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CONTINUED FRACTIONS AND ERGODIC THEORY

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Let x be an irrational number between 0 and 1,

(1) $$x = [\,0;\, a_1, a_2, a_3, \ldots\,]$$

its expansion as a regular continued fraction and $\left(\frac{p_n}{q_n}\right)_{n=1}^{\infty}$ the corresponding sequence of convergents. As is well-known

(2) $$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{\,2}} \quad .$$

Define $\theta_n(x) := q_n \mid q_n x - p_n \mid$. Hence, by (2), the sequence $\theta_n(x)$, $n = 0,1,2,\ldots$ is for every fixed x a sequence in the unit interval. The well-known theorem of Vahlen states that for every x and every $n \in \mathbb{N}$

$$\min \{ \theta_{n-1}(x), \theta_n(x) \} < 0.5 \quad .$$

It was conjectured by H. W. Lenstra Jr. and proved by Bosma, Jager and Wiedijk [1] that for almost all x, in the sense of Lebesgue, the sequence $\theta_n(x)$, $n = 0,1,2,\ldots$ is distributed in the unit interval according to the density function f, where f is given by

$$f(a) = \begin{cases} (\log 2)^{-1} & , 0 \le a \le 0.5 \\ (\log 2)^{-1}\,(a^{-1} - 1) & , 0.5 \le a \le 1 \end{cases} \quad .$$

In [2] a theorem was proved which contains the two above mentioned results as special cases. It reads

THEOREM 1

For all x *the two-dimensional sequence* $(\theta_{n-1}(x), \theta_n(x))$, $n=1,2,\ldots$ *is a sequence in the triangle with vertices* (0,0), (1,0) *and* (0,1). *For almost all* x *this sequence is distributed over this triangle according to the density function* f, *where*

$$f(a,b) = (\log 2)^{-1}(1-4ab)^{-1/2}.$$

The main purpose of this paper is to give a shorter and simpler proof of this theorem than the original one in [2]. At the basis is again the following fundamental result of Sh. Ito, H. Nakada and S. Tanaka, [4] and [5].

THEOREM 2 (Ito, Nakada, Tanaka)

Denote the set of irrational numbers between 0 *and* 1 *by* Ω *and put* $\mathbf{\Omega} := \Omega \times [0,1]$. *Let* **B** *be the set of all Borel subsets of* $\mathbf{\Omega}$, *and* μ *the measure induced on* **B** *by the density function* $(\log 2)^{-1}(1+xy)^{-2}$. *Finally, let the operator* $\boldsymbol{T}: \mathbf{\Omega} \to \mathbf{\Omega}$ *be defined by*

$$\boldsymbol{T}(x,y) = (Tx, (a_1+y)^{-1})$$

where, if x *is given by* (1), Tx *is defined as*

$$Tx := [0; a_2, a_3, \ldots] = \frac{1}{x} - \left[\frac{1}{x}\right]. \tag{3}$$

Then $(\mathbf{\Omega}, \mathbf{B}, \mu, \boldsymbol{T})$ *forms an ergodic system.*

From this theorem we derive the following

THEOREM 3

For almost all irrational numbers x *the two-dimensional sequence* $\left(T^n x, \frac{q_{n-1}}{q_n}\right)_{n=1}^{\infty}$ *is distributed over the unit square according to the density function* g, *with* $g(x,y) = (\log 2)^{-1}(1+xy)^{-2}$.
Here $T^n x$ *is defined inductively by* $T(T^{n-1}x)$ *with the* T *from* (3).

Proof.

Denote by A that set of numbers $x \in \Omega$ for which the sequence $\left(T^n x, \frac{q_{n-1}}{q_n}\right)_{n=1}^{\infty}$ is not distributed according to the density function $(\log 2)^{-1}(1+xy)^{-2}$.

In view of the well-known relation

$\frac{q_{n-1}}{q_n} = [0; a_n, a_{n-1}, a_{n-2}, \ldots, a_1]$ we see that $\mathcal{T}^n(x,0) = \left(T^n x, \frac{q_{n-1}}{q_n}\right)$.

