数理解析研究所講究録
第 616 巻 1987 年 30-45

5. Gauge Fields and Quaternion Structure

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(Dedicated to Professor M. Obata for his sixtieth birthday)

1. The aim of this lecture is to discuss geometry of the moduli spaces of Yang-Mills connections over a four-manifold with a quaternion structure.

We let (M, h) be a compact connected Riemannian 4-manifold with covariantly constant almost complex structures $\{I_i\}_{i=1,2,3}$ satisfying $I_1 I_2 = - I_2 I_1 = I_3$. Only a complex flat 2-torus and a Ricci flat K 3 surface are such spaces.

Before dealing with Yang-Mills connections over such space we exhibit its basic properties. Each almost complex structure I_i defines a covariantly constant 2-form θ_i on M; $\theta_i(X,Y) = h(I_i X, Y)$, $i=1,2,3$. The base metric h induces the metric on $\Lambda^k = \Lambda^k(T^*M)$, $k=1,2$ for which we use the same symbol. The base space M carries the canonical orientation compatible with the quaternion structre $\{I_i\}$. The base metric together with this orientation gives the Hodge operator $*$: $\Lambda^2(M) \longrightarrow \Lambda^2(M)$ which is involutive. So the bundle Λ^2 splits into $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ ($\Lambda^2_\pm$ is the subbundle of self-dual (or anti-self-dual) 2-forms). Then over M Λ^2_+ is trivial, in fact we have the decomposition;

$$\Lambda^2_+ = \mathbb{R}\,\theta_1 \oplus \mathbb{R}\,\theta_2 \oplus \mathbb{R}\,\theta_3.$$

Now let P be a smooth principal bundle over M with an arbitrary compact simple Lie group G. Fix a positive number $\ell > 2$. The set of all L^2_ℓ-connections on P is denoted by $\mathcal{A} = \mathcal{A}_P$. Denote by $\hat{\mathcal{A}} = \hat{\mathcal{A}}_P$ the subset consisting of irreducible connections on P. $\mathcal{A}$ is an affine space with model vector space $\Omega^1(\mathfrak{g}_P)_\ell = \{ L^2_\ell$ 1-forms with values in $\mathfrak{g}_P \}$; $\mathcal{A} = A + \Omega^1(\mathfrak{g}_P)_\ell$ for some smooth fixed connection ($\mathfrak{g}_P = P\times_{Ad}\mathfrak{g}$ is the adjoint bundle for the Lie algebra $\mathfrak{g}$ of G). $\hat{\mathcal{A}}$ is dense and open in $\mathcal{A}$ relative to the L^2_ℓ-norm.

Denote by $\mathcal{G} = \mathcal{G}_P$ the group of $L^2_{\ell+1}$-gauge transformations. $\mathcal{G}$ acts on $\mathcal{A}$ smoothly by $g(A) = g^{-1}dg + g^{-1}.A.g$. $\tilde{\mathcal{G}} = \mathcal{G}/_{Z(G)}$ acts freely on $\hat{\mathcal{A}}$ where $Z(G)$ is the center of G.

A connection A is ASD (anti-self-dual) if and only if its curvature $F = F(A) = d A + 1/_2 [A\wedge A]$ satisfies the ASD equations

$$F + *F = 0. \qquad (1.1)$$

Since $F(g(A)) = Ad(g^{-1})F(A)$, the solution space $\mathcal{A}^-$ of the ASD equations (1. 1) is invariant under the gauge action so that we have the quotient $\mathcal{M} = \mathcal{A}^- / \tilde{\mathcal{G}}$ which parametrizes the set of gauge equivalence classes of solutions. We call it the moduli space of ASD connections.

For geometric structure of the moduli space over a general compact oriented Riemannian 4-manifold (M, h) we have already the following theorem.

THEOREM (Atiyah, Hitchin & Singer [1]) The moduli space $\mathcal{M}$ of anti-self-dual connections is a smooth Hausdorff manifold with singularities. The dimension of the smooth part $\dim_{\mathbb{R}} \mathcal{M}$ is given by $p_1(\mathfrak{g}_P \otimes \mathbb{C})[M] - \dim_{\mathbb{R}} G\,(1 - b^1 + b^+)$ where b^1 is the first Betti number and b^+ is the dimension of the space $H^2_+(M)$ of self-dual harmonic 2-forms on M.

