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Nonlinear Problems in Geometry

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Contents

I. Conformal Geometry (Yamabe Problem)

Prescribing Curvature

Given M^2 : compact 2-dimensional manifold without boundary

& K : function on M.

Seek a metric g on M so that K = Gauss curvature of g.

Rem. By Gauss-Bonnet theorem

$$\iint_M K dA_g = 2\pi\chi(M).$$

So we have a sign condition on K. For example, on S^2, we have $\chi(S^2) = 2$ so that K must be positive somewhere.

When we seek the unknown metric g, we often pick some metric

g_0 and try to deform it to the desired metric g.
The most simplest deformation is "pointwise conformal" one.

Def. We say g & g_0 are pointwise conformal if there exists some positive function ρ on M such that $g = \rho g_0$.

Or more generally,
for (M, g), (N, g_0) : Riemannian manifolds $\varphi : M \longrightarrow N$ a diffeomorphism.
We say φ is a conformal map if for some positive function ρ on M we can write $g = \rho\varphi^*(g_0)$.
So $id : (M, g) \longrightarrow (M, g_0)$ is conformal map if and only if g & g_0 are pointwise conformal.

It turns out one should write $\rho = e^{2u}$ for some function u on M^2.
So the equation which we want to solve is

$$(1) \qquad \Delta_0 u = K_0 - Ke^{2u}$$

where Δ_0 : Laplacian of g_0
K_0 : Gauss curvature of g_0.

Special Case
Want g with $K \equiv$ constant
(According to the sign of $\chi(M)$, we can assume $K \equiv +1, 0,$ or -1.)
(The Uniformization theorem for Riemann surfaces states that there is a conformal map giving a metric with $K \equiv$ const.

example for (S^2, g_0) : given metric solve with $K \equiv 1$. By Uniformization theorem, (1) must have a solution. But no direct proof is known.

Higher Dimensional Case

Given

(M^n, g_0) : compact n-dimensional Riemannian manifold

S : function on M.

Seek a metric g pointwise conformal to g_0 so that S = scalar curvature of g.

We set $g = \rho g_0 = u^{4/(n-2)} g_0$ for $u > 0$, the equation is

(2) $$Lu \equiv -\gamma \Delta_0 u + S_0 u = S u^{\alpha}$$

where $\gamma := \frac{4(n-1)}{n-2}$ $\alpha := \frac{n+2}{n-2}$

Δ_0 : Laplacian of g_0

S_0 : scalar curvature of g_0.

Special Case

$S \equiv$ constant

This case is called "Yamabe Problem".

When we discuss P.D.E., there are two cases, the "easy" case and the difficult case.

If P.D.E. is

$$T(u) \equiv F(x, u, Du, D^2u) = 0$$

where $F(x, z, p_i, r_{ij})$ is $\mathcal{C}^1$-func. and $(\frac{\partial F}{\partial r_{ij}})$ is a pos.

definite matrix (so the equation is an elliptic type) then the easy case is

$$\frac{\partial F}{\partial z} < 0 \quad (\text{or} \ \leq 0 \ \text{when we consider the equ. on } \Omega \ (\text{domain}) \text{ with bdry}).$$

In the easy case, we can prove the uniqueness theorem.
i.e. if $u, v \in C^2(M)$ are solutions, so $T(u) = 0$, $T(v) = 0$, then $u \equiv v$ on M.
For the proof one uses the mean value theorem to write

$$0 = T(v) - T(u) = \int_0^1 \frac{d}{d\lambda} T(u + \lambda(v - u))d\lambda$$

$$= \Sigma a_{ij} w_{ij} + \Sigma b_j w_j + cw,$$

where $w = v - u$, the matrix a_{ij} is positive definite, and $c < 0$. Since $c < 0$, the function w can not have a positive maximum or negative minimum, so $w \equiv 0$.

Example

$$Lu \equiv \Delta u + cu = f(x)$$

$$(F(x, u, Du, D^2u) = \Delta u + cu - f)$$

$$\left(\frac{\partial F}{\partial r_{ij}}\right) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \qquad \frac{\partial F}{\partial z} = c.$$

The easy case is if $c < 0$. If $c = 0$ or $c > 0$ then it can occur that $\text{Ker } L \neq 0$.
So the eigen value of operator on Δ occurs only on $\{t \in \mathbb{R} : t \geq 0\}$.

Example

For equation (1) easy case is $K < 0$.

For equation (2) easy case is $S < 0$.

Returning to equation (1), in general there are three cases depending on the sign of $\bar{K}_0 = (\int_M K_0 dA_0)/\text{Area}(M)$. It is simplest to make a change of variable and let z be a solution of $\Delta_0 z = K_0 - \bar{K}_0$. Then $v = u - z$ satisfies

$$\Delta_0 v = \bar{K}_0 - (Ke^{2z})e^{2v},$$

which is of the some form as (1), but with $K_0 \equiv$ const. Thus, without loss of generality we will assume $K_0 \equiv$ const. in (1).

1) The case $K_0 < 0$ is easiest to discuss, so we ignor it. (cf. [KW1])

2) Next simplest is the case $K_0 = 0$.

If $K \equiv 0$, then (1) is linear equation, so we can solve it easily.

So we assume $K \not\equiv 0$, then (1) has a solution if and only if

$$\begin{cases} K : \text{changes sign} \\ \int_M K dA_0 < 0. \end{cases}$$

(Proof of necessity)

For the proof of sufficiency see [KW1].

Integrate the equation (1),

$$\begin{aligned} \int_M \Delta_0 u dA_0 &= \int_M K_0 dA_0 - \int_M Ke^{2u} dA_0 \\ &= - \int_M Ke^{2u} dA_0. \end{aligned}$$

On the other hand, left side equals 0 by divergence theorem.

So right side integral equals 0, K must changes sign. Since $K_0 = 0$, we can rewrite equation (1) as $K = -(\Delta_0 u)e^{-2u}$ so, after an integration by parts we obtain

$$\therefore \int_M K dA_0 = -\int_M e^{-2u}\Delta_0 u dA_0$$
$$= -\int_M 2e^{-2u} \times |du|^2 dA_0 < 0. \qquad //$$

Case 3°) $K_0 > 0$.

This case has some terrible troubles.

Th. On (S^2, canonical metric)

a) (Kazdan-Warner [KW1]) $\exists K > 0$ function on S^2 so that there is no solution for (1).

b) (Moser, Chen [Mo][Ch1][Ch2]) Let $G \subset O(3)$ subgroup, F_G the fixed point set,

$$F_G := \{x \in S^2 : g\cdot x = x \quad \text{for} \quad \forall g \in G\}$$
$$m_G := \max_{x\in F_G} K(x).$$

If either $m_G \leq 0$ (or F_G is empty) or $m_G > 0$ but $\Delta K(p) > 0$ for some $p \in F_G$ with $K(p) = m_G$ then equation (1) has a solution.

Higher Dimensional Case

There are three cases:

Let λ_1 be the lowest eigenvalue of $L \equiv -\gamma\Delta_0 + S_0$.

Case 1) $\lambda_1 < 0$ easy case

2) $\lambda_1 = 0$ semi-hard

3) $\lambda_1 > 0$ hard.

It is easy to reduce these cases where

1)' $S_0 < 0$

2)' $S_0 \equiv 0$

3)' $S_0 > 0$.

In fact, take φ so that $L\varphi = \lambda_1 \varphi$

We can assume $\varphi > 0$. If we set $\tilde{g}_0 := \varphi^{\frac{4}{n-2}} g_0$, then $\tilde{S}_0 = \lambda_1 \varphi^{1-\alpha}$,

so $\tilde{S}_0$ has the same sign as λ_1.

Say $S_0 > 0$ (then $\lambda_1 > 0$). Then

$$\int_M \varphi \cdot Su \, dV = \int_M \varphi \cdot Lu dV$$

$$= \int_M L\varphi \cdot u dV = \lambda_1 \int_M \varphi \cdot u dV > 0.$$

So S must be positive somewhere.

From now on we assume for simplicity that $S > 0$ everywhere

(Yamabe case : $S \equiv 1$).

Standard method to prove the existence of a solution uses the

calculus of variations.

Seek a critical point of

$$Y(u) = \frac{\frac{1}{2}\int_M [\gamma|\nabla u|^2 + S_0 u^2] dV}{(\int_M Su^{\alpha+1} dV)^{2/(\alpha+1)}}$$

or equivalently

$$\tilde{Y}(u) = \frac{1}{2}\int_M [\gamma|\nabla u|^2 + S_0 u^2] dV$$

with the contrait

$$\int_M Su^{\alpha+1} dV = 1.$$

Solutions of (2) are the critical points of Y.