Further it follows from the definition of $\mathcal{T}$ that for all $x \in \Omega$ and all pairs y and $y' \in [0,1]$, the sequence $(\mathcal{T}^n(x,y) - \mathcal{T}^n(x,y'))$, n = 1, 2, 3, ... is a null-sequence. Hence, if $\mathcal{A} := A \times [0,1]$, then for every pair $(x,y) \in \mathcal{A}$, the sequence $\mathcal{T}^n(x,y)$ is not distributed according to the density function $(\log 2)^{-1}(1+xy)^{-2}$. Now if A had, as a one- dimensional set, a positive Lebesgue measure, so had $\mathcal{A}$ as a two- dimensional set. But this would be in conflict with theorem 2. ♦

With this simple consequence of the theorem of Ito, Nakada and Tanaka it is now easy to prove theorem 1 as follows. We have, see [2], (2.1) and (2.2):

(4) $$\theta_{n-1}(x) = \frac{q_{n-1}}{q_n}\left(1 + \frac{q_{n-1}}{q_n} T^n x\right)^{-1}, \qquad \theta_n(x) = T^n x \left(1 + \frac{q_{n-1}}{q_n} T^n x\right)^{-1}.$$

In view of this we consider the function

$$F : (x,y) \to \left(\frac{y}{1+xy}, \frac{x}{1+xy}\right), \; xy \neq -1.$$

It is easily verified that F maps the interior of the unit square bijectively onto the interior of the triangle from theorem 1. Put $a := y(1+xy)^{-1}$, $b := x(1+xy)^{-1}$. The determinant of Jacobi, J, of F equals $(xy-1)(1+xy)^{-3}$.

For almost all x the sequence $\left(T^n x, \frac{q_{n-1}}{q_n}\right)_{n=1}^{\infty}$ is distributed according to the density function g from theorem 3. Hence, for almost all x the sequence $\left(F\left(T^n x, \frac{q_{n-1}}{q_n}\right)\right)_{n=1}^{\infty}$

which is in view of (4) the sequence $(\theta_{n-1}(x), \theta_n(x))$, $n = 1, 2, 3, \ldots$ is distributed over the interior of the triangle with vertices (0,0), (1,0) and (0,1) according to the density function $g\,|\,J\,|^{-1}$.

Now

$$g(x,y)\,|\,J\,|^{-1} = (\log 2)^{-1}\,\frac{1+xy}{1-xy} = (\log 2)^{-1}\left(\left(\frac{1-xy}{1+xy}\right)^2\right)^{-1/2} =$$

$$= (\log 2)^{-1}\left(\frac{(1+xy)^2-4xy}{(1+xy)^2}\right)^{-1/2} = (\log 2)^{-1}(1-4ab)^{-1/2} = f(a,b)\,.$$

◆

In [2], several properties of the sequence $(\theta_n(x))$, $n = 1,2,3,\ldots$ were given as corollaries of theorem 1. We mention one more.

COROLLARY. *Let* $\lambda \geq 0$. *Then for almost all* x *one has*

$$\lim_{n\to\infty} n^{-1}\,\#\{\,j;\ 1\leq j\leq n,\ \theta_{j-1}(x) < \lambda\theta_j(x)\,\} = \begin{cases} (2\log 2)^{-1}\log(1+\lambda) & ,\ 0\leq\lambda\leq 1 \\ 1-(2\log 2)^{-1}\log(1+\lambda^{-1}) & ,\ 1\leq\lambda \end{cases}.$$

Proof.

It follows from (4) that the condition $\theta_{j-1}(x) < \lambda\theta_j(x)$ is equivalent with $\frac{q_{n-1}}{q_n} < \lambda\,T^n x$. Hence, by theorem 3, we have, when $0\leq\lambda\leq 1$, for almost all x

$$\lim_{n\to\infty} n^{-1}\,\#\{\,j;\ 1\leq j\leq n,\ \theta_{j-1}(x) < \lambda\theta_j(x)\,\} =$$

$$= \frac{1}{\log 2}\int_0^1\left(\int_0^{\lambda x}\frac{dy}{(1+xy)^2}\right)dx = \frac{1}{2\log 2}\int_0^1\frac{2\lambda x}{1+\lambda x^2}\,dx = \frac{1}{2\log 2}\log(1+\lambda).$$

The case $1\leq\lambda$ follows immedeately from the case $0\leq\lambda\leq 1$. ◆

Final remark.

In [3], C. Kraaikamp extended the method of [2] to the nearest integer continued fraction and Hurwitz's singular continued fraction and obtained several interesting results. This author has now also obtained the results from [3] by the method of the present paper (oral communication). He has also applied the method to general α-expansions. His results will be published in due course.

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