REMARK. The Pontrjagin number $p_1(\mathfrak{g}_P \otimes \mathbb{C})[M]$ is calculated for each group as follws; $4nk$, $G = SU(n)$; $4(n-2)k$, $G = Spin(n)$; $4(n+1)k$, $G = Sp(n)$; $16k$, $G = G_2$; $36k$, $G = F_2$; $48k$, $G = E_6$; $72k$, $G = E_7$; $120k$, $G = E_8$, where k is the index of the bundle P ([3]).

2. The Riemannian structure on $\mathcal{M}$ and main theorems.

Since the Hodge operator depends on the conformal structure, the moduli space must reflect geometric properties of the base space. In fact we can define a natural Riemannian structure on the moduli space. Since $\mathcal{A}$ is affine, the tangent space $T_A \mathcal{A} = \Omega^1(\mathfrak{g}_P)_\ell$. On this tangent space an inner product is well defined by

$$
\begin{aligned}
(\beta, \gamma) &= \int_M ((-tr) \otimes h)(\beta, \gamma)\, dv \\
&= \int_M (-tr)(\beta \wedge {*\gamma}),
\end{aligned}
\tag{2.1}
$$

$\beta, \gamma \in \Omega^1(\mathfrak{g}_P)_\ell$ for $G = SU(n)$. For a general G

we must replace in (2. 1) -tr by some adjoint invariant inner product. This inner product is gauge invariant. On the other hand $\mathcal{M}$ is a subspace in the ambient space, that is, in the space of gauge orbits of connections $\mathcal{B} = \mathcal{A}/\tilde{\mathcal{G}}$ on P. The inner product (,) descends on $\hat{\mathcal{B}} = \hat{\mathcal{A}}/\tilde{\mathcal{G}}$ and its restriction on the smooth part of $\mathcal{M}$ provides a Riemannian structure there. Then we have the following theorem when we assume that the base space is Kähler.

THEOREM ([7],[8]) Let $\mathcal{M}^s$ be the smooth part of the moduli space $\mathcal{M}$ of ASD connections on a fixed bundle $P \longrightarrow M$. Then it admits an integrable almost complex structure for which the canonical Riemannian structure is Kähler. Its complex dimension is $\dim_{\mathbb{C}} \mathcal{M}^s = 1/_2\, p_1(\mathfrak{g}_P \otimes \mathbb{C}) - \dim_{\mathbb{R}} G\,(1 - q + p_g)$, where q, the irregularity of $M = 1/_2\, b^1$ and p_g, the geometric genus, is given by $\dim_{\mathbb{C}} H^0(M; \mathcal{O}(K_M))$.

REMARKS (i) If a Kähler surface (M, h) has positive scalar curvature or the line bundle K_M is holomorphically trivial, then $\mathcal{M}^s = \mathcal{M} \cap \hat{\mathcal{B}}$, i.e., $\mathcal{M}^{sing} = \mathcal{M} \setminus \mathcal{M}^s$ consists only of reducible ASD connections ([7]).

(ii) The anti-self-duality of connections is equivalent to the stability of holomorphic vector bundles over an algebraic surface. So, $\mathcal{M} \cap \hat{\mathcal{B}}$ is in one to one correspondence with the moduli $\mathcal{M}_{st}$ of stable vector bundles with corresponding rank and Chern classes.

The theorem says that if the base space holonomy is unitary, then the moduli space holonomy is also unitary. So we can consider the following problem:

Suppose (M, h) has holonomy in group $SU(2) = Sp(1)$. Is the holonomy of the Riemannian structure $\langle\, ,\, \rangle$ on $\mathcal{M}$ symplectic ?

This problem can be written as

Suppose that the base space is hyperkähler. Is it true that the moduli space is also hyperkähler ?

DEFINITION A manifold (N, g) is hyperkähler if there exists a quaternion structure $\{I_i\}_{i=1,2,3}$ on N which is covariantly constant with respect to the Levi-Civita connection.

A hyperkähler manifold is Ricci flat Kähler and has a holomorphic symplectic structure so that the canonical line bundle K_N is holomorphically trivial.