As a variation we take $u_t := u + t\eta$ (η : any function on M)

$$\frac{d}{dt} Y(u_t)|_{t=0} = c_1 \left\{ \int_M \gamma(\nabla u, \nabla \eta) + S_0 u\eta dV - c_2 \int_M Su^{\alpha} \cdot \eta dV \right\}$$

for some positive consts c_1, c_2.

So Euler-Lagrange equation of Y is $-\gamma\Delta_0 u + S_0 u = c_2 Su^{\alpha}$.

Let $v = c_3 u$ for an appropriate $c_3 > 0$ then v is a solution of (2). //

Let $\sigma(M, g_0, S) = \inf\{Y(u) : u \in H_1(M),\ u \geq 0,\ \&\ u \not\equiv 0\}$.

Because $Y(u)$ is always positive, $\sigma(M, g_0, S) \geq 0$.

We take minimizing sequence u_j (i.e. $Y(u_j) \searrow \sigma(M, g_0, S)$) with $\int_M Su_j^{\alpha+1} dV = 1$.

We want a convergent subsequence.

Since $S_0 > 0$, then

$$Y(u_1) \geq Y(u_j) = \frac{1}{2}\int_M (\gamma|\nabla u_j|^2 + S_0 u_j^2) dV$$

$$\geq c \int_M (|\nabla u_j|^2 + u_j^2) dV$$ for some $c > 0$ const. independent of j.

Thus $\|u_j\|_{H_1} \leq$ const. independent of j.

So we can take a subsequence (we use the same letter u_j) so that u_j converges weakly in H_1 to some $u \in H_1$.
By Sobolev imbedding theorem $H_1 \hookrightarrow L_p$ is cpt embedding if $p < 2n/(n - 2)$.
So $u_j \xrightarrow[L_p]{} u$ for $p < 2n/(n - 2)$.
Thus if $\alpha + 1 < 2n/(n - 2)$ we get $\int_M Su^{\alpha+1}dV = 1$.
Especially $u \geq 0$ & $u \not\equiv 0$ so $Y(u) \geq \sigma(M, g_0, S)$.
On the otherhand, by the lower semicontinuity of Y with respect to the weak H_1-convergence,

$$Y(u) \leq \lim_{j\to\infty} Y(u_j) = \sigma(M, g_0, S).$$

Thus u is a minimizing function of Y. But unfortunately in our situation

$$\alpha + 1 = \frac{n + 2}{n - 2} + 1 = \frac{2n}{n - 2}$$

so this does not quite work.

<u>Theorem</u> ($S_0 > 0$, $S > 0$).

0) $\sigma(M, g_0, S)$ is a conformal invariant.

1) (Trudinger, Aubin [Au]).

$$\sigma(M^n, g_0, S) \leq \frac{1}{(\max S)^{2/(\alpha+1)}}\sigma(S^n, \text{canonical}, 1)$$

(we will call the right-hand side of this the "standard constant".)

2) (Trudinger, Aubin [Au]).

If strict inequality $<$ holds above then the equation (2) has a solution.

3) (Schoen, Escobar [Sc], [ES]).

If (M^3, g_0) is not conformally equivalent to (S^3, can)

$\Longrightarrow$ strict inequality holds in 1) (So has a solution.).

4) (Schoen, Escobar [ES]).

If (M^n, g_0) is locally conformally flat (i.e. the Weyl tensor is identically zero) and not simply-connected and satisfies "annoying condition"

$\Longrightarrow$ strict inequality holds and there exists a solution.

* Annoying Condition

There exists a point $p \in M$ where S has its max and $(D^j S)(p) = 0$ for $|j| \leq n - 2$.

(So $n = 3$ $\Longrightarrow$ this condition is always satisfied; Yamabe case $\Longrightarrow$ this condition is always satisfied)

5) (Aubin [Au]).

If (M^n, g_0) is somewhere not locally conformally flat (i.e. Weyl tensor $\not\equiv 0$), and $n \geq 6$

$\Longrightarrow$ strict inequality holds and there exists a solution.

6) (Kazdan [KW2]).

On (S^n, g_0) with g_0 conformal to the canonical metric.

a) $\exists\, S > 0$ so that (2) has no solution.

b) If $S \not\equiv$ const and if there exists a solution, this solution can not minimize Y.

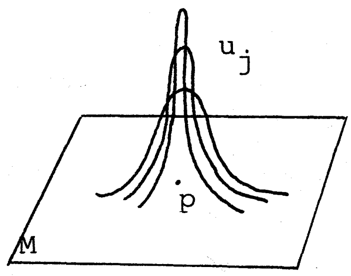
Hints about proof

Bad case is if $\int_M S u_j^{\alpha+1} dV = 1$ but $u_j \xrightarrow[H_1]{} 0$ (weakly).

This appears when u_j "bubbles off".

i.e. $u_j(p) \longrightarrow +\infty$ at some pt $p \in M$ and elsewhere u_j becomes smaller and smaller with $\int_M Su_j^{\alpha+1}dV = 1$. (See figure.)

figure



We examine what happens in these phenomena, thus we can prove above theorems.

Part 1)

$$u_\varepsilon(x) := \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{n-2}{2}} \qquad \text{in } \mathbb{R}^n.$$

Using a normal coordinate around p we lift u_ε to M. Then by calculation

$$Y(u_\varepsilon) = \frac{\sigma(S^n, \text{can}, 1)}{S(p)^{2/(1+\alpha)}} + \text{higher order terms in } \varepsilon$$

we get the result, if we take $S(p) = \max S$ and $\varepsilon \longrightarrow 0$.

Part 2)

Show $\int_M Su^{\alpha+1}dV = 1$, enough to show $\int_M Su^{\alpha+1}dV > 0$ (divide u by some const) or show $u_j \longrightarrow u$ in strongly H_1 or simply show $u \not\equiv 0$, one can give many different proofs, but we skip this because of lack of time.

Part 5)

Let u_ε be as in Part 1).

Show $Y(u_\varepsilon) = \text{std} - \text{const} + (\quad)\varepsilon^2 + \ldots$

$\uparrow$ neg. such as $-\|W\|^2$.

Part 3) Want a clever φ so that $Y(\varphi) < \text{std const}$.
4)

In Part 1) we use u_ε such that Lu_ε = Dirac's δ + higher order.

To get more precise result, Schoen used G = Green's function of $L \equiv -\gamma\Delta_0 + S_0$ (since $S_0 > 0$, L is invertible so we set $G = L^{-1}$

$$Lv = f \implies v = Gf$$

$$v(x) = \int_M G(x, y) f(y)\,dy. \Big)$$

Schoen showed the following using the positive mass theorem.

$n = 3$

$$G(x, y) = \frac{1}{|x - y|} + A + \text{higher order terms in } |x - y|$$

and $A \geqq 0$

$A = 0$ if and only if $M \underset{\text{conf.}}{\simeq} S^3$.

Part 6)

a) If u is a solution of (1) $\implies \int_{S^n} (\nabla S \cdot \nabla F) u^{\alpha+1} dV = 0$

for any 1st order spherical harmonic $F : -\Delta F = nF$.

$\Big($ On (S^n, can) i.e. $S^n \subset \mathbb{R}^{n+1}$

$F = (\text{linear func. in } \mathbb{R}^{n+1})|_{S^n} = (a_1x_1 + \cdots + a_{n+1}x_{n+1})|_{S^n}$. $\Big)$

If S is monotonic in the direction of F, there exist no solution.

<u>Example</u> S = F + const, F = linear func.

