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3次元多様体のG－手術

G-surgery of three dimensional manifolds

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1. Introduction

In this paper G will denote the cyclic group of ordrer $\leqq 2$. G-actions on manifolds should be understood to be smooth.

Let X and Y be compact, connected, oriented and 3-dimensional G-manifolds, and let $f : X \longrightarrow Y$ be a degee one G-map. We are sometimes required to make f a $\mathbb{Z}_{(p)}$-homology equivalence, p prime. This paper is concerned with the problem. Our main results are Theorems 3.3 and 3.7. As an application of Theorm 3.7, we can show the following theorem.

Theorem A. <u>There exist one fixed point actions of</u> A_5 <u>on</u> S^6, <u>where</u> A_5 <u>is the alternating group on</u> 5 <u>letters and</u> S^6 <u>is the standard sphere of dimension</u> 6.

We will, however, give the proofs of Theorems 3.7 and A at another opportunity.

It seems interesting that, in the case $|G| = 2$, we get an obstruction group $W_3(R[G], \Gamma(G))$ different from the Wall group $L_3(R[G])$ (it may happen that $W_3(R[G], \Gamma(G))$ is isomorphic to $L_3(R[G])$). The reason for it is that our G-framed normal map $f : X \rightarrow$

Y has the non-empty fixed point sets X^G and Y^G.

The idea of this paper originates from not only Wall's surgery theory [16] but also Dovermann - Petrie's equivariant surgery theory [5] and [6].

In Section 2 we define some Witt groups in which we construct G-surgery obstructions. In Section 3 we define G-normal maps and G-framed normal maps and we state our main results. Sections 4 and 5 are devoted to the construction of the G-surgery obstruction and to the proof of Theorems 3.3.

In the present paper, the orientation of the boundary ∂X of an oriented manifold X is given so that, on the boundary,

$$\text{the outward normal direction} + \text{the orientation of } \partial X = \text{the orientation of } X.$$

2. The Witt groups

Our general reference of this section is [2].

Let Λ be an associative ring with 1 and with an involution $^{-}$ satisfying $\bar{1} = 1$, $\overline{a+b} = \bar{a} + \bar{b}$ and $\overline{ab} = \bar{b}\,\bar{a}$ for $a, b \in \Lambda$. $M_n(\Lambda)$ denotes the set of n×n-matrices whose entries are in Λ. Let Γ be an additive subgroup of Λ such that

(Γ1) $\{a + \bar{a} \mid a \in \Lambda\} \subset \Gamma \subset \{a \in \Lambda \mid a = \bar{a}\}$, and

(Γ2) $a\Gamma\bar{a} \subset \Gamma$ for all $a \in \Lambda$.

Such Γ is called a <u>ring parameter</u> of Λ.

For an element (x_{ij}) of $M_n(\Lambda)$, $(x_{ij})^*$ is defined to be $(\bar{x}_{ji})$. In the following an arbitrary element in $M_{2n}(\Lambda)$ is often written in the form:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{with} \quad A = (a_{ij}),\ B = (b_{ij}),\ C = (c_{ij}) \ \text{ and } \ D = (d_{ij}),$$

where i and j run from 1 to n. Let $SU_n(\Lambda, \Gamma)$ be the group of non-singular matrices in $M_{2n}(\Lambda)$ which satisfy

(1) $$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} D^* & -B^* \\ -C^* & A^* \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \quad \text{and}$$

(2) the diagonal coefficients of BA^* and DC^* lie in Γ.

We denote by $TU_n(\Lambda, \Gamma)$ the subgroup of $SU_n(\Lambda, \Gamma)$ which consists of the elements with $B = 0$. We put

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then σ belongs to $SU_1(\Lambda, \Gamma)$ for any ring parameter Γ of Λ. We have the standard stabilizers $j_{n,n+1} : SU_n(\Lambda, \Gamma) \longrightarrow SU_{n+1}(\Lambda, \Gamma)$ definded by

$$j_{n,n+1}(x) = \begin{pmatrix} A & & B & \\ & 1 & & \\ C & & D & \\ & & & 1 \end{pmatrix}$$

$$\text{for} \quad x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{in} \quad SU_n(\Lambda, \Gamma).$$

With respect to the standard stabilizers, we define $SU(\Lambda, \Gamma) = \varinjlim_n SU_n(\Lambda, \Gamma)$ and $TU(\Lambda, \Gamma) = \varinjlim_n TU_n(\Lambda, \Gamma)$. We denote by $RU(\Lambda, \Gamma)$ the subgroup of $SU(\Lambda, \Gamma)$ which is generated by $TU(\Lambda, \Gamma)$ and σ. It is well-known (see [2, Corollary 3.9]) that $RU(\Lambda, \Gamma)$ includes the commutator subgroup $CU(\Lambda, \Gamma) = [SU(\Lambda, \Gamma), SU(\Lambda, \Gamma)]$ of $SU(\Lambda, \Gamma)$.