Over a compact hyperkähler 4-manifold the moduli space of ASD connections has dimension $\dim_{\mathbb{R}} \mathcal{M} = p_1(\mathfrak{g}_P \otimes \mathbb{C})[M] - 4\,\varepsilon(M) \dim_{\mathbb{R}} G$ where $\varepsilon(\text{torus}) = 0$ and $\varepsilon(\text{K3 surface}) = 1$. So the dimension is divisible by 4. On the other hand we have another circumstantial evidence for the problem:

THEOREM (Mukai [14]) Let M be a complex 2-torus or a K 3 surface. If M is algebraic, then the moduli space of stable sheaves has a holomorphic symplectic structure.

We have actually the following affirmative answer.

THEOREM([9]). Let $P \longrightarrow M$ be a principal bundle with a compact simple Lie group G and $\mathcal{M}$ the moduli space of ASD connections on P. Then the smooth part $\mathcal{M}^s$ of $\mathcal{M}$ is hyperkähler if the base space is hyperkähler.

REMARKS (i) The theorem holds for an arbitrary compact simple Lie group, for instance $G = SO(n)$.

(ii) Let A be an ASD connection over a compact hyperkähler 4-manifold. If it is irreducible, then a neighborhood at [A] gives a smooth structure on the moduli space $\mathcal{M}$. So, the smooth part $\mathcal{M}^s$ coincides with $\mathcal{M} \cap \hat{\mathcal{B}}$

(iii) Einstein-Hermitian metrics on a holomorphic vector bundle are defined over a compact Kähler manifold. In terminology of U(n)-principal bundle we can formulate the Einstein-Hermitian connection on the bundle, equivalent to the definition of Einstein-Hermitian metric ([9]).

We have for Einstein-Hermitian connections

THEOREM([9]) The moduli space of irreducible Einstein-Hermitian connections over a compact hyperkähler 4-manifold admits a hyperkähler structure.

(iv) The framed moduli space $\tilde{\mathcal{M}}$ of anti-instantons over the standard 4-sphere S^4 is a smooth complex manifold if the rank of G is sufficiently large relative to the index k of the bundle ([3]). There is a one to one correspondence between $\tilde{\mathcal{M}}$ and the moduli space of based anti-instantons over S^4 at the north pole ∞. Since the ASD equations are conformally invariant and $S^4 = \mathbb{R}^4 \cup \{\infty\}$ is the conformal compactification, we obtain

THEOREM ([10]) The moduli space $\mathcal{M}_\infty$ of based anti-instantons over the 4-sphere carries a quaternion structure induced naturally from $\mathbb{R}^4$ which yields a hyperkähler structure on $\mathcal{M}_\infty$.

(v) Besides the Euclidean $\mathbb{R}^4$ there are many nontrivial examples of open (complete) hyperkähler 4-manifold, Eguchi-Hanson metric, Taub-Nut metric, Multi-center Taub-Nut metrics and ones recently constructed by Kronheimer by using the momentum map ([12]). We can discuss the moduli spaces of ASD connections modulo based gauge transformations. These spaces carry a hyperkähler structre.

3. The momentum map and the proof of the main theorem.

There are two ways for the proof of the main theorem. One is the momentum map method due to originally Marsden-Weinstein([13,11,6,12]). Another is the Hodge decomposition method together with the Kuranishi map. In [9] we developed a proof of the theorem by adopting the latter method. While the caluculation is not so simple in the latter case, we can get explicitly the Riemannian curvature tensor in terms of the Green operators associated with the Laplacians of the deformation elliptic compleces.

Here we will use the momentum map to show the theorem.

Each almost complex structure I_i of the base space M induces naturally an endomorphism on Λ^1 and hence an endomorphism on the product bundle $\Lambda^1 \otimes \mathfrak{g}_P$ for which we use the same symbol. So the space $\hat{\mathcal{A}}$ of irreducible connections carries almost complex structures $\{I_i\}_{i=1,2,3}$, since $T_A \hat{\mathcal{A}} = \Omega^1(\mathfrak{g}_P)_\ell$. Then we have a skew symmetric bilinear form ω_i on $\Omega^1(\mathfrak{g}_P)_\ell$;

$$\omega_i(\alpha, \beta) = (I_i \alpha, \beta), \quad \alpha, \beta \in \Omega^1(\mathfrak{g}_P)_\ell. \tag{3. 1}$$

Note that every gauge transformation is symplectic with respect to each ω_i.

