数理解析研究所講究録
第 577 巻 1985 年 18-37

FOLIATION CYCLES
FROM THE SINGULAR HOMOLOGICAL POINT OF VIEW

Yoshihiko MITSUMATSU 三松 佳彦
(University of Tokyo) (東大・理)

§0. INTRODUCTION.

In this exposition, we try to consider foliation cycles as cycles of some kind of singular homology theory to define the simplicial volume (or Gromov's invariant) of foliation cycles. The main purpose of this is to prove the inequality of Milnor-Sullivan-Gromov type (see [4]), which is mentioned in §4. Throughout this paper, let $(M,\mathcal{F})$ denote a codimension q and dimension p smooth foliation $\mathcal{F}$ on a closed n-manifold M whose tangent bundle $\tau\mathcal{F}$ and normal bundle $\nu\mathcal{F}$ are both oriented, where $n=p+q$.

The homology theory we use is that of Radon measures of compact supports. In [8], Thurston made use of this homology to prove that $vol(X)=v_3\|X\|$ where X is a hyperbolic 3-manifold, $\|X\|$ is its Gromov's invariant, and v_3 is the volume of the ideal hyperbolic 3-simplex. In §1, we review the definition of this homology theory and give a proof of Theorem 1.1 which asserts that the usual singular homology and the compact support Radon measure homology for smooth manifolds are isomorphic and isometric.

In §2, we consider the foliated version of Radon measure homology. We claim that foliation cycles are in this homology. The key to this claim is the existence of fundamental cycles of foliations (Proposition 2.2). Of course, we give the definition of fundamental cycles of foliations, but the proof of the existence is postponed to the forthcoming paper [5].

Then we can define the simplicial volume of foliation cycles. Roughly speaking, the simplicial volume $\|C\|$ of a foliation cycle C (which corresponds to a transverse invariant measure μ) should be something like $\int_{L\in\mathcal{F}}\|L\|d|\mu|$.

In §3, we give two kinds of examples, i.e., bundle foliations and foliated bundles. Fundamental cycles are constructed explicitly instead of showing the existence of fundamental cycles for them, and expected formulas of the simplicial volume of foliation cycles of these two cases are shown, which give some reality to the spirit $\|C\| = \int_{L\in\mathcal{F}}\|L\|d|\mu|$.

§1. RADON MEASURE HOMOLOGY AND GROMOV'S INVARIANT.

Let us first review the Radon measure homology theory of smooth manifold X (see Thurston [8]). We set

$$\Sigma^k(X) = \{C^r\text{-singular } k\text{-simplex of } X\} \text{ with uniform } C^r\text{-topology } (0 \leq r \leq \infty)$$

and

$$\begin{aligned} C_k(X) &= \text{the free } \mathbb{R}\text{-module on } \Sigma^k(X) \\ &= \{\text{finite linear combination of Dirac measures (=point measures) on } X\}, \end{aligned}$$

so that the inclusion as a subcomplex

$$\alpha : (C_*(X), \partial) \longrightarrow (S_*(X), \partial) = (C_*(X), \mu) \text{ for } r=0$$

the usual singular chain complex of X

is obtained and induces the isomorphism between the homologies

$$\alpha_* : H_*(C_*(X), \partial) \xrightarrow{\ \simeq\ } H_*(X;\mathbb{R})$$

because of the usual smoothing argument. Here, we remark that the simplicial semi-norms on both homologies coincide and identify them.

Next, we set

$$RC_k(X) = \{\text{Radon measure on } \Sigma^k(X) \text{ with compact support}\}.$$

$RC_k(X)$ is a normed space by the usual dual norm $\|\cdot\|$ to bounded continuous functions. A Radon measure μ is decomposed as $\mu=\mu_+-\mu_-$ uniquely where μ_+ and μ_- are positive Radon measures and the measure $|\mu|$, i.e., the absolute value of μ is defined as

$$|\mu| = \mu_+ + \mu_- .$$

Then the norm $\|\mu\|$ may also be defined as

$$\|\mu\| = |\mu|(\Sigma^k(X))$$

for $\mu\in RC_k(X)$.