Definition 2.1. The _Witt group of dimension three_ $W_3(\Lambda, \Gamma)$ is defined to be the quotient group $SU(\Lambda, \Gamma)/RU(\Lambda, \Gamma)$. In the case where (1) Λ is commutative, (2) $-$ is trivial and (3) $\Gamma = \Lambda$, we use $W_3(\Lambda)$ instead of $W_3(\Lambda, \Gamma)$.

Lemma 2.2. _If_ Λ _is a commutative local ring with the trivial involution, then_ $W_3(\Lambda, \Gamma) = 0$ _for any ring parameter_ Γ.

We omit the proof.

It is important to estimate the commutator subgroup $CU(\Lambda, \Gamma)$ of $SU(\Lambda, \Gamma)$ for applications. We introduce a result of Bak [2].

For an integer k with $1 \leqq k \leqq n$ and for $x \in \Gamma$, we define an element $\varepsilon_k(x)$ of $TU_n(\Lambda, \Gamma)$ by (1) $a_{ii} = 1 = d_{ii}$, $1 \leqq i \leqq n$, (2) $c_{kk} = x$ and (3) all the other entries are 0. If $1 \leqq h, k \leqq n$ and $h \neq k$ and $x \in \Lambda$, then we define $\varepsilon_{hk}(x)$ (resp. $\kappa_{hk}(x)$) $TU_n(\Lambda, \Gamma)$ by (1) $a_{ii} = 1 = d_{ii}$, $1 \leqq i \leqq n$, (2) $c_{hk} = x = \bar{c}_{kh}$ (resp. $-\bar{a}_{kh} = x = d_{hk}$) and (3) all the other entries are 0. $FU(\Lambda, \Gamma)$ is the subgroup of $RU(\Lambda, \Gamma)$ generated by (1) σ, (2) $\varepsilon_k(x)$, (3) $\varepsilon_{hk}(x)$ and (4) $\kappa_{hk}(x)$, where n varies over all positive integers, and h, k and x take all possible values. Then Corollary 3.9 of [2] asserts:

Lemma 2.4. _It holds that_ $CU(\Lambda, \Gamma) \subset FU(\Lambda, \Gamma)$.

Remark 2.5. Hence for any $x \in SU(\Lambda, \Gamma)$ we have $TU(\Lambda, \Gamma)xFU(\Lambda, \Gamma) = xRU(\Lambda, \Gamma)$ as subsets of $SU(\Lambda, \Gamma)$. This is a key point of surgery theory.

We define $\sigma_k \in SU_k(\Lambda, \Gamma)$ by (1) $a_{ii} = 1 = d_{ii}$ if $i \neq k$, (2) $b_{kk} = 1 = -c_{kk}$ and (3) all the other entries are 0. We denote the stabilized element in $SU_n(\Lambda, \Gamma)$, $n \geqq k$, from σ_k again by σ_k. It holds that

(2.6) $\kappa_{hk}(x) = \sigma_k{}^3\varepsilon_{hk}(x)\sigma_k$.

Proposition 2.7. $FU(\Lambda, \Gamma)$ _is generated by_ σ_k , $\varepsilon_k(x)$ _and_ $\varepsilon_{hk}(x)$, _where_ h, k _and_ x _vary over all possible values_.

Let R be a commutaitve ring with 1. For a finite group G, we denote by R[G] the group ring of G with coefficient ring R. R[G] has the involution - satisfying $(ag)^- = ag^{-1}$ for $a \in R$ and $g \in G$.

In the rest of this paper we let G be {1} the trivial group or C_2 the group of order two.

Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ at a prime p. Hereafter R denotes one of $\mathbb{Z}$ and $\mathbb{Z}_{(p)}$.

Remark 2.8. (1) In the case where $1/2 \in R$, the ring parameter Γ of R[G] is unique, that is $\Gamma = R[G]$. In the case we have $W_3(R[G], \Gamma) = W_3(R[G])$. (2) Let $L_3^h(R[G])$ be the Wall group of R-homology equivalence and of dimension $3 + 4n \geqq 7$ (see [1]). Then we have $L_3^h(R[G]) = W_3(R[G], 2R[G])$.