Let A be an irreducible connection on a bundle $P \xrightarrow{G} M$. Now we would like to define a momentum map $\mu = (\mu_1, \mu_2, \mu_3): \hat{\mathcal{A}} \longrightarrow \Omega^0(\mathfrak{g}_P)^{\oplus 3}$. The self-dual curvature $F_+(A) = 1/_2(F + *F)$ belongs to $\Omega^2_+(\mathfrak{g}_P) = \bigoplus_{i=1}^{3} (\Omega^0(\mathfrak{g}_P) \otimes \theta_i)$. Denote by $F_i = F_i(A) \in \Omega^0(\mathfrak{g}_P)$ the θ_i- component of F_+; that is, $F_+(A) = \sum_{i=1}^3 F_i \otimes \theta_i$ ($F_i = |\theta_i|^{-2} h(F_+, \theta_i)$, here the map $h : (\Lambda^2 \otimes \mathfrak{g}_P) \times \Lambda^2 \longrightarrow \mathfrak{g}_P$ is the natural bilinear bundle map given by contraction with the base metric h). We define a map μ which should be a momentum map by

$$\mu_i : \hat{\mathcal{A}} \longrightarrow (\Omega^0(\mathfrak{g}_P))^* = \text{the dual of } \Omega^0(\mathfrak{g}_P),$$

$$< \mu_i(A), \phi > = |\theta_i|^2 \int_M (-\mathrm{tr})(F_i(A), \phi)\, dv, \tag{3. 2}$$

$\phi \in \Omega^0(\mathfrak{g}_P)$, $i = 1, 2, 3$.

In the left hand side $< \, , \, >$ is the dual pairing.

The moduli space $\mathcal{M} \cap \hat{\mathcal{B}}$ of irreducible ASD connections on P is then described as $\mathcal{M} \cap \hat{\mathcal{B}} = \mu^{-1}(0)/\tilde{\mathcal{G}}$.

We show the following

THEOREM (a) The map $\mu = (\mu_1, \mu_2, \mu_3)$ defined at (3. 2) is a momentum map, namely it satisfies

(i) $< d(\mu_i)_A(\beta), \phi > = \omega_i(\beta, \nabla_A \phi), \quad \phi \in \Omega^0(\mathfrak{g}_P),$ $\beta \in \Omega^1(\mathfrak{g}_P)_\ell$. (3.3)

(b) Moreover (ii) μ is $\tilde{\mathcal{G}}$-equivariant;

$$\mu_i(g^*(A)) = ad(g^{-1})^* \mu_i(A), \quad g \in \tilde{\mathcal{G}}, \ i = 1, 2, 3, \qquad (3.4)$$

(iii) the zero set $\mu^{-1}(0)$ is a submanifold of $\hat{\mathcal{A}}$ and at each $A \in \mu^{-1}(0)$ the tangent space to $\mu^{-1}(0)$ coincides with Ker ($d\mu_A$) and

(iv) gauge transformations $\tilde{\mathcal{G}}$ act freely on $\mu^{-1}(0)$ and at each A in $\mu^{-1}(0)$ there exists a slice $S_A \subset \mu^{-1}(0)$ for the action of $\tilde{\mathcal{G}}$.

(c) There exist symplectic forms $\{ \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3 \}$ on the reduced phase space $\mathcal{M} \cap \hat{\mathcal{B}} = \mu^{-1}(0)/\tilde{\mathcal{G}}$ satisfying $\pi^*\tilde{\omega}_i = j^* \omega_i$, $i = 1, 2, 3$, where $j : \mu^{-1}(0) \longrightarrow \hat{\mathcal{A}}$ is the canonical embedding and $\pi : \mu^{-1}(0) \longrightarrow \mathcal{M} \cap \hat{\mathcal{B}}$ is the natural projection.

It is derived from this theorem that if we let $\tilde{\nabla}$ be the Levi-Civita connection of the canonical Riemannian metric on $\mathcal{M} \cap \hat{\mathcal{B}}$ then $\tilde{\omega}_i$'s are covariantly constant ($\tilde{\nabla} \tilde{\omega}_i = 0$)

since we have observed that each almost complex structure on $\mathcal{M} \cap \hat{\mathcal{B}}$ induced from each I_i is integrable ([8]).
Hence $\mathcal{M}^s = \mathcal{M} \cap \hat{\mathcal{B}}$ equipped with the canonical Riemannian metric is hyperkähler.