Because the face operators

$$\partial_i : \Sigma^k(X) \longrightarrow \Sigma^k(X) , \qquad i = 0,\dots,k,$$

are continuous and Radon measures are covariant against continuous maps, we obtain a continuous boundary operator

$$\partial = \Sigma_{i=0}^{k}(-1)^k\partial_{i\#} : RC_k(X) \longrightarrow RC_{k-1}(X)$$

as usual. For a homology class c, the simplicial volume $\|c\|$ is defined to be inf $\{\|z\|$; z is a cycle which represents c$\}$. Hence we obtain the isometric inclusion of chain complex

$$\iota : (C_*(X), \partial) \longrightarrow (RC_*(X), \partial) .$$

THEOREM 1.1. The inclusion ι induces the isometric isomorphism

$$\iota_* : H_*(X) \xrightarrow{\simeq} RH_*(X) ,$$

where $RH_*(X)$ denotes the homology of $(RC_*(X),\partial)$ which is called the Radon homology of X.

Thurston's main purpose to introduce Radon homology is the following.

DEFINITION 1.2 (Thurston's new definition of Gromov's invariant, [8]). For a closed manifold M, Gromov's invariant $\|M\|$ is defined as the simplicial semi-norm of the fundamental homology class [M] in the Radon homology $RH_*(M)$.

COROLLARY 1.3. Gromov's invariants old and new coincide.

PROOF of Theorem 1.1. Let us take a Riemannian metric on X and fix it.

Step I. Let $I(x)$ be the injectivity radius at $x \in X$, and $U(I(x)/2,x)$ be the open metric ball of radii $I(x)/2$ centered at x. Then we choose countablly many x_i's (i=1,2,3,...) so that $\{U_i = U(I(x_i)/2,x_i)\}$ is a locally finite covering of X. Also we choose a partition of unity $\{\varphi_i ; i=1,2,3,\ldots\}$ which is subordinate to the covering $\{U_i\}$.

A Radon measure μ on $X=\Sigma^0(X)$ with compact support is a 0-cycle. First of all, let us try to deform μ to an ordinary 0-cycle which is homologuous to μ in $(RC_*(X),\partial)$. Let $H=\{h_{i,t}\}$ be a family of deformation retracts of U_i to x_i along geodesics, i.e.,

$$h_{i,t} = \exp_{x_i}(xt)\exp_{x_i}^{-1} : U_i \longrightarrow U_i, \quad t \in [0,1]$$

so that

$$h_{i,0}(U_i) = \{x_i\}, \quad h_{i,1} = Id_{U_i},$$

and

$$h_{i,t}(x_i) = x_i, \qquad \text{for any } t \in [0,1].$$

Let us re-index U_i's so that for some N, $U_i \cap Supp(\mu) \neq \phi$ iff $i \leq N$. Then we devide μ into Radon measures $\varphi_i\mu$ on U_i $(i=1,\ldots,N)$ and set

$$r_i = h_{i,0\#}\mu = \int_{U_i} d\mu_i .$$

Identifying any singular 0-simplex with the point of its image, we define an ordinary 0-cycle $H_\#(\mu)$ as

$$H_\#(\mu) = \Sigma_{i=1}^N r_i x_i .$$

Then we show that $H_\#(\mu)$ is homologuous to μ. Let Λ_i be a continuous map

$$\Lambda_i : \dot{U}_i \longrightarrow \Sigma^1(X)$$

defined as

$$\left[\Lambda_i(x)\right](t) = h_{i,t}(x), \quad t \in [0,1] = \Delta^1 .$$

We set $\lambda_i = \Lambda_{i\#}\mu$ and define a Radon measure λ as

$$\lambda = \Sigma_{i=1}^{N} \lambda_i .$$

Then we see

$$\partial\lambda = \Sigma_{i=1}^{N}(h_{0\#}\mu_i - \mu_i) = H_{\#}(\mu) - \mu$$

and hence $H_{\#}(\mu)$ and μ are homologuous in $(RC_*(X), \partial)$. Therefore the homomorphism

$$\iota_* : H_0(X) \longrightarrow RH_0(X)$$

is surjective.