(If we choose const appropriately so that S > 0.)

b) We show if $\exists\, u_0$: minimizing func. $Y(u_0) \leq Y(u)$ for all u, then

$\Longrightarrow S \equiv \text{const}.$

∵ By Part 1)

$$Y(u_0) = \sigma(S^n, g_0, S) \leq \frac{\sigma(S^n, \text{std}, 1)}{(\max S)^{2/(\alpha+1)}}.$$

On the other hand, if $S \not\equiv$ const

$$\int_{S^n} S u_0^{\alpha+1} dV < \max S \int_{S^n} u_0^{\alpha+1} dV.$$

So

$$Y(u_0) = \frac{\frac{1}{2}\int_{S^n} \gamma|\nabla u_0|^2 + S_0 u_0^2 dV}{\left(\int_{S^n} S u^{\alpha+1} dV\right)^{\frac{2}{\alpha+1}}}$$

$$> \frac{1}{(\max S)^{\frac{2}{\alpha+1}}} \frac{\frac{1}{2}\int_{S^n} \gamma|\nabla u_0|^2 + S_0 u_0^2 dV}{\left(\int_{S^n} u_0^{\alpha+1} dV\right)^{\frac{2}{\alpha+1}}}.$$

From the definition of $(S^n, g_0, 1)$

$$\geq \frac{1}{(\max S)^{\frac{2}{\alpha+1}}} \underbrace{\sigma(S^n, g_0, 1)}_{\sigma(S^n,\ \mathrm{std},\ 1)} .$$

These two inequalities on $Y(u_0)$ contradict each other. //

Quite recently Bahri and Coron [BC] show the following result using the Morse theory.

On (S^3, std)

Assume S has non-degenerated critical pts $\{y_1, \ldots, y_k\}$ with $\Delta S(y_j) \neq 0$ for $\forall j$.

Let k_j = Morse index of S at y_j.

If $\sum_{j:\ \Delta S(y_j)<0} (-1)^{k_j} \neq -1$, there exists a solution.

Sketch of the proof

Show bubbling off appears only at critical points of S.

Seeking critical point of Y, they use a finite dimensional analysis at these finite points $y_1, \ldots, y_k$.

Essential difficulty is to analyze loss of compactness.

(i.e. For Y : Yamabe's functional Palais-Smale condition C fails.)

II. Counting Lattice Points

Lax-Phillips : The Asymptotic Distribution of Lattice Points in Euclidean and Non-Euclidean Spaces, Journal of Functional Analysis Vol. 46 (1982), pp. 280-350.

The Purpose of this section is to show the very easy idea used

in the above paper.
Although their results hold in all-dimensional Euclidean spaces and hyperbolic spaces, we treat only $\mathbb{R}^3$ to save troublesome techniques.
The Main idea is same in the other situation.

Let Γ be a lattice in $\mathbb{R}^3$

i.e. Γ : discrete subgroup of euclidean motion group and has no fixed points.

Let $M = \mathbb{R}^3/\Gamma$ be the quotient manifold, assumed compact.
Lattice points are the orbit of $o \in \mathbb{R}^3$ under Γ.
We set

$$N(s;\, x) \equiv N(s) := \#\,\{\tau \in \Gamma : |x - \tau(o)| < s\}$$

= the number of lattice points contained in the ball of radius = s and center = x.

Theorem

$$N(s) = \frac{4\pi s^3}{3|M|} + O(s^{3/2}) \qquad \text{as} \quad s \longrightarrow \infty$$

where $|M|$ = vol of M.

Idea : Put firecrackers at all lattice points and explode them all at the same time.
Listen carefully and count explosions which you can hear.
In mathematical words solve the wave equation with "little bombs at lattice points" as the initial

condition.

And investigate the solution.

First we set initial condition f.

Pick $h \in C_0^\infty(|x| < 1)$

s.t. $h \geqq 0, \int h dx = 1$

$$h_\alpha(x) := \frac{1}{\alpha^3} h\left(\frac{x}{\alpha}\right) \in C_0^\infty(|x| < \alpha)$$

Thus
$$\int_{|\tau.z-x|<T} h_\alpha(z)dz = \begin{cases} 1 & \text{if } |x - \tau.0| \leqq T - \alpha \\ 0 & \text{if } |x - \tau.0| \geqq T + \alpha \end{cases} \quad (1)$$

Let $f(x) := \sum_{\tau \in \Gamma} h_\alpha(\tau^{-1}x)$

So f is invariant under the action of Γ.

Wave equation:

$$u_{tt} = \Delta u$$

with initial condition

$$\begin{cases} u(x, 0) = 0 \\ u_t(x, 0) = 4\pi f(x) \end{cases}$$

For any f the solution is given by the basic formula

$u(x, t) = \frac{1}{t} \int_{|y|=t} f(x + y)dA$ clearly, if f is invariant

under Γ, then so is $u(x, t)$ as a function of x.

Conservation of Energy

Let $E(t) := \frac{1}{2}\int_M (u_t^2 + |\nabla u|^2)dx$

Since $\nabla(u_t \nabla u) = u_t \Delta u + \nabla u . \nabla u_t$, then by the devergence theorem (since M has no boundary)

$$\frac{dE}{dt} = \int_M u_t . u_{tt} + \nabla u . \nabla u_t dx$$

$$= \int_M u_t (u_{tt} - \Delta u) dx$$

$$= 0$$

So E must be constant, i.e. "energy is conserved". This will allow us to estimate $E(t)$ in terms of the initial data. We have also proved the uniqueness theorem. Namely if we have two solutions u_1, u_2, $v := u_1 - u_2$ is also a solution of wave equation with initial data $v(x, 0) = 0$ & $v_t(x, 0) = 0$. By conservation of energy, energy of v is constant, but at time $t = 0$ energy of $v = 0$, so energy of v is zero at all time. Then v must be zero. Consequently, our solution u is the unique invariant solution.

Let $I(T, \alpha) := \int_0^T tu(x, t)dt.$

Then from the formula above for the solution,

$$= \int_{|y|<T} f(x + y)dy = \int_{|y-x|<T} f(y)dy$$

$$= \sum_\tau \int_{|y-x|<T} h_\alpha(\tau^{-1}.y)dy = \sum_\tau \int_{|\tau.z-x|<T} h_\alpha(z)dz$$

$\therefore$ By (1) $N(T-\alpha) \leqq I(T,\alpha) \leqq N(T+\alpha)$

$\therefore$ $I(T-\alpha,\alpha) \leqq N(T) \leqq I(T+\alpha,\alpha)$

So to estimate $N(T)$, we estimate $I(T,\alpha)$

Main Lemma

$$I(T,\alpha) = \frac{4\pi}{3}\frac{T^3}{|M|} + 0\left(\frac{T}{\alpha}\right)$$

$$I(T \pm \alpha, \alpha) = \frac{4\pi T^3}{3|M|} + 0(T^2\alpha) + 0\left(\frac{T}{\alpha}\right)$$

Pick α so that $T^2\alpha = \frac{T}{\alpha}$ i.e. $T^2\alpha = T^{3/2}$

We get the Main theorem.

(Proof of Main Lemma)

Let $\bar{u}(t) := \frac{1}{|M|}\int_M u(x,t)dx$ = average of u over M.

then $$(\bar{u})_{tt} := \frac{1}{|M|}\int_M u_{tt}(x,t)dx$$

$$= \frac{1}{|M|}\int_M \Delta u(x,t)dx = 0$$

$\therefore$ $\bar{u}(t) = at + b$

$b = \bar{u}(0) = 0$

$a = \bar{u}_t(0) = \frac{4\pi}{|M|}\int_M f(x)dx = \frac{4\pi}{|M|}$

$\therefore$ $\bar{u}(t) = \frac{4\pi}{|M|}t$

Write $u(x,t) = \frac{4\pi}{|M|}t + v(x,t)$

$$\begin{cases} \int_M v(x,t)dx = 0, \quad v_{tt} = \Delta v \\ v(x,0) = 0 \end{cases}$$

Moreover

$$I(T, \alpha) = \int_0^T t \left[\frac{4\pi}{|M|} t + v(x, t)\right] dt$$

$$= \frac{4\pi T^3}{3|M|} + \int_0^T tv(x, t)dt$$

We set $V(x, T) := \int_0^T tv(x, t)dt$

and show $|V| = 0(\frac{T}{\alpha})$

We use the Gargliardo-Nirenberg inequality.