Definition 2.9. We define the ring parameter $\Gamma(G)$ by $\Gamma(G) = 2R[G]$ if $G = \{1\}$ and $\Gamma(G) = \{ 2a + bg \mid a \in R \text{ and } b \in R\}$ if $G = C_2$.

By definition we have $W_3(R[G], \Gamma(G)) = L_3^h(R[G])$ if $G = \{1\}$.

Proposition 2.10. <u>It holds that</u> (1) $W_3(\mathbb{Z}_{(p)}[G]) = 0$ <u>for any prime</u> p <u>and</u> (2) $W_3(\mathbb{Z}_{(2)}[G], \Gamma) = 0$ <u>for any ring parameter</u> Γ.

Proof. This follows from Lemma 2.2.

3. G-normal maps

Let G be $\{1\}$ or C_2. In the present section we give the definition of a G-normal map $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ and a G-framed normal map $(f, b) : (X, \partial X; TX) \longrightarrow (Y, \partial Y; f^*\xi)$.

Let Y be a compact, 1-connected, oriented, 3-dimensional G-manifold with boundary ∂Y (possibly $\partial Y = \phi$).

Remark 3.1. The case where Y is the unit disk $D(V)$ of a real G-module V is important for applications. The reader may restrict the following arguments to the case $\pi_2(Y) = 0$.

If $G = C_2$, then we require that

(N1) Y^G is not empty nor Y, and each elemant of G preserves the orientation of Y.

This implies that each connected component of Y^G has dimension one. Let X be

(N2) a compact, connected, oriented, 3-dimensional G-manifold

and X_1 a G-subcomplex of X with $\dim X_1 \leqq 1$ with respect to some smooth G-triangulation of X.

Definition 3.2. We call a G-map $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ a G-*normal map* if the following (N3) - (N5) are satisfied:

(N3) $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ has degree one.

(N4) $f|\partial X : \partial X \longrightarrow \partial Y$ is a G-homotopy equivalence.

(N5) If $G = C_2$, then $f^G : (X^G, \partial X^G) \longrightarrow (Y^G, \partial Y^G)$ is a homotopy equivalence.

We put $X_s = \phi$ if $G = \{1\}$ and $X_s = X^G$ if $G = C_2$.

If a G-normal map $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ with a G-subcomplex X_1 of X is given, then by an argument similar to [5] and [6] we can give an element $\sigma(f)$ of $W_3(R[G])$ with the property:

Theorem 3.3. *If* $\sigma(f) = 0$ *in* $W_3(R[G])$, *then* $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ *can be converted to a* G-*normal map* $f' : (X', \partial X') \longrightarrow (Y, \partial Y)$ *with the following conditions* (S1) - (S3), *by* G-*surgery relative to the set* $\partial X \cup X_1 \cup X_s$:

(S1) $\partial X' = \partial X$, $X'_1 = X_1$ *and* $X'_s = X_s$.

(S2) $f'|\partial X' \cup X'_1 \cup X'_s = f|\partial X \cup X_1 \cup X_s$.

(S3) $f' : X' \longrightarrow Y$ *is an* R-*homology equivalence*.

Remark 3.4. The correspondence : a G-normal map $f \longmapsto \sigma(f)$, may be multivalued. When we construct $\sigma(f)$, we are required to make various choices. Thus $\sigma(f)$ may depend on the choices for the construction. Theorem 3.3 says that if we have $\sigma(f) = 0$ from one series of the choices, then f can be converted to nice f'. In other words, a G-normal map f corresponds to a subset $\{\sigma(f)\}$ of $W_3(R[G])$; if $\{\sigma(f)\}$ implies 0, then f can be converted to nice f'. Nevertheless, since most of $W_3(R[G])$'s vanish by Proposition 2.10, we are not too nervous about the ambiguity.

Let ξ be a real G-vector bundle over Y of fiber-dimension 3. If $b : TX \oplus \underline{V} \longrightarrow f^*\xi \oplus \underline{V}$, V a real G-module, is a G-vector bundle isomorphism (covering the identity map on X), then we roughly write $b : TX \longrightarrow f^*\xi$ and call it a <u>stable</u> G-<u>vector</u> <u>bundle</u> <u>isomorphism</u>.

Definition 3.5. We call a pair (f, b) of a G-normal map $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ and a stable G-vector bundle isomorphism $b : TX \longrightarrow f^*\xi$, a G-<u>framed</u> <u>normal</u> <u>map</u>.

Remark 3.6. Since we want to do G-surgery relative to the singular set, the normal part of bundle data of the singular set is needless.

