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Buchsbaum Rings and Cone Singularities

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Let $S=k[X_0,\dots,X_N]$ be the polynomial ring. Let us regard the polynomial ring S as a graded ring in which the degree of X_j is one for every $0\le j\le N$. Let $\mathfrak{m}=\oplus_{d\ge1} S_d$. Let M be a finitely generated graded S-module of dimension n+1. Then M is called a Buchsbaum S-module (resp. a quasi-Buchsbaum S-module) if $M_\mathfrak{m}$ is a Buchsbaum $S_\mathfrak{m}$-module (resp. a quasi-Buchsbaum $S_\mathfrak{m}$-module). That is, M is a quasi-Buchsbaum module if and only if $\mathfrak{m}H^i_\mathfrak{m}(M)=0$ for any $0\le i\le n$. On the other hand, M is a Buchsbaum module if and only if the difference $l(M/qM)-e_q(M)$ is an invariant for any homogeneous parameter ideal $q(\subset\mathfrak{m})$ for M, where l, respectively e, denotes length, respectively multiplicity of q.

In this paper, we search for the condition that a graded S-module M is a Buchsbaum module. In particular, our purpose is to clarify the difference between Buchsbaum property and quasi-Buchsbaum property.

For this, we introduce the notion of quasi-Buchsbaum modules of type r in Section 1. Then we give a criterion (Theorem 1.7) for it, which explicitly describes "Surjectivity Criterion" through spectral sequence. In Corollary 1.10, we give a sufficient condition that M is Buchsbaum. It is described only by the vanishing of its local cohomologies. This generalizes a result of Goto-Watanabe [4, Proposition 3.1]. Also, the theorem enables us to calculate the length of $\mathrm{Ext}_S^{n+1}(k, M)$ in Corollary 1.12.

In Section 2, we investigate the divisors of Segre products. In particular, we construct very concrete examples of quasi-Buchsbaum but not Buchsbaum rings.

§1. Quasi-Buchsbaum modules of type r

Definition 1.1. M is a quasi-Buchsbaum module of type r if, for every homogeneous system $f_1, \dots, f_{r-1}$ of parameters, $M/(f_1, \dots, f_k)M$ is a quasi-Buchsbaum module for every $k \le r-1$.

Remark 1.2. In the above definition, we have only to take elements of degree one as homogeneous systems of parameters, by Goto-Suzuki [6, Theorem 1.9] or Suzuki [19, Theorem 3.6].

Remark 1.3. M is a quasi-Buchsbaum module of type n+1 if and only if M is a Buchsbaum module, by Stückrad-Vogel [14].

Let U_i be an open set $D_+(X_i)$ in Proj S for $0 \le i \le N$. Then $\mathfrak{U} = \{U_i\}_{0 \le i \le N}$ is an open covering of Proj S. Let us consider a Čech complex $C^\cdot = C^\cdot(\mathfrak{U}; \tilde{M})$, where $\tilde{M}$ is the sheaf associated to the graded S-module M on Proj S. Then we define a complex

$L^{\cdot}=(0 \longrightarrow M \xrightarrow{\varepsilon} C^{\cdot}[-1])$, where ε is a natural map. (e.g. See Godement [2].)

Let $K.$ be the Koszul complex associated to $\langle X_0,\dots,X_N\rangle$, that is, $K_p=\wedge^d(\bigoplus_{k=0}^{N} Ae_k)=\bigoplus_{0\le i_1<\dots<i_p\le N} Ae_{i_1}\wedge\dots\wedge e_{i_p}$, where $\langle e_0,\dots,e_N\rangle$ is a free basis.