Step II. If it should be possible to generalize the above argument to higher dimensional chains, we would be fortunately able to construct a chain homotopy between $(C_*(X), \partial)$ and $(RC_*(X), \partial)$, but it seems difficult to take a 'locally finite' nice covering on $\Sigma^k(X)$ $(k>0)$. Therefore we have to do cycle by cycle. Let μ be a Radon measure on $\Sigma^k(X)$ with compact support $(k>0)$. We define the support of the faces of μ as follows.

$$S_k = \mathrm{Supp}(\mu) \subset \Sigma^k(X),$$

$$S_{k-1} = \cup_{j=0}^{k} \partial_j(S_k) \subset \Sigma^{k-1}(X) ,$$

and inductively

$$S_{i-1} = \cup_{j=0}^{i} \partial_j(S_i) \subset \Sigma^{i-1}(X) , \; i=1,\dots,k.$$

We need another one. Let S be the total support of μ defined as

S = the image of the evaluation map $\Delta^k \times S_k \longrightarrow X$.

Of course S_i's and S are compact. Now we take a finite covering

$$\{V(0,i)=U(\varepsilon,x_i) \ ; \ i=1,\ldots,N_0\}$$

of S and a partition of unity $\{\varphi_i\}$ which is subordinate to $\{V(0,i)\}$ where $\varepsilon=2^{-1}\min\{I(x);x\in S\}$. Then, inductively we construct a finite covering of S_l $(l=0,\ldots,k)$ and a partition of unity as follows. Let us assume that S_{l-1} is covered by

$$\{V(l-1,i) = U(\varepsilon,\sigma_i^{l-1}) \ ; \ i=1,\ldots,N_l-i\}$$

and $\{\varphi(l-1)\}$ is a partition of unity where $U(\varepsilon,\sigma^m)$ is an ε-neighbourhood (w.r.t. the uniform C^0-topology) of a singular r-simplex σ^m in $\Sigma^m(X)$ so that we have the deformation retract of $U(\varepsilon,\sigma^m)$ to σ^m along geodesics. Now, for each $j=0,\ldots,l$, $S_l \cap \partial_j^{-1}\overline{V(l-1,i)}$ is compact and hence there exists a finite covering $\{U(\varepsilon,\sigma_{i,j}^l)\}$ of it. Take a partition of unity $\{\psi_{i,j}\}$which is subordinate to $\{U(\varepsilon,\sigma_{i,j}^l)\}$. Then we define $\varphi_{i,j}$ as

$$\varphi_{i,j} = \psi_{i,j} \times (\varphi_i \ \mu_j)$$

and re-index $\{U(\varepsilon,\sigma_{i,j}^l)\}$ and $\{\varphi_{i,j}\}$ as $\{U(\varepsilon,\sigma_i^l)\}$ and $\{\varphi_i\}$ $(i=1,\ldots,N_l)$ and set

$$V(l,i) = U(\varepsilon,\sigma_i^l) \quad \text{and} \quad \varphi(l,i) = \varphi_i \ .$$

Then $\{V(l,i);i=1,\ldots,N_l\}$ and $\{\varphi(l,i)\}$ are the desired objects.

Now we have the family of deformation retracts

$$H = \{h_{i,t} \; ; \; i=1,\ldots,N_k\}$$

where $h_{i,t}$ is the deformation retracts of $V(k,i)$ defined by the same method as before and define an ordinary k-chain $H_{\#}(\mu)$. If we devide $\Delta^k \times [0,1]$ into (k+1)-pieces of (k+1)-simplexes

$$\langle y_0 \times \{0\},\ldots,y_l \times \{0\}, y_l \times \{1\},\ldots,y_k \times \{1\}\rangle \;, \quad l = 0,\ldots,k$$

where $\Delta^k = \langle y_0,\ldots,y_k\rangle$, we can construct a Radon measure $\lambda \in RC_{k+1}(X)$ and we see

$$\mu\lambda = H_{\#}(\mu) - \mu \;.$$

Also we see easily that if μ is closed, so is $H_{\#}(\mu)$. Therefore we see that $\iota_*: H_*(X) \longrightarrow RH_*(X)$ is surjective.