If a G-framed normal map $(f, b) : (X, \partial X; TX) \longrightarrow (Y, \partial Y; f^*\xi)$ is given, then we can give an element $\sigma(f, b)$ of $W_3(R[G], \Gamma(G))$ with the property:

Theorem 3.7. If $\sigma(f, b) = 0$ in $W_3(R[G], \Gamma(G))$, then the G-framed normal map $(f, b) : (X, \partial X; TX) \longrightarrow (Y, \partial Y; f^*\xi)$ can be converted to a G-framed normal map $(f', b') : (X', \partial X'; TX') \longrightarrow (Y, \partial Y; f'^*\xi)$ satisfying the conditions (S1) - (S3) in Theorem 3.3, by G-surgery relative to $\partial X \cup X_1 \cup X_s$.

4. Definition of $\sigma(f)$ in $W_3(R[G])$

In this section we show how the algebraic object $\sigma(f) \in W_3(R[G])$ is obtained from a G-normal map $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ with a subcomplex X_1 , $\dim X_1 \leqq 1$, defined in Section 3. We show it only in the case $G = C_2$ and $R = \mathbb{Z}$. The reader will easily analogize it for the case $G = \{1\}$ and $R = \mathbb{Z}$.

Remark 4.1. If we use $\sigma(f, R)$ instead of $\sigma(f) \in W_3(R[G])$ to make the coefficient ring clear (for gereral R), then $\sigma(f, R)$ is the image of $\sigma(f, \mathbb{Z})$ by the natural homomorphism from $W_3(\mathbb{Z}[G])$ to $W_3(R[G])$.

In the following G is C_2 with generator g and the coefficient ring of homology groups is $\mathbb{Z}$. Let $f : (X, \partial X) \longrightarrow (Y, \partial Y)$ be a G-normal map with $X_1 \subset X$.

It holds that $\dim Y^G = 1 = \dim X^G$. Take connected components $Y^G{}_*$ of Y^G and $X^G{}_*$ of X^G such that $f| : (X^G{}_*, \partial X^G{}_*) \longrightarrow (Y^G{}_*, \partial Y^G{}_*)$ has degree one. Fix a point x_* in $\operatorname{Int} X^G{}_*$, and take a G-invariant closed disk $\overline{D}$ with the center x_*. Take a point y_* in $\operatorname{Int} Y^G{}_*$ and a G-invariant closed disk $\overline{D}'$ with the center y_*. Further fix a point y_{**} in $\partial \overline{D}'^G$.

We denote the field of complex numbers by $\mathbb{C}$, the unit circle of $\mathbb{C}$ by S^1 and the closed unit disk of $\mathbb{C}$ by D^2.

Let $h_1, \ldots, h_n : S^1 \longrightarrow X - (\partial X \cup X_1 \cup X^G \cup \overline{D})$ be embeddings which generate $H_1(X)$ and lie in general position. Since the normal bundle of h_i is trivial, we have an orientation preserving embedding $\overline{f}_i : S^1 \times D^2 \longrightarrow X - (\partial X \cup X_1 \cup X^G \cup \overline{D})$ such that $\overline{f}_i(x, 0) = h_i(x)$ for $x \in S^1$. We put $U_i = \operatorname{Im} \overline{f}_i$ and $\overline{U}_i = U_i \cup gU_i$. We suppose that $U_1, \ldots, U_n, gU_1, \ldots, gU_n$ are mutually disjoint. Take a closed tube T_i connecting $\overline{D}$ with U_i in $X - (\partial X \cup X_1 \cup X^G \cup \operatorname{Int} \overline{D} \cup \bigcup_{i=1}^n \operatorname{Int} \overline{U}_i)$ such that $T_1, \ldots, T_n, gT_1, \ldots, gT_n$ are mutually disjoint. We set

$$\overline{T}_i = T_i \cup gT_i,$$

$$\overline{U} = \overline{D} \cup \bigcup_{i=1}^n (\overline{T}_i \cup \overline{U}_i),$$