Since $\int_M Vdx = 0$

$$|V| \leqq C\|\nabla V\|^{1/2}\|\Delta V\|^{1/2} \quad \text{in } L^2(M)\text{-norm}$$

<u>Claim</u>

1) $\|\Delta V\| = 0(T\alpha^{-\frac{3}{2}})$

2) $\|\nabla V\| = 0(T\alpha^{-\frac{1}{2}})$

We only prove 1). The proof of 2) is similar, but a little more technical.

$$\Delta V = \int_0^T t\Delta v dt = \int_0^T tv_{tt} dt$$

$$= \int_0^T [(tv_t)_t - v_t] dt$$

$$= Tv_t(x, T) - \int_0^T v_t dt$$

$$\therefore \|\Delta V\| \leqq T\|v_t\|\Big|_{t=T} + \int_0^T \|v_t\| dt$$

By conservation of energy, some $v(N, 0) = 0$, for any $t \geqslant 0$,

$$\|v_t\|^2 \leq 2E(t) = \int_M (v_t^2 + |\nabla v|^2)dx$$

$$= 2E(0) = \int_M v_t^2|_{t=0} \, dx = \|v_t\|^2|_{t=0}$$

On M

$$u_t(x, 0) = \frac{4\pi}{\alpha^3} h\left(\frac{x}{\alpha}\right)$$

So $$|u_t(x, 0)| \leq \begin{cases} \text{const} \times \alpha^{-3} & \text{for } |x| < \alpha \\ 0 & \text{elsewhere} \end{cases}$$

Now $u_t = \text{const} + v_t$ & $\int_M v_t dx = 0$ (i.e. $v_t \perp \text{const}$)

so by the Pythagorean theorem

$$\int_M v_t^2 dx \leq \int_M u_t^2 dx = \int_{|x|<\alpha} u_t^2 dx$$

$$\leq \left(\frac{\text{const}}{\alpha^3}\right)^2 \frac{4\pi}{3}\alpha^3 = \frac{\text{const}}{\alpha^3}$$

$$\|\Delta v\| = O(T\alpha^{-\frac{3}{2}})$$ //

IV. Unique Continuation Property

<u>Model case</u>

If $u \in \mathcal{C}^2(\Omega)$ $\quad \Delta u = 0$ in Ω(connected)

$u \equiv 0$ in some nonempty open set $\Omega_0 \subset \Omega$

$\Longrightarrow$ $u \equiv 0$ in Ω

<u>Problem</u> : Generalize this to solutions of more general linear elliptic equations & systems!

(In the Laplace equation case we use
u : analytic)

1957 Aronszajn & Cordes

Proved "Yes" in some cases independently.

1960 Pliš gave an example "No"

Until now, all proofs are complicated and not much know for systems. We give a simple proof, covering "all" cases of geometric interest.

Th.[K2] $u = (u^1, \dots, u^N)$ satisfies the elliptic system

$L_u \equiv \Delta_g u + \sum A_j(x)\frac{\partial u}{\partial x_j} + Bu = 0$ in Ω (connected)

where Δ_g: Laplaciau of metric $g \in \mathcal{C}^1(\Omega)$

A_j, B: N × N matrices $A_j \in \mathcal{C}^1(\Omega)$

$B \in \mathcal{C}^0(\Omega)$

& $A_j^* = A_j$ (self-adjoint)

⟹ unique continuation property holds

In fact we can prove if u has a zero of infinite order at x_0, u must be zero in a neighborhood of X_0.

Applications

1. Say $-\Delta u = \lambda u$ u: eigenfunc. of Δ

⟹ $u \not\equiv 0$ on any open set

⟹ if $\Omega_1 \subsetneqq \Omega_2$, $\lambda_1(\Omega_2) < \lambda_1(\Omega_1)$

where $\lambda_1(\Omega_j)$ is the smallest eigenvalue of $-\Delta$ with Dirichlet condition on $\partial\Omega_j$

⊙ $\lambda_1(\Omega_j) = \inf\left\{ \int_{\Omega_j} |\nabla \varphi|^2 dx : \varphi \in \mathring{H}_1(\Omega_j), \int_{\Omega_j} |\varphi|^2 dx = 1 \right\}$

We take an eigenfunction u for $\lambda_1(\Omega_1)$

i.e. $-\Delta u = \lambda_1(\Omega_1)u$ in Ω_1

$u = 0$ on $\partial\Omega_1$

& $$\int_{\Omega_1} |u|^2 dx = 1$$

$$\int_{\Omega_1} |\nabla u|^2 dx = \lambda_1(\Omega_1)$$

We extend u to $\overset{\circ}{H}_1(\Omega_2)$ putting zero in $\Omega_2 \setminus \Omega_1$

$$\int_{\Omega_2} |\nabla u|^2 dx = \lambda_1(\Omega_1), \quad \int_{\Omega_2} u^2 dx = 1$$

By definition of $\lambda_1(\Omega_2)$ $\quad \lambda_1(\Omega_2) \leqq \lambda_1(\Omega_1)$

If "=" holds, then u is also an eigenfunction for $\lambda_1(\Omega_2)$

i.e. $$\begin{cases} -\Delta u = \lambda_1(\Omega_2) u & \text{in } \Omega_2 \\ u = 0 & \text{on } \partial\Omega_2 \end{cases}$$

But this is contradiction to the unique continuation property, since $u \equiv 0$ in $\Omega_2 \setminus \Omega_1$ hence $u \equiv 0$ in Ω_2 //

2. Let $F : M \longrightarrow N$ be a harmonic map.

i.e. F is a critical point of energy integral

$$E(F) = \int_M |dF|^2 dx$$

thus F satisfies

$$-d^* dF = \Delta_g F^i + g^{\alpha\beta}(x)\, {}^N\Gamma^i_{jk}(F(x)) \frac{\partial F^i}{\partial x^\alpha} \frac{\partial F^k}{\partial x^\beta} = 0 \quad \text{in } M$$

where

d^* : adjoint op. of d

(x^α) : coord. of M

(y^i) : 〃 N

Δ_g : Laplaciau of (M,g)

${}^N\Gamma^i_{jk}$: Christoffel symbol of (N,h)

Say $F(\Omega_0) = \{p\}$

for some nonempty open set $\Omega_0 \subset M$

$\Longrightarrow F(M) = p$ (constant map)

(∵) Since dF satisfies $\Delta dF = 0$

(where Δ is Laplacian for 1-form)

and $dF|_{\Omega_0} = 0$,

so by unique continuation theorem $dF \equiv 0$ in M

F must be constant map. //

Problem : If we take S: submfd of N instead of p,

can we say $F(\Omega_0) \subset S \Longrightarrow F(M) \subset S$?

<u>Sketch of proof</u>

Idea: Say $H(t) = ct^k$ + higher order $(c \neq 0)$

then k is characterized by

$$k = \lim_{t \to 0} \frac{tH'(t)}{H(t)}$$

More generally, if $H(t) > 0$ for $0 < t < T$ and $\frac{tH'(t)}{H(t)} \leq \text{const}$,

then H has a zero of finite order at $t = 0$.

For simplicity

Say $N = 1$

$$g_{ij} = \delta_{ij},\ A_j = 0 \quad B = 0$$

(i.e. We treat classical case $\Delta u = 0$, but do not use the analyticity of u)

Let $v(x, t) := u(tx)$

$$H(t) := \frac{1}{t^{n-1}} \int_{|x|=t} u^2 dA = \int_{|x|=1} v^2 dA$$

$$D(t) := \iint_{|x|<1} |\nabla v|^2 dV = \int_{|x|=1} v\frac{\partial v}{\partial r} dA$$

(since $\Delta v = 0$)

We have $t\frac{\partial v}{\partial t} = r\frac{\partial v}{\partial r}$ where $r = |x|$

$$H' = 2\int_{|x|=1} v\frac{dv}{dt} dA = \frac{2}{t}\int_{|x|=1} v\frac{\partial v}{\partial r} dA$$

$$DH' = \frac{2}{t}\left\{\int_{|x|=1} v\frac{\partial v}{\partial r} dA\right\}^2 \leqq \frac{2}{t}\left(\int_{|x|=1} v^2 dA\right)\left(\int_{|x|=1} \left(\frac{\partial v}{\partial r}\right)^2 dA\right)$$

(by Schwarz)

But $D' = 2\iint_{|x|<1} \nabla v . \nabla \frac{\partial v}{\partial t} dV$

$$= 2\int_{|x|=1} \frac{\partial v}{\partial t}\frac{\partial v}{\partial r} dA = \frac{2}{t}\int_{|x|=1} \left(\frac{\partial v}{\partial r}\right)^2 dA$$

So we get $DH' \leqq HD'$

i.e. $0 \leqq HD' - DH'$

If $u \not\equiv 0$ in a neighborhood of 0

$H(t) > 0$ for all t

So $\frac{d}{dt}\left(\frac{D}{H}\right) \geqq 0$ from above

if $t < T$

$\frac{D(t)}{H(t)} \leqq \frac{D(T)}{H(T)}$ = const independent of t.

But $\frac{D(t)}{H(t)} = \frac{tH'(t)}{2H(t)}$

$\therefore$ $H(t)$ has a zero of finite order.

hence also u.