Step III. To show the injectivity of ι_*, we only have to remark that in Step II if μ lies in $C_k(X)$ from the first, also λ lies in $C_{k+1}(X)$. If two ordinary cycles μ_1 and $\mu_2 \in C_k(X)$ are homologuous in $RC_*(X)$, i.e., there exists a Radon measure $\lambda \in RC_{k+1}(X)$ such that

$$\partial\lambda = \mu_1 - \mu_2 \;,$$

we apply the above procedure at (k+1)-stage to λ and that at k-stage to μ_1 and μ_2, then we obtain

$$\partial H_{\#}(\lambda) = H_{\#}(\mu_1) - H_{\#}(\mu_2) \;, \qquad H_{\#}(\lambda) \in C_{k+1}(X)$$

and there exist λ_1 and $\lambda_2 \in C_{k+1}(X)$ such that

$$\partial\lambda_i = H_{\#}(\mu_i) - \mu_i \;, \qquad \text{for } i = 1,\, 2 \;.$$

Therefore, μ_1 and μ_2 is homologuous in $(C_*(X),\partial)$ and the injectivity of ι_* is accomplished.

Step IV. It remains to show that ι_* is isometric, however, this is fairly easy. The fact that ι is an isometric inclusion implies that ι_* is norm-decreasing. On the other hand, in above constructions, clearly we see

$$\|H_{\#}(\mu)\| \leqq \|\mu\| .$$

This fact implies that ι_* is norm-increasing and we finish the proof of Therem 1.1. Q.E.D.

REMARK. 1). We can detect the cycles of $RH_*(X)$ by the pairing with smooth forms if we work in C^r-category $(r\geqq1)$. For example, such a method works well in the proof of Proposition 3.3.

2). If the obstruction mentioned in the beginning of Step II might be resolved, we would be able to prove the following problem in the same way. Let us loosen the condition that the support of μ is compact into that μ is totally L^1, i.e.,

$$|\mu|(\Sigma^k(X)) < \infty$$

in the definition of the Radon homology theory, and let ${}^1RH_*(X)$ denote the homology of the chain complex of such Radon measures.

PROBLEM. ${}^1RH_*(X) \simeq {}^1H_*(X)$?

Where ${}^1H_*(X)$ denotes the ℓ^1-homology of X (see [8]).

§2. FOLIATION CYCLES AS SINGULAR SIMPLICIAL CYCLES.

In this section, we consider the foliated version of §2.

Let $\Sigma^k(M,\mathcal{F})$ be the subspace $\bigcup_{L\in\mathcal{F}}\Sigma^k(L)$ of $\Sigma^k(M)$ with the induced topology, i.e., the set of singular k-simplices each of which has its image in a single leaf. Restricting everything to this subspace, we obtain subchain complexes

$$\begin{array}{ccc}(C_*(M;\mathcal{F}),\ \mu) & \subset & (C_*(M),\ \mu)\\ \cap & & \cap\\ (RC_*(M;\mathcal{F}),\ \mu) & \subset & (RC_*(M),\ \mu)\end{array}$$

and the induced maps on homologies

$$\begin{array}{ccc}H_*(M;\mathcal{F}) & \longrightarrow & H_*(M)\\ \iota_*\downarrow & & \|\iota_*\\ RH_*(M;\mathcal{F}) & \longrightarrow & RH_*(M)\ .\end{array}$$

Our interest is concentrated on $RH_*(M;\mathcal{F})$. It is trivial that

$$H_*(M;\mathcal{F}) = \bigoplus_{L\in\mathcal{F}} H_*(L)\ .$$

In this foliated version, the induced homomorphism

$$\iota_*\ :\ H_*(M;\mathcal{F}) \longrightarrow RH_*(M;\mathcal{F})$$

is not isomorphic any longer. Foliation cycles which is not supported on finitely many compact leaves are lives outside the image of ι_*.