Now we must to show the above theorem. We prove at first that the differential of μ satisfies (3.3).
For any $\beta \in T_A \hat{\mathcal{A}} = \Omega^1(\mathfrak{g}_P)_\ell$

$$< (d\mu_i)_A(\beta), \phi > = |\theta_i|^2 \int_M (-tr)((d_A\beta)_i, \phi)dv$$

$$= \int_M (-tr \otimes h)(d_A\beta, \phi\otimes\theta_i)dv.$$

Here the integrand is calculated as

$$(-tr \otimes h)(d_A\beta, \phi\otimes\theta_i)dv = (-tr)(d_A\beta\Lambda*(\phi\otimes\theta_i))$$

$$= (-tr)(d_A\beta\Lambda(\phi\otimes\theta_i)) = d(-tr(\beta\Lambda(\phi\otimes\theta_i))) +$$

$$(-tr)(\beta\Lambda(\nabla_A\phi\Lambda\theta_i)).$$

Hence

$$<(d\mu_i)_A(\beta), \phi > = - \int_M (-tr)(\beta\Lambda**(\nabla_A\phi\Lambda\theta_i))$$

$$= - \int_M (-tr\otimes h)(\beta,*(\nabla_A\phi\Lambda\theta_i))dv.$$

If we use the following formula which is easily obtained, then we have (3. 3).

FORMULA. $*(\alpha\wedge\theta_i) = -I_i(\alpha), \quad \alpha \in \Lambda \otimes \mathfrak{g}_P, \quad i = 1, 2, 3.$ (3.5)

The $\tilde{\mathcal{G}}$ -equivariance of μ is derived since the curvature $F(A)$ is $\tilde{\mathcal{G}}$ -equivariant and the inner product on $\Omega^1(\mathfrak{g}_P)_\ell$ is invariant under the action of $\tilde{\mathcal{G}}$. To prove (iii) and (iv) we can make use of the slice lemma argument and also the fact that the second cohomology space H^2_A vanishes for each irreducible ASD connection A. But we omit the detail. The statement (c) is a direct consequence of [11].

4. Further remarks on the theorems.

The base space is assumed first to be a general compact oriented 4-manifold. We defined before the canonical Riemannian structure $\langle\ ,\ \rangle$ on the smooth part $\mathcal{M}^s$ of the moduli space of ASD connections on a certain bundle. The Riemannian curvature tensor R of this metric can be calculated ([8]). Namely, for tangent vectors $X, Y \in T_{[A]}\mathcal{M}^s$ ($\cong H^1_A$, the first cohomology space) the value of the tensor, $\langle R(X,Y)Y,X\rangle$ is written as

$$\langle R(X,Y)Y,X\rangle = 3\,(\{X,Y\}, G_A\{X,Y\}) - ([X\wedge Y]^+, G_A[X\wedge Y]^+) + ([X\wedge X]^+, G_A[Y\wedge Y]^+). \tag{4.1}$$

Here $(\ ,\ )$ is the inner product given at (2.1) and the linear map G_A denotes the Green operator of the Laplacian associated to an ASD connection A. The definition of the bilinear mappings $\{\ ,\ \}$ and $[\ \Lambda\ ]^+$ is given in [8].

Now let the base space (M, h) be a compact hyperkähler 4-manifold and $P \longrightarrow M$ a bundle with a compact simple G. We saw in [9] that each tangent space H^1_A is invariant under the operation of the quaternion structure $\{I_i\}_{i=1,2,3}$ and hence the space H^1_A becomes a Hamiltonian vector space. Note that each I_i is an isometry with respect to $(\ ,\ )$. Then by some observation we derive from (4.1) the following curvature identity.

THEOREM([9]).

$$\langle R(X,Y)Y,X\rangle + \sum_{i=1}^{3} \langle R(X,I_iY)I_iY,X\rangle = 0, \quad X, Y \in H^1_A \qquad (4.2)$$

From this we can immediately show that the Ricci curvature of the smooth part of the moduli space vanishes.

The formula (4.2) asserts moreover that the left hand side of it represents a "quaternionic" bisectional curvature of 1-dimensional $\mathbb{H}$-linear subspaces $V_X = \{X, I_1X, I_2X, I_3X\}$ and V_Y and this bisectional curvature vanishes.

The above formula gives further Riemannian geometrically a strong restriction to the smooth part $\mathcal{M}^s$. So the following problem still remains:

Does every hyperkähler manifold satisfy this curvature identity ?