$$X_0 = X - \operatorname{Int} \overline{U} \quad \text{and}$$

$$Y_0 = Y - \mathrm{Int}\ \overline{D}'.$$

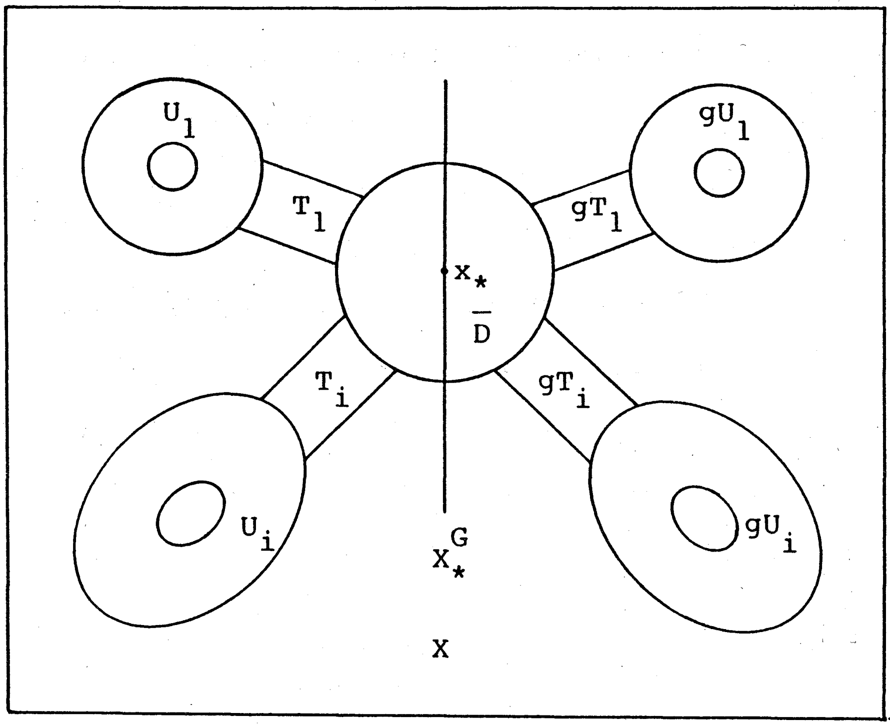


Figure 1.

Deform $f : X \longrightarrow Y$ by a G-homotopy of G-normal map : $(X, \partial X) \longrightarrow (Y, \partial Y)$ so that $f(T_i) = \{y_{**}\} = f(U_i)$, $f(\overline{U}) = \overline{D}'$ and $f(X_0) = Y_0$.

If $h : H_n(A, A') \longrightarrow H_n(B, B')$ is a homomorphism, then we put $K_n(h) = \mathrm{Ker}\ h$. If there is no confusion, we use $K_n(A, A')$ instead of $K_n(h)$.

The restrictions of f give the kernels

$$K_2(\bar{U}, \partial\bar{U}) = \mathrm{Ker}[H_2(\bar{U}, \partial\bar{U}) \longrightarrow H_2(\bar{D}', \partial\bar{D}')],$$

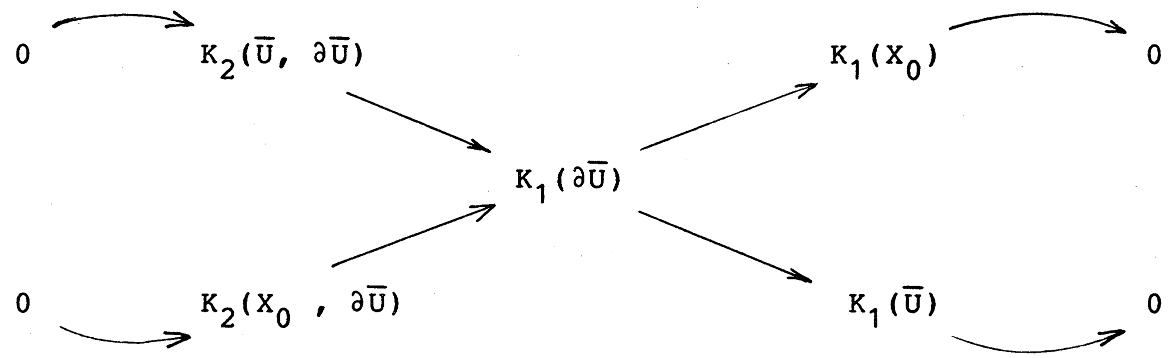
$$K_2(X_0, \partial\bar{U}) = \mathrm{Ker}[H_2(X_0, \partial\bar{U}) \longrightarrow H_2(Y_0, \partial\bar{D}')],$$

$$K_1(\partial\bar{U}) = \mathrm{Ker}[H_1(\partial\bar{U}) \longrightarrow H_1(\partial\bar{D}')],$$

$$K_1(X_0) = \mathrm{Ker}[H_1(X_0) \longrightarrow H_1(Y_0)] \quad \text{and}$$

$$K_1(\bar{U}) = \mathrm{Ker}[H_1(\bar{U}) \longrightarrow H_1(\bar{D}')].$$

These make the exact sequence:



We define e_i, $f_i : S^1 \longrightarrow \partial U_i$ by $e_i(x) = \bar{f}_i(1, x)$ and $f_i(x) = \bar{f}_i(x, 1)$ for $x \in S^1$. We call e_i (resp. f_i) the <u>meridian</u> (resp. <u>longitude</u>) of $\bar{f}_i$. $K_1(\partial\bar{U})$ is isomorphic to $\bigoplus_{i=1}^{n} H_1(\partial\bar{U}_i)$.