Thus we have proved

If u has a zero of infinite order at x_0, u must be zero in a neighborhood of x_0. //

IV. Compact Surfaces with Constant Mean Curvature

$x : \Sigma^2 \hookrightarrow R^3$: immersed surface

H := the mean curvature of $\Sigma = \frac{k_1+k_2}{2}$

where k_1, k_2 : principal curvatures of Σ^2.

We treat surfaces with $H \equiv$ const.

Lemma Every compact surface has a point where k_1 and $k_2 > 0$

So const. is not zero in our situation.

Question (H. Hopf)

Are round spheres the only compact soap bubbles?

(i.e. compact surfaces immersed in $\mathbb{R}^3$ with constant mean curvature)

Some Answers

About 1900 (Liebmann)

Yes if surface is convex

About 1905 (H. Hopf)

Yes if surface is S^2.

(This holds only assuming that x is an immersion.)

1958 (A.D. Alexandroff)

Yes if surface is embedded.

(this also works for hypersurfaces in $\mathbb{R}^{n+1}$)

1981 (W.Y. Hsiang [HTY])

No $\exists$ nonstandard $S^3 \hookrightarrow \mathbb{R}^4$

with $H \equiv$ const

This shows that the natural generalization of Hopf's theorem to higher dimensions is false.

1984 (H. Wente [We])

No $\exists\, T^2 \looparrowright \mathbb{R}^3$ with $H \equiv$ const

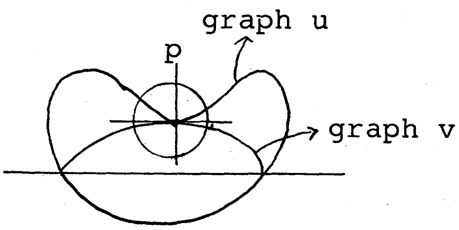
In H. Wente's constraction, he uses the special property of T^2 as rectangle in $\mathbb{R}^2$, so far we don't know whether there are higher genus counter examples.

Alexandroff's idea

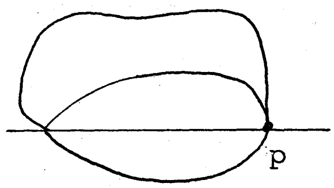
Lemma Say Σ has a plane of symmetry in every direction, then M = round sphere

⊙ Center of gravity lies on a plane of symmetries.
So it is characterized as the common point of all the planes.
Since any rotation can be viewed as product of two symmetries leaving the center fixed, we get all the rotations.
So Σ is invariant under the action of $O(n)$.
Thus Σ = round sphere.

To prove the Alexandroff's theorem, we use maximum principle for elliptic equation.
Fix a direction and move plane which has normal of that direction until the reflection of Σ intersects outside of
Then either



or



case 1 case 2

For example for the first case, if we represent Σ and reflection of Σ as graphs of functions u and v (resp.)

around p (common point of Σ and reflection of Σ).

Then u and v both satisfy the same elliptic equation since Σ has constant mean curvature, and we have $u \geq v$ in n.b.d of p and $u(p) = v(p)$.

So by maximum principle we conclude $u \equiv v$.

Thus Σ and reflection of Σ are coincide. The boundary point version of the maximum principle used in the second case.

By Lemma Σ is a round sphere.

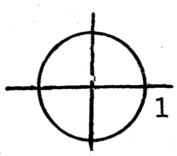
A more detailed discussion can be found in Spivak's book[Sp], Vol. 4.

Alexandroff's technique also applies to many other situations (see Gidas-Ni-Nirenberg[GNN])

for example if $u \geq 0$ satisfies

$\Delta u = f(u)$ in $|x| < 1$

$u = 0$ on $|x| = 1$,



then the only solutions are radial.

<u>Hsiang's Idea</u>

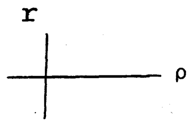
Look for "surfaces of revolution".

Here, hypersurface of revolution means one that is invariant under a large subgroup of the orthogonal group.

For $O(4)$ uses the subgroup $O(2) \times O(2) = \left\{ \left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right] \in O(4) \right\}$

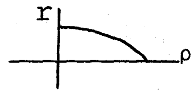
If we view $\mathbb{R}^4$ as $\mathbb{C} \times \mathbb{C} = \{(z,w)\}$

the group action is multiplying $(e^{i\theta}, e^{i\varphi})$

and $\mathbb{R}^4/_{O(2)\times O(2)}$ = 

$(r.\rho)$. $r > 0,\ \rho > 0$

Our hyper sphere is given by picking a <u>clever</u> curve in $(r.\rho)$ - plane and using group action.



Here we mean "clever" as satisfying two conditions.

1) H = const (this corresponds to that (r, ρ) satisfies a ordinary differential equation.)

2) The constructed surface is compact. (This corresponds to that the curve touches the axes.)

Background for H. Wente

$X : \Sigma^2 \hookrightarrow \mathbb{R}^3$ immersion

ξ : unit normal vector.

We take a isothermal coord (u, v)

thus $g = ds^2 = E(du^2 + dv^2)$

2nd fundamental form

$II = -dX.d\xi = Ldu^2 + 2Mdudv + Ndv^2$

Formulas

Gauss curvature $K = -\frac{1}{2E}\Delta \log E$

Mean curvature $H = \frac{L+N}{2E}$

Codazzi equation

$$\begin{cases} L_v - M_u = HE_v \\ M_v - N_u = -HE_u \end{cases}$$

Or equivalently

$$(\frac{L-N}{2})_u + M_v = EH_u = 0 \quad (\text{since} \quad H \equiv \text{const})$$

$$(\frac{L-N}{2})_v - M_u = -EH_v = 0$$

Let $\varphi(\zeta) := (\frac{L-N}{2}) - \sqrt{-1}M$ where $\zeta = u + \sqrt{-1}v$

So φ is holomorphic quadratic differential,

In the case $\Sigma \approx S^2$ (homeo) (H.Hopf case)

$\varphi \equiv 0$

$M \equiv 0$, $L \equiv N$ i.e. $II = L \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ i.e. $k_1 = k_2$

Thus Σ is everywhere umbilic and hence is a round sphere.

Let T^2 : 2-dim. torus.

$X : T^2 \hookrightarrow \mathbb{R}^3$ immersion

(u, v) : we fix global isothermal coordinates on T^2

$g = ds^2 = E(u,v)(du^2 + dv^2)$: metric on T^2

$II = Ldu^2 + 2Mdudv + Ndv^2$: the second fundamental form.

Let $\varphi := \frac{L-N}{2} - iM$ and $\zeta = u + iv$

as before. Then φ is an analytic function of ζ, and is doubly periodic. So by Liouville theorem $\varphi \equiv$ const. and if const $\equiv 0$ then the surface must be a round sphere, so const is not zero. By rotation of coordinate (i.e. diagonalize the 2nd fundamental form at one point) $M \equiv 0$, $L - N \equiv$ const. Moreover by a homothety and changing E by a factor

We may assume $\begin{cases} M \equiv 0 \\ L - N = -1 \\ H = 1/2 \end{cases}$

Write $E = e^{2w}$. Then

$$\begin{cases} k_1 - k_2 = \frac{L-N}{E} = -e^{-2w} \\ k_1 + k_2 = 2H = 1, \end{cases}$$

so that

$$\begin{cases} k_1 = e^{-w}\sinh w \\ k_2 = e^{-w}\cosh w \end{cases}$$

And since $M \equiv 0 \quad k_1 = \frac{L}{E} \quad k_2 = \frac{N}{E}$

so $$\begin{cases} L = e^{w}\sinh w \\ N = e^{w}\cosh w \end{cases}$$

By the Gauss equation we get

$$\Delta w + Ke^{2w} = 0 \quad \text{where} \quad K = k_1 k_2 \text{ : Gauss curvature}$$

so

$$(1) \qquad \boxed{\Delta w + \sinh w \ \cosh w = 0}$$

i.e. $\Delta(2w) + \sinh(2w) = 0$ (Sinh - Gordon equation)

Now we go backwards

Solve (1) for doubly periodic w, this gives us II (i.e. L,M,N). And Gauss equation and Codazzi equation (in our case since mean curvature = const. Codazzi equation is satisfied) are satisfied.

Use the Bonnet's theorem (fundamental theorem of surfaces) for $\forall \Omega$: open set, if the Gauss and codazzi equations are satisfied, then $\exists 1 X : \Omega \longrightarrow \mathbb{R}^3$

with II as its second fundamental form.