In this way, we have a double complex $K^{\cdot\cdot}=\mathrm{Hom}(K.,L^{\cdot})$. Now let us consider two "stupid" filtrations, that is,

$$'K_t=\sum_{p\ge t} K^{p,q} \text{ and } ''K_t=\sum_{q\ge t} K^{p,q}.$$

As usual, we get the spectral sequence for each filtered complex:

$$'E_1^{p,q} = \mathrm{Ker}\ d''^{p,q}/\mathrm{Im}\ d''^{p,q-1} \Longrightarrow H^{p+q}(K^{\cdot\cdot})$$

$$''E_1^{p,q} = \mathrm{Ker}\ d'^{p,q}/\mathrm{Im}\ d'^{p-1,q} \Longrightarrow H^{p+q}(K^{\cdot\cdot})$$

Lemma 1.4. *Under the above conditions, we have the following things.*

(1) $'E_1^{p,q}=\wedge^p(\bigoplus_{k=0}^{N} H_m^q(M)e_k^*)$,

where $\langle e_0^*,\dots,e_N^*\rangle$ *is the dual basis of* $\langle e_0,\dots,e_N\rangle$, *and*

(2) $''E_1^{p,q} = \begin{cases} \mathrm{Ext}_S^p(k,M) & \text{if } q=0 \\ 0 & \text{otherwise.} \end{cases}$

This implies that $H^{p+q}(K^{\cdot\cdot}) = \mathrm{Ext}_S^{p+q}(k,M)$.

From now on we will treat the first filtration. So we write $E_r^{p,q}$ for $'E_r^{p,q}$.

Lemma 1.5. *The spectral sequence* $\{E_r^{p,q}\}$ *does not depend on the choice of coordinates* $\langle X_0, \ldots, X_N\rangle$.

Now then we will study the relation between the spectral sequence and Buchsbaum property.

Proposition 1.6. *Notations being above, the following conditions are equivalent.*

(a) M *is a quasi-Buchsbaum* S*-module.*

(b) $d_1^{p,q} : E_1^{p,q} \longrightarrow E_1^{p+1,q}$ *is a zero map for* $q \leq n$.

(c) $d_1^{0,q} : E_1^{0,q} \longrightarrow E_1^{1,q}$ *is a zero map for* $q \leq n$.

Proof. By (1.4.1.), $E_1^{p,q} = \wedge^p(\bigoplus_{k=0}^{N} H_{\mathfrak{m}}^q(M)e_k^*)$. From the construction of the double complex $K^{\cdot\cdot}$, we see that

$d_1^{p,q}(e_{i_1}^* \wedge \ldots \wedge e_{i_p}^*) = \sum_{j=0}^{N} X_j e_j^* \wedge e_{i_1}^* \wedge \ldots \wedge e_{i_p}^*$. Thus the assertion follows from the definition of quasi-Buchsbaum module.

Suppose that M is a quasi-Buchsbaum module and that every subset I of $\{X_0, \ldots, X_N\}$ such that $\#I = n+1$ makes a system of parameters of M. Setting $\overline{M} = M/X_jM$, we have the following exact sequence:

$$0 \longrightarrow (0:X_j)_M \longrightarrow M \xrightarrow{\cdot X_j} M \longrightarrow \bar{M} \longrightarrow 0 .$$

Since $H^q_{\mathfrak{m}}((0:X_j)_M)=0$ for $q\geq 1$ and M is a quasi-Buchsbaum module, we have the following short exact sequence:

$$0 \longrightarrow H^{q-1}_{\mathfrak{m}}(M) \longrightarrow H^{q-1}_{\mathfrak{m}}(\bar{M}) \longrightarrow H^q_{\mathfrak{m}}(M) \longrightarrow 0 .$$

for $1\leq q\leq n$. Thus we have the following commutative diagram with exact raws:

(1.7.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{q-1}_{\mathfrak{m}}(M) & \longrightarrow & H^{q-1}_{\mathfrak{m}}(\bar{M}) & \xrightarrow{\alpha} & H^q_{\mathfrak{m}}(M) & \longrightarrow & 0 \\ & & \downarrow \phi & (1) & \downarrow \bar{\psi} & (2) & \downarrow \phi' & & \\ 0 & \longrightarrow & H^{q-1}_{\mathfrak{m}}(M) & \xrightarrow{\beta} & H^{q-1}_{\mathfrak{m}}(\bar{M}) & \longrightarrow & H^q_{\mathfrak{m}}(M) & \longrightarrow & 0 \end{array}$$