Now, let us start to construct foliation cycles in $C_*(M;\mathcal{F})$ and define the simplicial volume.

DEFINITION 2.1 (a fundamental cycle of a foliation). Let $\zeta_i=(N_i^q,\rho_i,f_i)$ be a triple of a smooth compact manifold N_i^q which may have piecewise smooth boundaries, a continuous map $\rho_i:\Delta^p\times N_i^q\longrightarrow M$ where Δ^p is the standard simplex, and a continuous $\mathbb{R}$-valued function f_i on N_i^q. A finite family $Z=\{\zeta_i\ ;\ i\in I\},(\#I<\infty)$ of such triples is called a fundamental cycle of a foliation $\mathcal{F}$ if it satisfies the following three conditions.

1) For any $i\in I$, the map

$$\rho_{i,y}\ :\ \Delta^p \longrightarrow M$$

has its image in a single leaf for any $y\in N_i$, and for any $x\in\Delta^p$ the map

$$\rho_i^x\ :\ N_i^q \longrightarrow M$$

is smooth and transverse to $\mathcal{F}$ where $\rho_{i,y}(x)=\rho_i^x(y)=\rho_i(x,y)$.

2) The support $\mathrm{supp}(f_i)$ is a codimension zero cornered submanifold of $\mathrm{Int}(N_i^q)$.

3) The restriction $Z|_L$ of Z to any leaf L of $\mathcal{F}$ is a fundamental cycle of L in the homology of locally finite chains where the restriction $Z|_L$ is given as follows.

$$Z|_L = \Sigma f_i(y)\rho_{i,y}\ ,$$

where the summation is taken over the set $\{y\ ;\ y\in N_i^q,\ \rho_{i,y}(x)\subset L,$

$i \in I\}$ and $\rho_{i,y}$ is considered as a singular p-simplex.

PROPOSITION 2.2. For any $(M,\mathcal{F})$, there always exists a fundamental cycle.

The proof is abbreviated here (see [5]).

Once we obtain a fundamental cycle Z of $(M,\mathcal{F})$, it is easy to construct a cycle in $RH_*(M;\mathcal{F})$. Each ρ_i determines a continuous map

$$\rho_i : N_i \longrightarrow \Sigma^p(M;\mathcal{F})$$

and μ is restricted to a Radon measure μ_i on N_i. Now, we have a Radon measure $f_i\mu_i$ on N_i for each i and obtain a chain $\mu Z=\sum_i \rho_{i*}(f_i\mu_i)$ in $RC_p(M;\mathcal{F})$. It is easy to check that μZ is a cycle and mapped to the foliation cycle C by the canonical homomorphism

$$\beta : RC_*(M;\mathcal{F}) \longrightarrow \Omega_*(M)$$

which is defined by the integration on simplices.

DEFINITION 2.3 (the simplicial volume of a foliation cycle C w.r.t. a fundamental cycle Z). The simplicial volume $\|\mu\cdot Z\|$ of a $C=C_\mu$ w.r.t. a fundamental cycle $Z=\{\zeta_i=(N_i^q,\rho_i,f_i)\ ;\ i\in I\}$ of $\mathcal{F}$ is defined as follows, if μ is positive.

$$\|\mu\cdot Z\| = \Sigma_{i\in I}\int_{N_i^q} |f_i(y)|d\mu_i(y) .$$

In general, μ is decomposed into positive measures μ_+ and μ_- so that $\mu=\mu_+-\mu_-$. Then, the simplicial volume $\|\mu\cdot Z\|$ of the simplicial cycle $\mu\cdot Z$ is defined as follows.

$$\|\mu\cdot Z\| = \||\mu|\cdot Z\| = \|\mu_+\cdot Z\| + \|\mu_-\cdot Z\| .$$

DEFINITION 2.4 (the simplicial volume or Gromov's invariant of foliation cycles). The simplicial volume $\|C\|$ of a foliation cycle C is defined as follows.