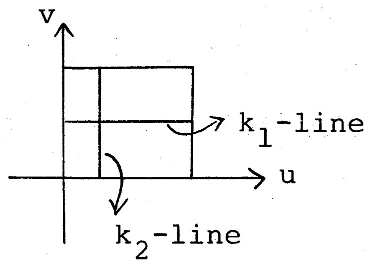
By uniqueness, we mean unique up to a rigid motion in $\mathbb{R}^3$.

Trouble This is only local. Does the surface close up? We just need a "clever" solution w.

We remark that since $M \equiv 0$ and $k_1 = \frac{L}{E}$ $k_2 = \frac{N}{E}$ the lines $u \equiv$ const and $v \equiv$ const are the lines of curvature.

We assume $v \equiv$ const are k_1 - lines of curvature. and $u \equiv$ const are k_2 - lines of curvature.

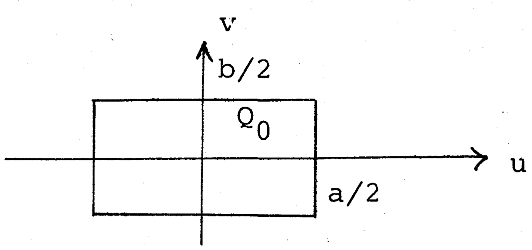
Note $k_1 < k_2$ (since $L - N = -1$).



So far, everything we have said is rather old and was known to many people. Wente's approach was to consider the special case where the fundamental region in $\mathbb{R}^2$ is a rectangle (in general it is a parallelogram) and seek solutions w that are positive in the rectangle

$$Q_0 = \{ |u| < a/2, |v| < b/2 \}$$

with $w = 0$ on $\partial\Omega_0$.



One extends such a solution to the larger rectangle

$$Q = \{ |u| < a, |v| < b \}$$

by reflecting the solution across the lines $|u| = a/2$, $|v| = b/2$ as on odd function. The fundamental region for T^2 is then Q. Wente shows that one can pick the parameters a and b so that the solution w yields an immersed torus.

Subsequently, Abresch [Ab] made a computer graph of Wente's solution and saw that the k_1 curvature lines looked planar. This led him to seek special solutions of (1) which have the additional property that the k_1 curvature lines are planar (i.e. torsion = 0). For this, the function w must also satisfy

$$w_{uv} - \frac{\cosh w}{\sinh w} w_u w_v = 0 \tag{2}$$

(Just last month, J. Spruck showed that in fact <u>every</u> positive solution w of (1) in Q_0 with $w = 0$ on ∂Q_0 <u>automatically</u> satisfies (2). His proof uses the key computation that if we let η denote the left side of (2), then η satisfies

$$\Delta\eta = \left\{ 1 + \frac{2|\nabla w|^2}{\sinh^2 w} \right\} \eta$$

in Q_0.)

Now it is quite easy to find the general solution of (2). One approach is to make the change of variables

$$Z = -\ln \tanh \frac{w}{2}$$

<u>Remark</u>.

How can we get above change of variable? For a system of equations

$$\Delta w^k + \sum_{i,j=1}^{N} c^k_{ij} \nabla w^i \cdot \nabla w^j = 0 \qquad k = 1,\ldots,N$$

We think C^k_{ij} as Christotfel symbols of some affine connection.

We want to choose new coordinates $Z^i = F^i(w',\ldots,w^N)$, $i = 1,\ldots,N$, to simplify this; for example $C^k_{ij} \equiv 0$ is the simplest case.

But we can do this if and only if the curvature determined by C^k_{ij} is zero everywhere. Note that Δ is the "Laplacian" of any metric, possibly with Lorentz signature etc. For our single equation, curvature is zero since the space is 1-dimensional. so we can take a change of variable so that $c^k_{ij} \equiv 0$. Then Z satisfies the simple single equation

$$Z_{uv} = 0$$

whose general solution is

$$\therefore Z = \varphi(u) + \psi(v)$$

For our problem, it is strangthforward to find the change of variables $Z = F(w)$

$$w = -\ln\tanh\frac{Z}{2} = -\ln\tanh\left(\frac{\varphi(u) + \psi(v)}{2}\right)$$

Next, plug this into (1) and find

(2) $(\varphi_{uu} + \psi_{vv})\sinh(\varphi+\psi) - (\varphi_u^2 + \psi_v^2 + 1)\cosh(\varphi+\psi) = 0$

By a computation (not difficult, but not obvious either) this implies that φ and ψ satisfy the first order ordinary differential equations

$$\varphi_u^2 = D\cosh 2\varphi - \gamma$$

$$\psi_v^2 = -D\cosh 2\psi - \delta$$

for some constants D and γ, δ with $\gamma + \delta = 1$

Thus, the problem of solving (1) is reduced to finding clever solutions of these ordinary differential equations.

Let $f = \varphi_u$ and $g = \psi_v$. From above we get that f and g are elliptic functions since they satisfy the equations

$$f'^2 = f^4 + (1 + \alpha^2 - \beta^2)f^2 + \alpha^2, \quad f(0) = 0, \ f'(0) = \alpha$$

$$g'^2 = g^4 + (1 + \beta^2 - \alpha^2)g^2 + \beta^2, \quad g(0) = 0, \ g'(0) = \beta$$

These elliptic functions depend on α and β. We can assume $\alpha, \beta > 0$. Note also that

(3) $$\cosh w = \frac{f_u + g_v}{1+f^2+g^2}$$

It remains to show that the constants α and β can be chosen cleverly so that the surface closes up. We shall only sketch the ideas.

First note from (3) that

$$\alpha + \beta = f_u(0) + g_v(0) = \cosh w(0) \geq 1.$$

Moreover, if $\alpha + \beta = 1$, then $w \equiv 0$, which does not give a surface that closes up. Thus $\alpha + \beta > 1$.

Next Abresch shows that the only values of $\alpha > 0$, $\beta > 0$, $\alpha + \beta > 1$ that yield solutions w that are positive in some rectangle Q_0 with $w = 0$ on ∂Q_0 are

$$|\alpha - \beta| < 1,$$

in which case the rectangles Q_0 are exactly those with $a^{-2} + b^{-2} \geq \pi^{-2}$.

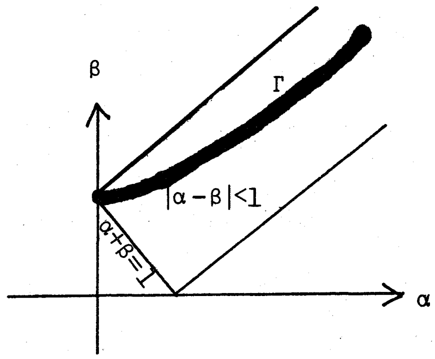
By reflection to the larger rectangle Q (see above) one obtains a solution w which is doubly periodic with fundamental region Q, having periods $2a$ and $2b$.

We still need to pick the parameters α, β so the surface closes up. It terms out that after one moves one period in the u coordinate, the k_1 curvature lines lie in parallel planes. We want the distance between these planes to be zero. This gives the condition that (α, β) be on the hyperbolic Γ

$$\Gamma = \{\beta^2 = (\alpha + q)^2 + 1 - q^2\}$$

where $q \in \mathbb{R}$ is chosen so that

$$\int_{-q}^{1} \frac{t\,dt}{\sqrt{(1-t^2)(t+q)}} = 0.$$



This gives $q = 0.652229\ldots$

Finally, we want the k_2 curvature lines to close up. One can show that the k_2 curvature line $X(0, v)$ lies in a plane as dv all periodic translates $X(0, v + 2b)$, $X(0, v + 4b)$, ...

Let θ_2 be the angle between the planes containing the k_2 curvature lines $X(0, v)$ and $X(0, v + 2b)$. If this angle θ_2 is a rational multiple of π, then eventually some plane $X(0, v + 2Nb)$ coincides with the plane containing $X(0, v)$ and the surface closes up. By a computation, $\theta_2 = \theta_2(\alpha, \beta)$ has the properties that it is monotone increasing along Γ, that

$$\theta_2(0, 1) = \pi, \quad \text{and} \quad \lim_{\alpha \to \infty} \theta_2(\alpha, \alpha + \varepsilon) = 2\pi$$

uniformly for $0 \leq \varepsilon \leq 1$. Thus, there are infinitely many points along Γ where θ_2 is a rational multiple of π. Each of these yields an immersed torus with constant mean curvature. These are the surfaces we wanted.