All 4-dimensional hyperkähler manifolds do indeed satisfy it.

With respect to this problem we have the result of M. Obata. He obtained the following fact in [15, Theorem 3. 1]. Let (N, g) be a hyperkähler manifold with a quaternion structure $\{I_i\}_{i=1,2,3}$. Then its Levi-Civita connection and hence its Riemannian curvature tensor can be expressed in terms of only derivatives of the components of I_2 with respect to the complex coordinates associated to I_1.

In the final part we should mention the moduli space of ASD connections with group $G = SO(3)$. Let $P \to M$ be an SO(3) bundle over a compact oriented 4-manifold. Then the moduli space of ASD connections on P has the virtial real dimension

$$\dim \mathcal{M} = -2\ell - 3(1 - b_1 + b^+),$$

where $\ell = p_1(P)[M]$. We remark that the second Stiefel-Whitney class $w_2(P) \in H^2(M;Z_2)$ and the first Pontrjagin class $p_1(P) \in H^4(M;Z)$ classify SO(3)-bundles over a 4-manifold M. If $w_2(P) = 0$, then P comes from an SU(2)-bundle and $p_1(P) = -4c_2(P)$. See [4] and [5] for basic references on SO(3)-bundles and the moduli space of SO(3)-connections.

THEOREM. Let M be a complex 2-torus or a K 3 surface.
(i) Suppose $P \to M$ be an SO(3)-bundle with $w_2(P) \neq 0$. If $p_1(P)[M]$ is odd, then P does not admit any ASD connection with respect to an arbitrary base metric on M.
(ii) On the complex 2-torus there exists an SO(3)-bundle P with $w_2 \neq 0$ and $p_1(P)[M] = -2$ such that relative to an appropriate flat Kähler metric the moduli space of ASD connections

on P (more precisely each of its connected component) carries a structure of complex flat 2-torus.

Proof. (i) Assume that one has an ASD connection on P. Then this connection must be irreducible because any reducible connection splits P into $L \oplus 1$ for a complex line bundle L and a rank one real trivial bundle 1, and hence $p_1(P)[M] = c_1(L)^2 [M]$ is even. Then the moduli space of ASD connections contains only irreducible ones. So with respect to a base Kähler metric defining the hyperkähler structure on M the moduli space carries a hyperkähler structure from the main theorem in sect. 2, while its virtual dimension is not divisible by four.

(ii) From the property of the intersection form of the complex 2-torus we have a complex line bundle L satisfying $c_1(L)^2 = p_1(P)$, $c_1(L)_{mod2} = w_2(P)$ and $c_1(L) \wedge [\omega_h] = 0$ (ω_h is the Kähler form of the standard flat metric h). Then the $SO(3)$-bundle associated to the vector bundle $L \oplus 1$ is equivalent with P and admits a reducible ASD connection with respect to h. So, if we denote $\mathcal{M}(g)$ by the moduli space of g-ASD connections on P for a base metric g on M, then $\mathcal{M}(h)$ is non-empty. Making use of the argument developed in [5 , Ch 3] we can show that for a generic base metric $\mathcal{M}(g)$ is a smooth manifold with no singularities. Now we choose a base Kähler metric h' on M so that $c_1(F) \wedge [\omega_{h'}] \neq 0$ for any holomorphic line bundle F satisfying $c_1(F)^2 = p_1(P)$ and $c_1(F)_{mod2} = w_2(P)$. Since the ASD equations $*_g F_A = - F_A$ depend smoothly on the base metric g, $\mathcal{M}(h')$ is non-empty. The condition $c_1(F) \wedge [\omega_{h'}] = 0$ then implies that $\mathcal{M}(h')$ consists of only irreducible h'-ASD connections. Since $c_1(M) = 0$, one has a Ricci flat

Kähler metric h_1 on M with $[\omega_{h_1}] = [\omega_{h'}]$ which yields a hyperkähler structure on the complex 2-torus M. Therefore $\mathcal{M}(h_1)$ is endowed with a hyperkähler structure and its dimension is $-2\ell = 4$. Moreover we can obtain that it is compact and also locally homogeneous, because the compactness argument in [4] is applicable and any nontrivial infinitesimal deformation of ASD connections is induced by the action of an infinitesimal isometries of (M, h_1) (see [2 , sect 2]). So, the moduli space $\mathcal{M}(h_1)$ is a complex flat 2-torus.

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