$$\|C\| = \inf\{\|\mu\cdot Z\| \ ; \ Z \text{ is a fundamental cycle of } \mathcal{F}\} .$$

We have seen that for the natural chain homomorphism β : $RC_*(M;\mathcal{F}) \longrightarrow \Omega_*(M)$,

$$\beta^{-1}(C_\mu) \cap \{\text{cycles in } RC_*(M;\mathcal{F})\} \neq \phi .$$

Let $R(C_\mu)$ denote this subset of $RC_*(M;\mathcal{F})$. Then, it seems quite reasonable to define Gromov's invariant $\|C_\mu\|_{new}$ as

$$\|C_\mu\|_{new} = \inf \{\|\lambda\| \ ; \ \lambda \in R(C_\mu)\} .$$

Of course we see $\|C_\mu\|_{new} \leqq \|C_\mu\|$.

PROBLEM. $\|C_\mu\|_{new} = \|C_\mu\|$,

for any foliation cycle C_μ.

§3 shows that this is true for bundle foliations and foliated bundles. Our definition is convinient to prove the inequality Theorem 4.1.

§3. EXAMPLES AND FORMULAS.

Instead of proving Proposition 2.2, we give examples of fundamental cycles for typical two cases (Example 3.1) below and show that the expected formulas for the simplicial volumes hold.

<u>EXAMPLE 3.1</u>. For the following two cases, fundamental cycles are easy to construct.

3.1.1). Bundle foliations.

Let $\pi:M^n \longrightarrow B^q$ be a smooth fibre bundle with its canonical fibre L^p and $\mathcal{F}$ be a foliation by the fibres of π, i.e., $\mathcal{F}=\{\pi^{-1}(b);b\in B\}$. Then take a finite open covering $U=\{U_j;j\in J\}$ of the base space B such that each restriction $\pi^{-1}\overline{U}_j \longrightarrow \overline{U}_j$ is trivial, and fix a product structure $\pi^{-1}U_j \cong U_j\times L$ on each of them. Next, take a partition of unity $\{\varphi_j\}$ which is subordinate to U and a fundamental cycle $\sum_{k\in K} r_k\sigma_k$ ($r_k\in\mathbb{R}$, σ_k's are singlar p-simplexes) of L. Then define $\rho_i:\Delta^p\times\overline{U}_j \longrightarrow M$ to be $\rho_i=\sigma_k\times\overline{U}_j$ for $i=(k,l)\in K\times J=I$. Then $Z=\{\zeta_i=(\overline{U}_i,\rho_i,\varphi_i)\}$ is a fundamental cycle of $\mathcal{F}$.

3.1.2). Foliated bundles.

Let $\pi:M \longrightarrow B^p$ a smooth fibre bundle, F^q be the canonical fibre of π, and $\mathcal{F}$ is a foliation on M which is transverse to the fibres of π. Then take a fundamental cycle $\Sigma_{i\in I} r_i\sigma_i$ of the base manifold B and a product structure $\Delta^p\times F$ on $\sigma_i{}^*(\pi:M \longrightarrow B)$. Define $\rho_i:\Delta^p\times F \longrightarrow M$ through the above product structure. Then $Z=\{\zeta_i=(F^q,\rho_i,f_i\equiv 1)\}$ is a fundamental cycle of $\mathcal{F}$. This easy construction has much importance as we will see below.

PROPOSITION 3.2. For a foliation cycle C of a bundle foliation as in Example 3.1,1), as is expected, the formula

$$\|C\| = |\mu|(B)\cdot\|L\| \tag{3.2.1}$$

holds, where $\mu=\mu_C$ is the corresponding invariant measure, which is naturally defined on B.