The above construction is clearly quite special.

V. Scalar Curvature on Non-compact Manifolds

In this section we assume (M^n, g) : is a non-compact Riemannian manifold and g is a <u>complete</u> metric. We denote $S_g \equiv S$ scalar curvature of (M, g).

§1. Topological obstraction

Certain M have no metrics g with $S_g > 0$

<u>Gromov-Lawson</u> [GL]

If M_0 is cpt and has a metric g with sectional curvature $R(M_0, g) \leq 0$, then

i) $M_0 \times \mathbb{R}$ has no non-flat g_1 with $S_{g_1} \geq 0$.

ii) $M_0 \times \mathbb{R}^2$ has no g_2 with $S_{g_2} \geq$ const > 0.

iii) $M_0 \times \mathbb{R}^3$ always has g_3 with $S_{g_3} \geq$ const > 0.

The last part is easy because the scalar curvature $S_{M\times N}$ of Riemannian product manifold $M \times N$ is $S_M + S_N$. So if we has a metric on R^3 which has sufficiently large scalar curvature, S_{g_3} is bounded from below by a positive constant .

<u>Example 1</u> $T^2 \times \mathbb{R}^2$ has no metric g with $S \geq$ const > 0.

<u>Example 2</u>  $\times \mathbb{R}$ has no metric with $S \geq 0$.

On the other hand in case $n \geq 3$.

Green-Wu [GW]. There are no obstructions to negative S. i.e. every non compact manifold has a complete metric with $S < 0$.

Bland-Kalka [BK]. For every non compact manifold M there exists a complete metric g with $S \equiv -1$.

For the proof enough to show the existence of complete metric g with $-b \leq S_g \leq -a < 0$.
Then using super & sub-solution method we can prove above.
In case $n = 2$
Cohn-Vossen showed "Gauss-Bonnet inequality"

$$\int_M KdA \leq 2\pi\chi(M)$$

for complete non-compact surfaces. We do not yet know any useful generalization of this to higher dimensions.

§2. Prescribing scalar curvature

Question Given S, find g so S = scalar curvature of g.
Simplest case (since there are no topological obstructions) : given $S < 0$ find g. Even here, we do not understand much.

One approach is to use conformal deformations. We take a nice complete metric g_0, and seek $g = pg_0$ with S = scalar curvature of g. If one has $0 < \alpha \leq p \leq \beta$, then g is also complete, but this assumption on p is much to strong.
Proving the completeness of g is perhaps the most difficult part; as we shall see, there may not be a complete metric.

§3. Pointwise conformal deformations

Given g_0, seek $g = pg_0$ for $p > 0$ with some property.

Simplest question Find p so g has constant scalar curvature.

(In compact case this is called the Yamabe problem, which was recently solved.)

$n = 2$. You can always do this (Uniformization theorem for Riemann surface).

$n \geq 3$. No. (Recently showed by Jin Zhiren [J]). We sketch his construction.

Find (M, g_0) not pointwise conformal to (M, g_1) with $S_{g_1} \equiv$ const.

Jin finds examples on any M which is obtained from a compact M_0 of dimension $n \geq 3$ by deleting a finite number of points: $M = M_0 \setminus \{p_1, \ldots, p_k\}$. The simplest case is of course $\mathbb{R}^n = S^n \setminus \{p_1\}$. To find the metric g, one begins with the known (and non-trivial) fact that any compact M_0 has a metric g_0 with scalar curvature $S_0 < 0$. Then one can easily find a conformal metric (there are many of them)

$$g = \varphi^{4/(n-2)} g_0 \qquad (\varphi > 0)$$

so that g is complete on M. This is our complete non-compact Riemannian manifold (M, g).

Theorem (Jin). This manifold (M, g) is not conformal to a complete metric g_1 with constant scalar curvature.

To prove this, we observe that if $g_1 = u^{4/(n-2)}g$, with $u > 0$, then the scalar curvature S_1 of g_1 is given by the standard formula

$$(4) \qquad Lu \equiv -\gamma\Delta_g u + Su = S_1 u^{\alpha},$$

where $\gamma = 4(n-1)/(n-2)$, and $\alpha = (n+2)/(n-2)$. If we let $v = u\varphi$, then we can write $g_1 = v^{4/(n-2)}g_0$ and the general formula (4) becomes

$$(5) \qquad L_0 v = -\gamma\Delta_0 v + S_0 v = S_1 v^{\alpha}, \qquad v > 0.$$

We will show that there is no constant S_1 for which (5) has a solution $v > 0$ on M so that g_1 is complete.

Step 1 There is no solution if $S_1 \geq 0$.

Here we do not need to use the completeness of g_1 or that $S_1 \equiv$ constant, only the fact that one can not conformally deform from negative scalar curvature to positive (or zero) scalar curvature. On a compact manifold this is just an integration by parts. In the noncompact case we use the special fact that M is obtained from M_0 by deleting a finite number of points.

One needs a technical device. Let λ_1 denote the lowest eigenvalue of the linear operator L_0 on (M_0, g_0) and observe that $S_0 < 0$ implies that $\lambda_1 < 0$ (in fact, the only known construction of g_0 actually first shows that $\lambda_1 < 0$, although this is irrelavent to us). Let $B_\rho(p_j)$, $j = 1,\ldots,k$ denote the g_0 geodesic balls of small radius ρ about p_j, let

$$M_\rho = M_0 - \bigcup_{j=1}^{k} B_\rho(p_j),$$

and let $\lambda_1^D(M_\rho)$ denote the lowest (Dirichlet) eigenvalue of L_0 on M_ρ with zero boundary condition.

<u>Lemma</u> If $n \geq 3$, then $\lim_{\rho\to 0} \lambda_1^D(M_\rho) = \lambda_1$.

The proof is not difficult. The main observation is the simple fact that if $\eta \in C^\infty(M_0)$ satisfies $0 \leq \eta \leq 1$, $\eta = 1$ in $M_{2\rho}$, $\eta = 0$ in $\bigcup B_\rho(p_j)$, and $|\nabla\eta| \leq c/\rho$, then $\eta \longrightarrow 1$ in the Sobolev space $H_1(M_0)$ for any dimension $n \geq 3$. If φ is the eigenfunction corresponding to λ_1, normalized so that $\|\varphi\|_{L^2} = 1$, then set $w = \eta\varphi$, so $w = 0$ on ∂M_ρ. Now because $\eta \longrightarrow 1$ in $H_1(M_0)$ then

$$\|w - \varphi\|^2_{H_1(M_0)} \leq c\rho^{n-2}.$$

This implies that in $L^2(M_\rho)$, if ρ is near zero then because $w = 0$ in $\bigcup B_\rho(p_j)$

$$\frac{\langle w, L_0 w\rangle_\rho}{\|w\|_\rho^2} = \frac{\langle w, L_0 w\rangle_0}{\|w\|_0^2} \leq \frac{\langle \varphi, L_0\varphi\rangle_0}{\|\varphi\|_0^2} + \varepsilon = \lambda_1 + \varepsilon .$$

The variational (Rayleigh-Ritz) characterization of $\lambda_1^D(M_\rho)$ then shows that if ρ is near zero (depending on ε), then

$$(6) \qquad \lambda_1^D(M_\rho) \leq \lambda_1 + \varepsilon.$$

On the other hand, if ψ is the eigenfunction for $\lambda_1^D(M_\rho)$, then we can extend ψ to M_0 by letting $\psi = 0$ on $M_0 - M_\rho$

so the variational characterization of λ_1 gives

$$(7) \qquad \lambda_1 \leq \frac{\langle \psi, L_0\psi \rangle_0}{\|\varphi\|_0^2} = \frac{\langle \psi, L_0\psi \rangle_\rho}{\|\psi\|_\rho^2} = \lambda_1^D(M_\rho)$$

Q.E.D.

