数理解析研究所講究録
第 598 巻 1986 年 219-235

On the unitarizability of irreducible representation of GL(n,k)

by

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Introduction.

Let k be a non-archimedean local field with the standard norm $|\ |$. Zelevinskii [2] parametrized all the irreducible smooth representations of $GL(n,k)$ using the multisets of segments of cuspidal representations. In the present paper we determine when the irreducible representations of $GL(n,k)$ have non-degenera Whittaker models in Zelevinskii's parametrization. We also study for degenerate Whittaker models.

Bernstein [1] gave a criterion of unitarizability of irreducible representations of $GL(n,k)$ along Zelevinskii's parameter. Applying his criterion, we find the unitarizability condition of irreducible representations of GL(2,k), GL(3,k), and of multiplicity free support.

In the final section we compute values of Zelevinskii's duality t and ascertain Bernstein's unitarizability conjecture for some special cases.

1. Zelevinskii's parametrization and Whittaker models.

If $(n_1, n_2, \ldots, n_r)$ is a partition of the number n and ρ_i is a irreducible representation of $GL(n_i,k)$, then we have the tensor product representation $\tau = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_r$ of $\prod_{i=1}^{r} GL(n_i,k)$, which is isomorphic to the block diagonal subgroup D of GL(n,k). The representation τ can be extended to the representation $\tilde{\tau}$ of the standard parabolic subgroup P of GL(n,k) by the canonical epimorphism $P \longrightarrow D$. We call the induced representation $\mathrm{Ind}_P^{GL(n,k)} \tilde{\tau}$ the product representation of ρ_i and denote it by $\rho_1 \times \rho_2 \times \cdots \times \rho_r$.

Let ρ be a cuspidal representation of GL(n,k) and α be a real number. We denote by $\nu^{\alpha}\rho$ the cuspidal representation defined by $g \longmapsto |\det g|^{\alpha} \cdot \rho(g)$ ($g \in GL(n,k)$). A finite set Δ is called a <u>segment</u> of length m if it is of the form $\Delta = \{\rho, \nu^1\rho, \nu^2\rho, \cdots, \nu^{m-1}\rho\}$, where ρ is a cuspidal representation of GL(n,k).

Let $\Delta_1 = \{\rho_1, \nu^1\rho_1, \cdots \nu^{m-1}\rho_1\}$, $\Delta_2 = \{\rho_2, \nu^1\rho_2, \cdots, \nu^{m'-1}\rho_2\}$ be segments. We say that Δ_1 and Δ_2 are <u>linked</u> if the union $\Delta_1 \cup \Delta_2$ is a segment different from Δ_1, Δ_2 . If Δ_1 and Δ_2 are linked and $\rho_2 = \nu^k\rho_1$ for some $k > 0$ then we say that Δ_1 <u>precedes</u> Δ_2 .

Let $\Delta = \{\rho, \nu^1\rho, \cdots, \nu^{m-1}\rho\}$ be a segment. Then the product representation $\rho \times \nu^1\rho \times \cdots \times \nu^{m-1}\rho$ is reducible if $m > 1$ and has a unique irreducible subrepresentation, which we denote by $\langle\Delta\rangle$

Let $a = \{\Delta_1, \Delta_2, \cdots, \Delta_r\}$ be a multiset of segments. (Each element of a multiset may have multiplicity. See [2] .) Suppose for each pair of indices i, j such that $i < j$, Δ_i does not precede Δ_j . Then the product representation $\langle\Delta_1\rangle \times \langle\Delta_2\rangle \times \cdots \times \langle$ has a unique subrepresentation. We denote it by $\langle a \rangle$.

We denote by $\mathcal{O}$ the set of all multisets of segments.

Let $a \in \mathcal{O}$. Call an <u>elementary operation</u> on the multiset a the replacement in it of linked segments Δ_1, Δ_2 by $\Delta_1 \cup \Delta_2$, $\Delta_1 \cap \Delta_2$. Further we can define an order $\leq$ in $\mathcal{O}$: $b < a$ if b may be obtained from a by a chain of elementary operations.

We denote by R_n the Grothendieck group of the abelian category of smooth $GL(n,k)$-modules of finite length. We regard $GL(0,k)$ as the trivial group.

We introduce the product $\times$ on $R = \bigoplus_{n=0}^{\infty} R_n$ by the induction functors (product representations)

$$R_n \times R_m \ni (\sigma, \tau) \longmapsto \sigma \times \tau \in R_{n+m}.$$

Then the algebra R is associative and commutative.

For a multiset $a = \{\Delta_1, \Delta_2, \cdots, \Delta_r\}$, $\pi(a)$ denotes the element $\langle\Delta_1\rangle \times \langle\Delta_2\rangle \times \cdots \times \langle\Delta_r\rangle$ of R.