for $1\leq q\leq n$, where ϕ, ϕ' and $\bar{\psi}$ are multiplication maps $\cdot X_i$'s. Since M is a quasi- Buchsbaum module, ϕ and ϕ' are zero maps. By snake lemma, we get a graded S-homomorphism of degree 2 $\psi : H^q_{\mathfrak{m}}(M) \longrightarrow H^{q-1}_{\mathfrak{m}}(M)$ such that $\psi\circ\alpha=\beta\circ\bar{\psi}$. Let us write $(X_i\wedge X_j)$ for this map ψ. Since $\psi=0$ is equivalent $\bar{\psi}=0$, we see that $\bar{M}$ is a quasi-Buchsbaum module if and only if $(X_i\wedge X_j)$ is a zero map for any i. In other words, M is a quasi-Buchsbaum module of type 2 if and only if and only if $(X_i\wedge X_j)$ are zero map for any i and j.

Comtinuing the similar steps, we have the following theorem.

Theorem 1.7. *Let* S *be the polynomial ring* $k(X_0,\dots,X_N)$.

Let M *be a finitely generated graded* S*-module with dimension* n+1. *Suppose that* M *is a quasi-Buchsbaum module of type* r−1. *Then*

$$(X_I)=(X_{i_1}\wedge\ldots\wedge X_{i_r}) : H^q_{\mathfrak{m}}(M) \longrightarrow H^{q-r+1}_{\mathfrak{m}}(M)$$

is well-defined.

Furthermore, M *is a quasi-Buchsbaum module of type* r *if and only if* (X_I) *defined above is a zero map for any* I.

Now there still remains to link between the spectral sequence $\{E^{p,q}_r\}$ and the maps (X_I)'s constructed above.

Lemma 1.8. *Let* M *be a quasi-Buchsbaum graded* S*-module of type* r−1 *and with dimension* n+1. *Then we have*

(1) $E^{p,q}_r=\wedge^p(\bigoplus_{k=o}^{N} H^q_{\mathfrak{m}}(M)e^*_k)$ *for any* $q\neq n+1$.

By (1), *we can write* $(d^{p,q}_r)_{J,K} : H^q_{\mathfrak{m}}(M) \longrightarrow H^{q-r+1}_{\mathfrak{m}}(M)$ *for the map from* e^*_K*-component to* $e^*_J\wedge e^*_K$*-component of the map* $d^{p,q}_r$.
Then we have

(2) $(d^{p,q}_r)_{J,K}=(-1)^{(r-1)(p+q-r)}(X_J)$, *where* (X_J) *is the map defined in* (1.7).

Then we have the following theorem.

Theorem 1.9. *Notations being above, the following conditions are equivalent.*

(a) M *is a quasi-Buchsbaum* S*-module of type* r.

(b) $d_s^{p,q} : E_s^{p,q} \longrightarrow E_s^{p+s,q-s+1}$ *is a zero map for* $s \leq r$ *and* $q \leq n$.

(c) $d_s^{0,q} : E_s^{0,q} \longrightarrow E_s^{s,q-s+1}$ *is a zero map for* $s \leq r$ *and* $q \leq n$.

Corollary 1.10. *Let* M *be a finitely generated graded* S*-module. Let us define*

$\mathfrak{S} = \{(i, \ell) \mid 0 \leq i \leq n,\ \ell \in \mathbb{Z},\ H_{\mathfrak{m}}^i(M)_\ell \neq 0\}$.

Suppose that $\mathfrak{S}$ *satisfies the following condition* (*).

(*) *For any* (i, ℓ) *and* (j, m) *of* $\mathfrak{S}$, *if* $i \geq j$, *then* $i+\ell+1 \neq j+m$.

