)$. Hence i_λ for each $\lambda \in D$ can be extended to $\overline{\Omega(\Gamma^\phi)} = \hat{\mathbb{C}}$ quasi-conformally. On the other hand, i_λ is a Γ^ϕ-compatible quasi-conformal mapping, i. e., $i_\lambda \circ \Gamma^\phi \circ (i_\lambda)^{-1}$ ia also a Kleinian group, and Γ^ϕ is finitely generated. Thus, the Beltrami differential of i_λ vanishes almost everywhere on $\Lambda(\Gamma^\phi)$ from Sullivan's ergodic theorem (Sullivan[6]). This implies that i_λ is conformal on $\hat{\mathbb{C}}$ for each $\lambda \in D$, namely i_λ is a Möbius transformation. Hence, the Schwarzian derivative $\{i_\lambda, z\} \equiv 0$ on $\mathbb{C}$. But this is absurd because $\{i_\lambda, z\} = \lambda(\psi_0 - \phi)(W_\phi^{-1}(z)) \cdot ((W_\phi^{-1})'(z))^2 \neq 0$ for $\lambda \neq 0$.

Therefore, we completes the proof of Theorem 1.

Proof of Corollary. We may assume that D contains the infinity. Let h be a conformal mapping of L onto D satisfying

$$h(z) = (z+i)^{-1} + O(|z+i|) \quad \text{as } z \to -i.$$

Then $\Gamma = h^{-1}\circ G\circ h$ is a finitely generated Fuchsian group of the first kind and $\{h, z\}$ is in $B_2(L, \Gamma)$ by Nehari's theorem (cf. [1]). So, if all f satisfying the codition of Corollary are schlicht on D, then $\{f\circ h, z\} = \{f, h(z)\}\cdot(h'(z))^2 + \{h, z\}$ is in $S(\Gamma)$, and $\{h, z\}$ is in Int $S(\Gamma)$. Hence, $\{h, z\}$ is in $T(\Gamma)$ from Theorem 1, that is, $h(L) = D$ is a quasi-disk.

Conversely, if D is a quasi-disk, then D has the Schwarzian derivative property (cf. [2]). Hence, all f satisfying the condition are schlicht on D.

3. Proof of Theorem 2.

Suppose that $H - H\cap\overline{T(\Gamma)}$ is not connected. Then there exists a bounded component if $H - H\cap\overline{T(\Gamma)}$ in H, say V, because $H\cap T(\Gamma)$ ia bounded in H. Obviously, $\hat{\partial}V \subset S(\Gamma)$ and therefore we can show that V is contained in $S(\Gamma)$ by the same argument as in the proof of [4] Theorem 2.

For a nono-parabolic element $\gamma\in\Gamma$, $(\mathrm{trace}\ \chi_\phi(\gamma))^2 - 4$ is analytic in $B_2(L, \Gamma)$ and does not vanish identically on H, because $H\cap T(\Gamma) \neq \emptyset$. Therefore, the set $\{\phi\in V : (\mathrm{trace}\ \chi_\phi(\gamma))^2 - 4 = 0\}$ is a nowhere dense subset of V, and by the same argument as in the proof of Theorem 1, we can take such a ϕ in V that $(\mathrm{trace}\ \chi_\phi(\gamma))^2 \neq 4$ for every non-parabolic element $\gamma\in\Gamma$. Since ϕ is in $S(\Gamma) - T(\Gamma)$, Γ^ϕ is a totally degenerate Kleinian group. By using Proposition (the λ-lemma) and Sullivan's ergodic theorem again as in the proof of Theorem 1 for a small disk centered at ϕ, we have a contradiction. Since we have already shown that

$$(H - H \cap \overline{T(\Gamma)}) \supset H \cap \partial T(\Gamma)$$

(Theorem 2'), we have

$$\hat{\partial}(H - H \cap \overline{T(\Gamma)}) = H \cap \partial T(\Gamma)$$

by a general relation $\hat{\partial}(H - H \cap \overline{T(\Gamma)}) \subset H \cap \partial T(\Gamma)$. Thus, we complete the proof of Theorem 2.

4. Remarks.

(1) Bers conjectured in [1] that every (finitely generated) b-group is a boundary group of a finitely genrated Fuchsian group of the first kind. As for regular b-groups, Abikoff showed that this conjectur is affirmative. Noting that every b-group is in $S(\Gamma)$ for a suitable Γ by a conformal mapping of L onto the invariant component, we verify that Theorem 1 implies that the set of b-groups which are not boundary groups of Teichmüller spaces, even if it is not empty, is not so large in a certain sense.

(2) By using the same argument as before, we have the following which shows the complexity of boundaries of Teichmüller spaces.

Theorem 3. For each ϕ corresponding to a totally degenerate group on $\partial T(\Gamma)$, there exists no complex manifold in $\overline{T(\Gamma)}$ containing ϕ.

Proof. If such a complex manifold $M \subset \overline{T(\Gamma)}$ exists, then there is a holomorphic injection f of the unit disk in $\mathbb{C}$ into $\overline{T(\Gamma)}$ with $f(0) = \phi$. Set $i_\lambda(z) = W_{f(\lambda)} \circ (W_\phi)^{-1}(z)$ on $\Omega(\Gamma^\phi)$ for $\lambda \in D$. By

:he same argument as in the proof of Theorem 1, we have $\{i_\lambda, z\} \equiv 0$ ɔn $\mathbb{C}$ for all $\lambda \in D$ and this yields a contradiction as before.

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