We put $\mathrm{Irr} = \bigcup_{n=0}^{\infty} \{\text{irreducible smooth representations of } GL(n,k)\}$.

Then we have the following

Theorem 1 (Zelevinskii [2]).

(1) $\mathcal{O} \ni a \longmapsto \langle a \rangle \in \mathrm{Irr}$ is bijective.

(2) $(\pi(a))_{a \in \mathcal{O}}$ is a Z-basis of R. In other words, R is the polynomial ring over Z in variables $\langle \Delta \rangle$ $(\Delta \in \mathcal{S})$, where $\mathcal{S}$ is the set of segments.

For the above correspondence (1), we have furthermore a proposition.

Proposition 2.

Let a be a multiset and π be the corresponding irreducible representation. Then the representation π has a non-degenerate Whittaker model if and only if the multiset a consists of one-point segments.

The proof depends on the arguments of derivatives of representati (see [2]).

We give a list of the correspondence between some classes of multisets and irreducible representations (see [1,2]).

the set of multisets consisting of a single segment	⟷	the set of representations whose restriction to P *) are still irreducible
$\cup$		$\cup$
of a single one-point segment	⟷	cuspidal rep.
$\cap$		$\cap$
of one-point segments $\{\rho\}, \{\nu^1\rho\}, \{\nu^2\rho\}, \cdots, \{\nu^{m-1}\rho\}$ $(m \geqq 1)$	⟷	quasi-square-integrable rep.**)
$\cap$		$\cap$
of one-point segments	⟷	rep. having non-degenerate Whittaker models

*) Here we denote by P the subgroup of GL(n,k) consisting of matrices whose final rows are 0, 0, 0, ... , 0, 0, 1.

**) A representation of GL(n,k) is called <u>quasi-square-integrable</u> if its matrix coefficients become square-integrable modulo the center of GL(n,k) after multiplying by a suitable character of GL(n,k).

For degenerate Whittaker models we have the following proposition. Let $a = \{\Delta_1, \Delta_2, \cdots, \Delta_r\} \in \mathcal{O}$, where Δ_i is a segment consisting of cuspidal representations of $GL(n_i, k)$. The <u>level</u> (of non-degeneracy) of the representation $\langle a \rangle$ is an integer $\sum_{i=1}^{r} n_i$.

<u>Proposition 3.</u> Let U be the subgroup of upper triangular matrices in $GL(n,k)$ and ψ be a non-trivial additive character of k. For a finite set S satisfying

$$\{ n-r+1, n-r+2, \ldots , n-1 \} \subset S \subset \{ 1, 2, 3, \ldots , n-1 \}$$

we denote by χ_S the character of U defined by

$$U \ni (u_{ij}) \longmapsto \psi\left(\sum_{i \in S} u_{i\,i+1} \right) \in \mathbb{C} \quad .$$

Then the level of any irreducible subrepresentation of the induced representation $\mathrm{Ind}_U^{GL(n,k)} \chi_S$ is greater than or equal to r.

Remark. Proposition 3 for r = n coincides with "only if" part of Proposition 2.

2. Bernstein's unitarizability criterion.

Let π be an irreducible representation of GL(n,k). We say the representation π is hermitian if there exists a GL(n,k)-invariant, non-degenerate sesquilinear form on the representation space of π . And we say π is unitarizable if there exists a GL(n,k)-invariant, positive-definite sesquilinear form. Any unitarizable representation is hermitian, but a hermitian representation is not necessarily unitarizable even if it is irreducible and its central character is unitary.

For a segment $\Delta = \{\rho, \nu^1\rho, \cdots, \nu^{m-1}\rho\}$,

we put $\nu^{\frac{1}{2}}\Delta = \{\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho, \cdots, \nu^{m-\frac{1}{2}}\rho\}$,

$$\Delta' = \{\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho, \cdots, \nu^{m-\frac{3}{2}}\rho\}.$$

By virtue of the latter half (2) of Theorem 1, the correspondence

$$\langle\Delta\rangle \longmapsto \langle\nu^{\frac{1}{2}}\Delta\rangle + \langle\Delta'\rangle$$

for segments can be extended to a ring endomorphism $\mathfrak{D}$. For a multiset $a = \{\Delta_1, \Delta_2, \cdots, \Delta_r\}$, we put $a' = \{\Delta'_1, \Delta'_2, \cdots, \Delta'_r\}$

For a representation π of GL(n,k), we denote by deg(π) the integer n and denote by e(π) the real number which satisfies the following equality $|\chi(\lambda)| = |\lambda|^{e(\pi)}$ ($\lambda \in k^{\times}$), where χ is the central character of π and the center of GL(n,k) is canonically identified with the multiplicative group $k^{\times}$.