Now we can complete Step 1. By the Lemma since $\lambda_1 < 0$ we can pick $\rho > 0$ so small that $\lambda_1^D(M_\rho) < 0$. Let ψ be the eigenfunction corresponding to $\lambda_1^D(M_\rho)$ and note that we may assume $\psi > 0$ in M_ρ (this is a general fact about the eigenfunction corresponding to the lowest Dirichlet eigenvalue).
If $v > 0$ is a solution of (5) on M then integrating by parts we have

$$\lambda_1^D(M_\rho)\langle \psi, v \rangle_\rho = \langle L_0 \psi, v \rangle_\rho$$
$$= \langle \psi, L_0 v \rangle_\rho + \gamma \int_{\partial M_\rho} (\psi \frac{\partial v}{\partial N} - v \frac{\partial \psi}{\partial N}),$$

where N is the unit outer normal on ∂M_ρ. Because $\psi > 0$ in M_ρ and $\psi = 0$ on ∂M_ρ, then $\partial \psi/\partial N \leq 0$ on ∂M_ρ so the integral on ∂M_ρ is non-negative. Hence (recall $\psi > 0$, $v > 0$).

$$0 > \lambda_1^D(M_\rho)\langle \psi, v \rangle_\rho \geq \langle \psi, L_0 v \rangle_\rho = \langle \psi, S_1 v^\alpha \rangle_\rho$$

This shows that $S_1 \geq 0$ is impossible and completes case 2.

<u>Case 2</u> There is no solution $v > 0$ of (5) on M with const $< S_1 <$ const < 0 such that $g_1 = v^{4/(n-2)} g_0$ is complete.

In fact, by a general removable singularity theorem (see [Av, Theorem 2.2]) any solution of (5) on a punctured disk $D\setminus\{p\}$ can be extended to all of D —— assuming const < S_1 < const < 0. But then in particular, v is a bounded function on M_0 so the metric g_1 on M can not be complete (to be complete, v must blow-up near the points $p_1, \dots, p$). Q.E.D.

For these special manifolds, we now see that there may not be complete conformal metrics with <u>constant</u> scalar curvature. Thus we pose the following question.
What is the best one can do by a pointwise conformal change of given metric?
We have almost no informations to this problem.
The only known informations for prescribing scalar curvature problem on non-compact manifold are

i) $(\mathbb{R}^n, \text{std})$ [Ni1], [Ni2]

i.e. $-\gamma\Delta_0 u = Su^\alpha$

ii) $(\mathbb{H}^n, \text{std})$ [AvM], [BK]

i.e. $-\gamma\Delta_0 u - u = Su^\alpha$.

They showed both existence and non-existence results.
As we said above, the most difficult part is to prove growth conditions on u insuring that g is complete.

For example

<u>n = 2</u>

$$\Delta u = -Ke^{2u} \quad \text{on } \mathbb{R}^2$$

condition : $\quad -\dfrac{C}{|x|^\ell} \leq K(x) \leq 0 \quad$ for some $\ell > 2$

then for every b satisfying $-\frac{(\ell - 2)}{2} < b < 0$ there is a solution u with the asymptotic behavior

$$u(x) = -b \log|x| + u_\infty + O(|x|^\gamma)$$

$$\text{as} \quad |x| \longrightarrow \infty$$

where u_∞ : constant

$\gamma > \max(-1, 2 - \ell - 2b)$

thus the metric $g = e^{2u} g_0$ is also complete.

§4. On manifolds with boundary

Converse of Gauss-Bonnet theorem

Gauss-Bonnet says

(M^2, g) : 2-dim. Riemannian manifold with bdry ∂M then

$$\iint_M K_g dA_g + \int_{\partial M} k_g ds = 2\pi\chi(M)$$

where K_g : Gauss curvature of (M, g)

k_g : geodesic curvature of $(\partial M, g)$

$\chi(M)$: Euler characteristic.

Converse Question:

Given any Ω: 2-form satisfying

α: 1-form

$$\iint_M \Omega + \int_{\partial M} \alpha = 2\pi\chi$$

$\exists$? g so that $\Omega = K_g dA_g$ $\alpha = k_g ds$.

Theorem (Nakamura [Na]). Yes.

Proof is simple. One picks a metric g_0 and seeks g pointwise conformal to g_0.

Other boundary value problems

Cherrier [Ch] treats the Neumann-problem.

VI. Fully Nonlinear Equations

Graphs with prescribed curvature

Given $\Omega \subset \mathbb{R}^n$

$$x_{n+1} = u(x) \quad (x \in \Omega) \ : \text{graph of } u$$

then the principal curvature of this with upper normal are the eigen-values of

$$[a_{i\ell}] = \frac{1}{w}\left[u_{i\ell} - \frac{u_i u_j u_j u_\ell}{w(1+w)} - \frac{u_\ell u_k u_{ki}}{w(1+w)} + \frac{u_i u_\ell u_j u_k u_{jk}}{w^2(1+w)^2}\right]$$

where $w = (1 + |\nabla u|^2)^{1/2}$

$$u_i = \frac{\partial u}{\partial x_i} \qquad u_{i\ell} = \frac{\partial^2 u}{\partial x_i \partial x_\ell}$$

One gets various curvatures by looking at symmetric functions of principal curvatures.

Ex. The mean curvature $\equiv H := \frac{1}{n} \operatorname{tr} a_i$

the scalar curvature $\equiv S := \sum_{i \neq j} k_i k_j$

the Gauss-Kronecker curvature $\equiv K := \det[a_{ij}]$.

Thus we pose the following question.

<u>Question</u> Given $\phi \in \mathcal{C}^\infty(\bar{\Omega})$ and $K \in \mathcal{C}^\infty(\bar{\Omega})$, does there exist a solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ such that

1) graph of u has a curvature K (such as mean curvature, scalar curvature).

2) $u = \phi$ on $\partial\Omega$?

Here we know almost nothing !!

<u>Note</u> That here we have restricted ourselves to hypersurfaces in $\mathbb{R}^{n+1}$.

If we treat general manifold M, the problem seems to be much harder.

We have two difficulties

i) One may need to assume Ω has certain geometric properties.

For example

Gauss-Kronecker curvature case ... Ω : convex, mean curvature case ... (mean curvature of $\partial\Omega$) $\geq \frac{n}{n-1}|H|$.

ii) Ω can not be too large relative to the prescribed curvature.

Typical situation

(1) $\int_\Omega K\,dx \leq \frac{\omega_{n-1}}{n}$ if K = Gauss curvature of graph u.

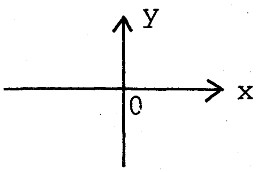
<u>Theorem</u> (Trudinger-Urbas [TU], [U])

<u>Assumption $K > 0$</u>

(i) if $<$ holds in (1) then there exist a solution.

(ii) if = holds in (1) then there exist a solution if and only if the surface is a hemisphere.

Embedding convex complete surfaces into $\mathbb{R}^3$



$(\mathbb{R}^2, g)$: complete metric not necessarily standard on $\mathbb{R}^2$ assume K = (Gauss curvature of g) > 0.

Problem (non-compact Weyl problem). Embedded this into $\mathbb{R}^3$ as a convex graph.

Theorem (Pogorelev). Yes.

There is a new proof is given by Corona [Co]. His proof involves solving the Monge-Ampère equation

$$(2) \qquad \frac{\det u''}{\det g} = K(1 - |\nabla u|^2) \qquad \text{on } \mathbb{R}^2,$$

where u'' is the Hessian of u in the g metric. This equation says that the metric $\hat{g} = g - (du)^2$ has Gauss curvature $\hat{K} \equiv 0$. The after a diffeomorphism, $\hat{g} = dx^2 + dy^2$ so

$$g = \hat{g} + (du)^2 = dx^2 + dy^2 + du(x, y)^2$$

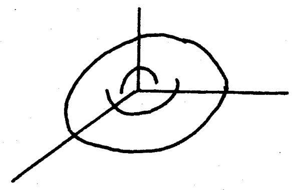
so $z = u(x, y)$ gives the embedding.

Key step in solving (2)

Solve the Dirichlet problem for $\det u'' = f(x)$ in $\Omega \subset \mathbb{R}^2$ where Ω : convex with respect to metric g Since $K > 0$, the equation (8) is of elliptic type.

We know almost nothing if K changes sign (in this case the P.D.E. is so called the mixed type, not of elliptic type).

Problem



Given $T^2 \hookrightarrow \mathbb{R}^3$ isometric embedding.
Say its Gauss curvature K_0 we remark that K_0 changes sign.
Given $K \approx K_0$ sufficiently near.
Fined a surface in $\mathbb{R}^3$ with Gauss curvature = K.
There is a similar problem for $(S^2, g_0) \hookrightarrow \mathbb{R}^3$. If one assumes the curvature K_0 of (S^2, g_0) is positive, then the equation is elliptic and is accessible to standard techniques (see [CNS]). However, for equations that are not elliptic, the only tool we have so far is the Nash-Moser implicit function theorem.

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