PROOF. The inequality '$\|C\|\leq|\mu|(B)\|L\|$' is the consequence of the construction in Example 3.1,1). Let us prove the converse. For any fundamental cycle $Z=\{\zeta_i=(N_i,\rho_i,f_i)\}$ of $\mathcal{F}$, $\pi:N_i\longrightarrow B$ is locally diffeomorphic, and f_i is supported on the interior of N_i so that we can consider f_i as a continuous function $\tilde{f}$ on B. Therefore, using $\tilde{f}$'s, we obtain

$$\|\mu\cdot Z\| = \Sigma_i\int_{N_i}|f_i|d\mu_i = \Sigma_i\int_B|\tilde{f}_i|d\mu = \int_B\Sigma_i|\tilde{f}_i|d\mu .$$

By the definition, for any $y\in B$,

$$\begin{aligned}\Sigma_{i\in I}|\tilde{f}_i(y)| &= \|Z|_{L_y}\| \\ &\geq \|L\| = \text{Gromov's invariant of the fibre L.}\end{aligned}$$

Therefore the converse is proven. Q.E.D.

PROPOSITION 3.3. In the case of a foliated bundle as in

Example 3.1,2), the following formula is satisfied.

$$\|C\| = |\mu|(F)\cdot\|B\|. \tag{3.3.1}$$

<u>PROOF</u>. For simplicity, we assume $|\mu|(F)=1$. It is easy to see $\|C\|\leq|\mu|(F)\cdot\|B\|$ as before. For any fundamental cycle Z of $\mathcal{F}$, we have the cycle $\mu Z\in RC_p(M;\mathcal{F})$ and $\pi_*(\mu\cdot Z)\in RC_p(B)$ which represents the fundamental class of B. From §1 and §2,

$$\|C\| \geq \||\mu|\cdot Z\| \geq \|\mu Z\| \geq \|\pi_*(\mu Z)\| \geq \|B\|$$

turns out. Q.E.D.

§4. AN APPLICATION.

In the case p=q=even, we can estimate the transverse Euler number $|\langle e(\nu\mathcal{F}), [C]\rangle|$ of a foliation cycle C by its simplicial volume $\|C\|$.

THEOREM 4.1. Let $(M,\mathcal{F})$, $\nu\mathcal{F}$, and C be as usual and p=q=even. Then we have the following inequality.

$$|\langle e(\nu\mathcal{F}), [C]\rangle| \leqq 2^{-p}(p+1)\|C\|.$$

For the proof, see [5]. This is motivated by the following well-known two theorems.

PROPOSITION 4.2 (Bott's vanishing theorem, see [1] and [2]). $\nu\mathcal{F}$ admits a $GL^{+}(q:\mathbb{R})$-connection which is flat along the leaves of $\mathcal{F}$.

REMARK. Such a connection is called a 'Bott connection' or a 'basic connection'.

PROPOSITION 4.3 (Milnor-Sullivan-Gromov-Smillie's inequality, see [3], [4], [7]). Let X be an oriented closed p-manifold and ξ be a flat oriented $\mathbb{R}^p$-vector bundle over X. Then we can estimate the euler number of ξ by Gromov's invariant $\|X\|$ as follows.

$$|\langle e(\xi), [X]\rangle| \leqq 2^{-p}\|X\|,$$

REFERENCES

[1] BOTT,R., On a topological obstruction to integrability. Proc. Symp. Pure Math., vol.XVI, Global Analysis, AMS, 127-131.

[2] BOTT,R., Lectures on characteristic classes of foliations. Springer Lect. Notes in Math. 279 (1972).

[3] GROMOV,M., Volume and bounded cohomology. Publ. Math. I.H.E.S. 56 (1982), 5-100.

[4] MILNOR,J.W. On the existence of a connection with curvature zero. Comment. Math. Helv. 32 (1958), 215-223.

[5] MITSUMATSU,Y., Self-intersections and transverse euler numbers of foliation cycles. Dissertation for Ph.D., University of Tokyo, (1984).

[6] MITSUMATSU,Y., Foliation cycles from the singular homological point of view. To appear.

[7] SULLIVAN,D., A generalization of Milnor's inequality concerning affine foliations and affine manifolds. Comment. Math. Helv. 51 (1976), 183-189.

[8] THURSTON,W.P., The geometry and topology of 3-manifolds. Preprint.