Then M *is a Buchsbaum module.*

Proof. Since $(E_1^{p,q})_\ell = \oplus H_{\mathfrak{m}}^q(M)_\ell$, $(E_1^{p+s,q-s+1})_{\ell+s} = \oplus H_{\mathfrak{m}}^{q-s+1}(M)_{\ell+s}$ and $d_s^{p,q}$ is a graded homomorphism of degree s, the assumption gives that $d_s^{p,q}$ is zero for any p, q $(\leq n)$ and s. By Theorem 1.9 (or (1.7)), M is a Buchsbaum module.

Remark 1.11. (1.10) is a generalization of Goto-Watanabe [4] and Schenzel [13, Theorem 3.1] and is "best possible" in the following sense. Unless $\mathfrak{S}$ satisfies (*), we cannot see only by the vanishing of its local cohomologies whether or not M is a Buchsbaum module. In fact, Goto abstractly and systematically constructed such examples in his papers [3, 5]. Moreover, his method gives the construction of quasi-Buchsbaum ring of type r but not of type r+1 by virtue of Evans-Griffith [1].

Now then, let us calculate the length of $\mathrm{Ext}_S^{n+1}(k, M)$ written by $r(M)$. By (1.9), we have the following result.

Corollary 1.12. Suppose that M is a Buchsbaum S-algebra, we have

$$\sum_{j=0}^{n}\binom{N+1}{n+1-j}\dim H_{\mathfrak{m}}^{j}(M)+1\leq r(M)\leq\sum_{j=0}^{n}\binom{N+1}{n+1-j}\dim H_{\mathfrak{m}}^{j}(M)+\mu(K_M),$$

where $K_M=\mathrm{Ext}_S^{N-n}(M, S)$ and $\mu(K_M)$ is the minimal number of generaters of K_M.

§2. Divisors on Segre products

Let k be a field. Let X be an arithmetically Cohen-Macaulay subscheme of $\mathbb{P}_k^N=\mathrm{Proj}\ S$, that is, its affine cone $C(X)=\mathrm{Spec}\ S/J$ is locally Cohen-Macaulay, where $J=\bigoplus_{\ell\in\mathbb{Z}}\Gamma(\mathcal{I}_{X/\mathbb{P}}(\ell))$. Let V be a subscheme of X such that $0<\dim V=n<\dim X$. Let A be the coordinate ring of V. The following proposition shows that the results of Section 1 can be applied to geometric case.

Proposition 2.1. *Under the above conditions, we have*

$$\tau_0^{n+1}\mathbb{R}\Gamma(\bigoplus_{\ell\in\mathbb{Z}}\mathcal{I}_{V/X}(\ell))\simeq\tau^{n+1}\mathbb{R}\Gamma_{\mathfrak{m}}(A)$$

in the derived category $D_h^+(S)$ *of complexes bounded below of* *graded S-modules.*

Example 2.2. Let $X=\mathbb{P}_k^r\times\mathbb{P}_k^s$ be Segre embedding in $\mathbb{P}=\mathbb{P}_k^{rs+r+s}$. Let V be a divisor of X corresponding to
$\mathcal{O}_X(a,b)=p_1^*\mathcal{O}_{\mathbb{P}_k^r}(a)\otimes p_2^*\mathcal{O}_{\mathbb{P}_k^s}(b)$.

(1) V is an arithmetically Cohen-Macaulay subscheme of $\mathbb{P}$ if and only if $a-r\leq b\leq a+s$.

(2) The following things are equivalent.

(a) V is an arithmetically Buchsbaum subscheme of $\mathbb{P}$.

(b) V is an arithmetically quasi-Buchsbaum subscheme of $\mathbb{P}$.

(c) $a-r-1\leq b\leq a+s+1$.

Proof. It is given, for example, by Goto-Watanabe[7], Stückrad-Vogel[18] or Schenzel[13, Proposition 5.1]. We will prove it, however, because our proof indicates the motivation of the next example.