$K_1(\partial\overline{U})$ is regarded as a $\mathbb{Z}[G]$-free module with the basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$. $K_2(\overline{U}, \partial\overline{U})$ and $K_2(X_0, \partial\overline{U})$ are treated as submodules of $K_1(\partial\overline{U})$ in the following. Obviously $K_2(\overline{U}, \partial\overline{U})$ has the $\mathbb{Z}[G]$-basis $\{e_1, \ldots, e_n\}$. Since $K_1(\overline{U})$ has the $\mathbb{Z}[G]$-basis $\{h_1, \ldots, h_n\}$, we have the splitting $s : K_1(\overline{U}) \longrightarrow K_1(\partial\overline{U})$ defined by $s(h_i) = f_i$. We regard $K_1(\overline{U})$ as a submodule of $K_1(\partial\overline{U})$ with respect to s. $K_1(X_0)$ and $K_2(X_0, \partial\overline{U})$ are $\mathbb{Z}[G]$-projective, hence $\mathbb{Z}[G]$-free. $K_2(X_0, \partial\overline{U})$ is a direct summand of $K_1(\partial\overline{U})$. Furthermore by the Poincaré duality and the universal coefficient theorem, $K_2(X_0, \partial\overline{U})$ has the half rank of $K_1(\partial\overline{U})$.

Since $K_1(\partial\overline{U}) = H_1(\partial\overline{U})$, we have the intersection form $\lambda_0 : K_1(\partial\overline{U}) \times K_1(\partial\overline{U}) \longrightarrow \mathbb{Z}$, hence we have the (equivariant) intersection form $\lambda : K_1(\partial\overline{U}) \times K_1(\partial\overline{U}) \longrightarrow \mathbb{Z}[G]$ defined by

$$\lambda(x, y) = \lambda_0(x, y) + \lambda_0(x, gy)g \quad \text{for} \quad x, y \in K_1(\partial\overline{U}).$$

Then λ satisfies $\lambda(x, ay) = a\lambda(x, y)$ and $\lambda(x, y) = -\lambda(y, x)$ for $x, y \in K_1(\partial\overline{U})$ and $a \in \mathbb{Z}[G]$. It holds that $\lambda(e_i, e_j) = 0 = \lambda(f_i, f_j)$, $\lambda(e_i, f_j) = \delta_{ij}$ and $\lambda(K_2(X_0, \partial\overline{U}), K_2(X_0, \partial\overline{U})) = 0$. Hence all $K_2(\overline{U}, \partial\overline{U})$, $K_1(\overline{U})$ and $K_2(X_0, \partial\overline{U})$ are hyperbolizers of $(K_1(\partial\overline{U}), \lambda)$.

There are $\mathbb{Z}[G]$-isomorphisms $\alpha : K_1(\partial\overline{U}) \longrightarrow K_1(\partial\overline{U})$ such that $\alpha(K_2(\overline{U}, \partial\overline{U})) = K_2(X_0, \partial\overline{U})$ and α preserve λ. Fix one of them. We write α in the form:

$$\alpha(e_1, \ldots, e_n, f_1, \ldots, f_n) = \Phi(\alpha)\ {}^t(e_1, \ldots, e_n, f_1, \ldots, f_n)$$

by a $2n\times 2n$-matrix $\Phi(\alpha)$. $\Phi(\alpha)$ belongs to $SU_n(\mathbb{Z}[G], \mathbb{Z}[G])$.

Remark 4.2. Another choice of a $\mathbb{Z}[G]$-isomorphism : $K_1(\partial\overline{U}) \longrightarrow K_1(\partial\overline{U})$ above, determines an element in $TU_n(\mathbb{Z}[G], \mathbb{Z}[G])\Phi(\alpha)$.

Definition 4.3. We define $\sigma(f)$ to be the coset $[\Phi(\alpha)]$ in $W_3(\mathbb{Z}[G]) = SU(\mathbb{Z}[G], \mathbb{Z}[G])/RU(\mathbb{Z}[G], \mathbb{Z}[G])$.

Here $\sigma(f)$ may depend not only on f but also on $\overline{f}_1, \ldots, \overline{f}_n$, etc.