Criterion 4 (Bernstein [1]). Let $a \in \mathcal{O}$. The representation $\langle a \rangle$ is unitarizable if and only if the following three conditions are satisfied:

(i) $\langle a \rangle$ is hermitian,

(ii) $\langle a' \rangle$ is unitarizable,

(iii) In the expression $\mathfrak{D}(\langle a \rangle) = \sum_{b \in \mathcal{O}} c_b \cdot \pi(b)$, coefficients c_b is zero for such a $b \in \mathcal{O}$ that deg($\langle b \rangle$) $>$ deg($\langle a' \rangle$) and e($\langle b \rangle$) $\leq$ (

3. Unitarizability condition for some representations.

Using Criterion 4, we can write down the following lists of nitarizability condition for irreducible representations of GL(2,k) ıd GL(3,k).

Case of GL(2,k).

Multiset consisting of	Unitarizabilty condition
$\{\rho\}$ $(\rho \in \mathcal{C}_2)$	$e(\rho) = 0$
$\{\mu, \nu'\mu\}$ $(\mu \in \mathcal{C}_1)$	$e(\mu) = -1/2$
linked segments $\{\mu\}, \{\nu'\mu\}$ $(\mu \in \mathcal{C}_1)$	$e(\mu) = -1/2$
non-linked segments $\{\mu_1\}, \{\mu_2\}$ $(\mu_1, \mu_2 \in \mathcal{C}_1)$	(i) $e(\mu_1) = e(\mu_2) = 0$, or (ii) μ_1 and μ_2 are hermitian contragredient each other and $-1/2 < e(\mu_1) < 1/2$, $-1/2 < e(\mu_2) < 1/2$.

Here $\mathcal{C}_n$ is the set of cuspidal representations of GL(n,k).

Case of GL(3,k).

Multiset consisting of	Unitarizability condition
$\{\rho\}$ $(\rho \in \mathcal{C}_3)$	$e(\rho) = 0$
$\{\rho_1\}, \{\rho_2\}$ $(\rho_1 \in \mathcal{C}_1,\ \rho_2 \in \mathcal{C}_2)$	$e(\rho_1) = e(\rho_2) = 0$
$\{\rho, \nu^1\rho, \nu^2\rho\}$ $(\rho \in \mathcal{C}_1)$	$e(\rho) = -1$
$\{\rho_1, \nu^1\rho_1\}, \{\rho_2\}$ $(\rho_1, \rho_2 \in \mathcal{C}_1)$	$e(\rho_1) = -1/2,\ e(\rho_2) = 0$
$\{\rho\}, \{\nu^1\rho\}, \{\nu^2\rho\}$ $(\rho \in \mathcal{C}_1)$	$e(\rho) = -1$
$\{\rho_1\}, \{\nu^1\rho_1\}, \{\rho_2\}$ $(\rho_1, \rho_2 \in \mathcal{C}_1,$ $\rho_2 \neq \nu^{-1}\rho_1, \nu^2\rho_1)$	$e(\rho_1) = -1/2,\ e(\rho_2) = 0$
$\{\rho_1\}, \{\rho_2\}, \{\rho_3\}$ $(\rho_1, \rho_2, \rho_3 \in \mathcal{C}_1,$ no pairs are linked $)$	$-1/2 < e(\rho_i) < 1/2,$ $\nu^{-2e(\rho_i)}\rho_i \in \{\rho_1, \rho_2, \rho_3\}$ $(i=1,2,3)$

We also examine unitarizability of composition factors of the product representation.

Proposition 5. Let ρ be a cuspidal representation. Then the product representation $\rho \times \nu^1\rho \times \cdots \times \nu^{m-1}\rho$ is of length 2^{m-1}.

If $e(\nu^{\frac{m-1}{2}}\rho) \neq 0$, then no composition factors are unitarizable.

If $e(\nu^{\frac{m-1}{2}}\rho) = 0$, then exactly two factors, corresponding to the multisets $\{\{\rho, \nu^1\rho, \nu^2\rho, \cdots, \nu^{m-1}\rho\}\}$ and $\{\{\rho\}, \{\nu^1\rho\}, \cdots, \{\nu^{m-1}\rho\}\}$, are unitarizable, and others are not unitarizable.

In order to apply the criterion, we explicitly calculate the value $e(\langle b \rangle)$ for some $b \in \mathcal{O}$.

4. Zelevinskii's duality and Bernstein's conjecture.

Let us consider another ring endomorphism t of R extending the correspondence

$$\langle \{\rho, \nu^1\rho, \cdots, \nu^{m-1}\rho\} \rangle \longmapsto \langle \{\{\rho\}, \{\nu^1\rho\}, \cdots, \{\nu^{m-1}\rho\}\} \rangle.$$

The endomorphism t is involutive and maps Irr into Irr, which we call duality after Zelevinskii. Bernstein states in [1] the following

<u>Conjecture 6.</u> Duality t maps irreducible unitarizable representations into unitarizable ones.