Now let us assume $a\geq b$. First of all, let us find the numbers $1\leq i<r+s$ and $\ell\in\mathbb{Z}$ satisfying $H^i(\mathcal{I}_{V/X}(\ell))=0$. By the way $\mathcal{I}_{V/X}$ is isomorphic to $\mathcal{O}_X(-a,-b)$. By Künneth formula, we have $H^i(\mathcal{I}_{V/X}(\ell))=0$ if and only if $i=r$ and $b\leq\ell\leq a-r-1$. This shows (1) and "(c) (a)" of (2). It is trivial that (a) implies (b). Thus it still remains to prove that if $a-r-1>b$, then V is not an arithmetically quasi-Buchsbaum subscheme of $\mathbb{P}$.

Let us take $\mathbb{P}^r\times\mathbb{P}^s=\mathrm{Proj}\ k[X_0,\ldots,X_r]\times\mathrm{Proj}\ k[Y_0,\ldots,Y_s]$.

Then we can show

$$\cdot X_j:\ H^r(\mathcal{O}_{\mathbb{P}^r}(\ell-a)) \longrightarrow H^r(\mathcal{O}_{\mathbb{P}^r}(\ell+1-a))$$

is surjective and

$$\cdot Y_j:\ H^0(\mathcal{O}_{\mathbb{P}^s}(\ell-b)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^s}(\ell+1-b))$$

is injective. On the other hand we see

$$H^r(\mathcal{I}_{V/X}(\ell)) \simeq H^r(\mathcal{O}_{\mathbb{P}^r}(\ell-a)) \otimes H^0(\mathcal{O}_{\mathbb{P}^s}(\ell-b)) .$$

Hence we have that $\bigoplus_{\ell\in\mathbb{Z}} H^i(\mathcal{I}_{V/X}(\ell))$ is not a k-vector space if $a-r-1>b$. This gives V is not an arithmetically Buchsbaum subscheme of $\mathbb{P}$.

In Example 2.2, we need not distinguish Buchsbaum rings from quasi-Buchsbaum rings. However, taking $X=\mathbb{P}^r_k\times\mathbb{P}^r_k\times\mathbb{P}^r_k$, we can find the examples clarifying the difference between them.

Theorem 2.3. *Let* $X=\mathbb{P}^r_k\times\mathbb{P}^r_k\times\mathbb{P}^r_k$ *be Segre embedding in* $\mathbb{P}=\mathbb{P}_k^{(r+1)^3-1}$. *Let* V *be a divisor of* X *corresponding to* $\mathcal{O}_X(a-r-1, a, a+r+1)$. *Then* V *is an arithmetically quasi-Buchsbaum subscheme of type* r *in* $\mathbb{P}$ *but not an arithmetically quasi-Buchsbaum subscheme of type* r+1.

Proof. For simplicity, we will prove only in case r=1. Let $\langle X, Y\rangle\times\langle Z, W\rangle\times\langle U, T\rangle$ be a coordinate of $X=\mathbb{P}^1_k\times\mathbb{P}^1_k\times\mathbb{P}^1_k$. By Künneth formula, we have

$$H^2(\bigoplus_{\ell\in\mathbb{Z}} \mathscr{I}_{V/X}(\ell)) = k\cdot 1 \otimes k\cdot\frac{1}{ZW} \otimes (k\cdot\frac{1}{U^3T} + k\cdot\frac{1}{U^2T^2} + k\cdot\frac{1}{UT^3})$$

and

$$H^1(\bigoplus_{\ell\in\mathbb{Z}} \mathscr{I}_{V/X}(\ell)) = (k\cdot X^2 + k\cdot XY + k\cdot Y^2) \otimes k\cdot 1 \otimes k\cdot\frac{1}{UT} \quad .$$

Then we see

$$(X\otimes Z\otimes U)\wedge(Y\otimes W\otimes T) \ : \ H^2(\bigoplus_{\ell\in\mathbb{Z}} \mathscr{I}_{V/X}(\ell)) \longrightarrow H^1(\bigoplus_{\ell\in\mathbb{Z}} \mathscr{I}_{V/X}(\ell))$$

is not zero. By Theorem 1.7, V is not arithmetically quasi-Buchsbaum of type 2. Further, it is easy to show V is arithmetically quasi-Buchsbaum. Thus the assertion is proved.

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