5. Proof of Theorem 3.3

We prove Theorem 3.3 only in the case $G = C_2$ and $R = \mathbb{Z}$. The proof of the case $G = C_2$ and $R = \mathbb{Z}_{(p)}$ needs further work. G. A. Anderson generalized Wall's surgery theory to the surgery theory with coefficient ring $\mathbb{Z}_{(p)}$. His technique [1, pp. 69 - 70] works well here, too.

For simplicity we use SU_n, TU_n, RU, ... instead of $SU_n(\mathbb{Z}[G], \mathbb{Z}[G])$, $TU_n(\mathbb{Z}[G], \mathbb{Z}[G])$, $RU(\mathbb{Z}[G], \mathbb{Z}[G])$, ... respectively.

Stabilization. We defined matrices $\sigma_k \in SU_k$ by (1) $a_{ii} = 1 = d_{ii}$ if $i \neq k$, (2) $b_{kk} = 1 = - c_{kk}$ and (3) all the other entries are 0. We denote the stabilized element in SU_n, $n \geqq k$, from σ_k in SU_k again by σ_k. We define Σ_n to be the product $\sigma_1 \sigma_2 \cdots \sigma_n \in SU_n$.

Take another orientation preserving trivial embedding $\bar{f}_{n+1} : S^1 \times D^2 \longrightarrow \text{Int } X_0$ such that $\text{Im } \bar{f}_{n+1}$ lies in a small neighborhood of $\bar{D}$. Consider the effect by adding $\bar{f}_{n+1}$ to $\{\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n\}$ on the matrix $\Phi(\alpha) \in SU_n$. We use $\Psi(f)$ instead of $\Phi(\alpha)$ in order to put stress on the map f. The matrix $\Psi(f) \in SU_n \subset SU_{n+1}$ is replaced by $\Psi(f)\sigma_{n+1}$. Let $f' : (X', \partial X') \longrightarrow (Y, \partial Y)$ be the resulted G-normal map by G-surgery of f along $\bar{f}_{n+1}$. We use the above $\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n$ and the dual of $\bar{f}_{n+1}$ to determine $\Psi(f') \in SU_{n+1}$. Then we have $\Psi(f') = \Psi(f)$ as matrices in SU_{n+1}.

Hence for the stabilized element $x \in SU_{n+k}$ from $\Psi(f) = \Phi(\alpha) \in SU_n$, we can get a G-framed normal map f'' with $\Psi(f'') = x$ by G-surgery of f relative to $\partial X \cup X_1 \cup X^G$.

It is easy to see that if $\Phi(\alpha) = \Psi(f)$ is Σ_n, then f is a homology equivalence. Since FU implies Σ_n, for the proof of Theorem 3.3 it is sufficient to show that

(5.3) if a matrix $x \in \Psi(f)RU$ is arbitrarily given, then we can obtain a G-framed normal map f' with $\Psi(f') = x$ by G-surgery of f relative to $\partial X \cup X_1 \cup X^G$.

By Remaks 2.5 and 4.2, the matrix x in (5.3) may be restricted to the case $x \in \Psi(f)FU$. We recall Proposition 2.7. Each element of FU is a product of matrices of type σ_k, $\varepsilon_k(y)$ and $\varepsilon_{hk}(y)$, where y lies in $\{1, -1, g, -g\}$. It is sufficient to find G-surgery corresponding to the matrix changes : $\Psi(f) \longmapsto \Psi(f)z$, where z are σ, $\varepsilon_k(y)$ and $\varepsilon_{hk}(y)$.

It is well known that G-surgery along $\bar{f}_k$ corresponds to the matrix change : $\Psi(f) \longmapsto \Psi(f)\sigma_k$.

Assertion 5.4. <u>By some replacement of $\bar{f}_k$, $\Phi(\alpha)$ is converted to $\Phi(\alpha)\varepsilon_k(\pm 1)$.</u>

Proof. We prove it in the case $\varepsilon_k(\pm 1) = \varepsilon_1(1)$.

Let $h_1' = h_1$ and define $\bar{f}_1' : S^1 \times D^2 \longrightarrow X$ by $\bar{f}_1'(x, y) = \bar{f}_1(x, x^{-1}y)$ for $x \in S^1$ and $y \in D^2$. Then the corresponding f_1' and e_1' are given by $f_1'(x) = \bar{f}_1'(x, 1) = \bar{f}_1(x, x^{-1})$ and $e_1'(x) = \bar{f}_1'(1, x) = \bar{f}_1(1, x) = e_1(x)$ for $x \in S^1$. Hence we have $e_1 = e_1'$ and $f_1 = e_1' + f_1'$ in $K_1(\partial\bar{U})$.