A partial answer to this conjecture is given by Bernstein himself in [1]. He proves that $t(\pi)$ is unitarizable if π is a unitarizable representation of the form $\pi = \pi(a) = \langle a \rangle$ $(a \in \Theta)$.

Combining the results of the previous section and the following propositions, we can ascertain the conjecture for representations of GL(2,k), GL(3,k), and for composition factors of the product representation $\rho \times \nu^1 \rho \times \cdots \times \nu^{m-1} \rho$.

<u>Proposition 7.</u> Let ρ_i be a cuspidal representation of $GL(n_i, k)$ $(i = 1, 2)$. We assume that the segments $\Delta_1 = \{\rho_1, \nu^1$ and $\Delta_2 = \{\rho_2\}$ are not linked. Then the product module $\rho_1 \times \nu^1 \rho_1 \times \rho_2$ is of length 2. Its composition factors are $\langle \{\{\rho_1, \nu^1\rho_1\}, \{\rho_2\}\} \rangle$ and $\langle \{\{\rho_1\}, \{\nu^1\rho_1\}, \{\rho_2\}\} \rangle$ which are dual (under t) of each other.

Let $\Delta = \{\rho, \nu^1\rho, \cdots, \nu^{m-1}\rho\}$ be a segment of length m. For an element $e = (e_1, e_2, \ldots, e_{m-1})$ in $\{1, -1\}^{m-1}$, we associate an equivalence relation $\underset{e}{\sim}$ on Δ in such a manner that $\nu^i\rho \underset{e}{\sim} \nu^{i+1}\rho$ if and only if $e_i = 1$. We denote by $\Delta(e)$ the set of equivalence classes with respect to the equivalence relation $\underset{e}{\sim}$. We regard naturally $\Delta(e)$ as an element in $\mathcal{O}$.

Proposition 8. Using the above notation, we have a bijection $e \longmapsto \langle\Delta(e)\rangle$ of $\{1, -1\}^{m-1}$ onto the set of all composition factors of the product $\pi = \rho \times \nu^1\rho \times \nu^2\rho \times \cdots \times \nu^{m-1}\rho$.

Duality t permutes the composition factors of π as the following manner:

$$t(\langle\Delta(e_1, e_2, e_3, \ldots, e_{m-1})\rangle) = \langle\Delta(-e_1, -e_2, \ldots, -e_{m-1})\rangle.$$

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Note: Three papers [2] , [3] and [4] form a trilogy on the representation theory of GL(n,k). The paper [3] is by itself an excellent introduction to the representation theory of reductive p-adic groups basing on the works of Harish-Chandra, Jacquet, Gel'fand and Kajdan.

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Note: This paper neatly summarizes the works [2] and [4] of Bernstein and Zelevinskii. He emphasizes the importance of Lemma 4.7 [4] in their works.

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Note: One of the main methods in [2] - [4] is based on studying the restriction of representations of GL(n,k) to the subgroup $P \subset GL(n,k)$ consisting of matrices with the last row 0, 0, ... , 0, 1. This is a method of Gel'fand and Kajdan [9] , [10]. Bernstein and Zelevinskii formulate this method in terms of functors.

[11] A. V. Zelevinskii, Representations of finite classical groups, Lec. Notes in Math., Vol. 869, Springer-Verlag, 1981.
Note: In this volume, Zelevinskii applies the technique developed in [2] - [4] for the investigation of representations of general linear groups over p-adic fields to the representation theory of the groups $GL(n,F_q)$.

[12] A. V. Zelevinskii, p-adic analog of the Kazhdan-Lusztig hypothesis, Funkt. Anal. i Prilozen., Vol. 15, No. 2, 9-21.
Note: In the reduction process of Bernstein's unitarizability criterion, we require a knowledge of the multiplicity matrix $m = (m_{ab})$, which describes the decomposition in Grothendieck group of induced representations into irreducible ones. In [12] Zelevinskii defined some polynomials $P_{ab}(q)$, analogous to the Kazhdan-Lusztig polynomials, and conjectured that $m_{ab} = P_{ab}(1)$.

[13] S. Kato, Open problems in algebraic groups, Conference on "Algebric groups and their representations" Held at Katata, August 29-September 3, 1983.
Note : In this proceeding Kato states some generalization of Zelevinskii's conjecture in [12].

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Note: In this paper Tadić proved the conjecture of Bernstein (see Conjecture 6 in the present paper) by a dexterous reduction.

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Added. When I sent this note to Professor N. Kawanaka, he kindly sent back to me the following preprint.

M. Tadić : Solution of the unitarizability problem for general linear group (non-archimedean case), preprint.