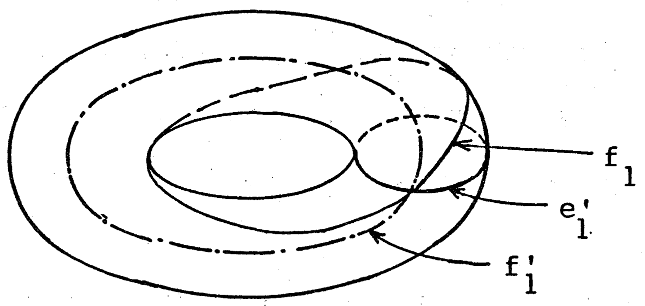


Figure 2.

Define a $\mathbb{Z}[G]$-isomorphism $\alpha' : K_1(\partial\bar{U}) \longrightarrow K_1(\partial\bar{U})$ by

$$\alpha'(e_1', e_2, \ldots, e_n, f_1', f_2, \ldots, f_n)$$

$$= \Phi(\alpha)\varepsilon_1(1)\ {}^t(e_1', e_2, \ldots, e_n, f_1', f_2, \ldots, f_n).$$

Then $\alpha'(K_2(\bar{U}, \partial\bar{U})) = K_2(X_0, \partial\bar{U})$ and α' preserves λ. This completes the proof.

Assertion 5.5. <u>By some replacement of</u> $\bar{f}_k$, $\Phi(\alpha)$ <u>is converted to</u> $\Phi(\alpha)\varepsilon_k(\pm g)$.

Proof. We prove it in the case $\varepsilon_k(\pm g) = \varepsilon_1(g)$.

Take a thin G-invariant closed tube $\bar{T}$ connecting T_1 with gT_1 along $\partial\bar{D}$ in X_0. Without loss of generality, we may assume $f(\bar{T}) = \{y_{**}\}$.

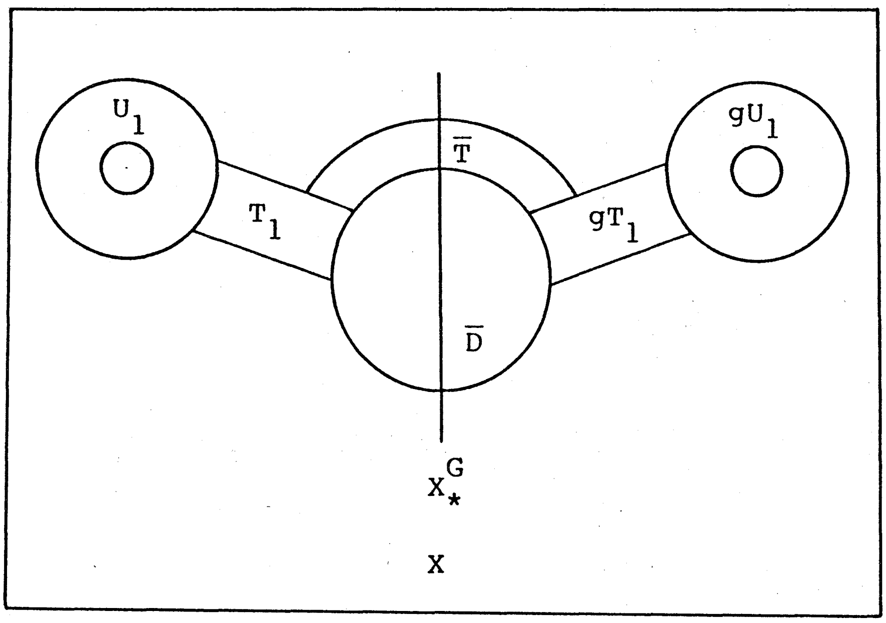


Figure 3.

We put $W = U_1 \cup T_1 \cup \bar{T} \cup gT_1 \cup gU_1$ $(\subset X)$. Denote by W'', $\bar{f}_1''$, e_1'' and f_1'' the copies of W, $\bar{f}_1$, e_1 and f_1 respectively. There is a G-diffeomorphism $\beta : W \longrightarrow W''$ satisfying

$\beta \circ e_1 = e_1''$ as maps and $\beta \circ f_1 = ge_1'' + f_1''$ in $H_1(\partial W'')$. We assume this here and continue the proof. The way to obtain such β will be shown after the completion of the left part of proof. Define $h_1' : S^1 \longrightarrow X$ and $\bar{f}_1' : S^1 \times D^2 \longrightarrow X$ by $h_1' = \beta^{-1} \circ h_1''$ and $\bar{f}_1' = \beta^{-1} \circ \bar{f}_1''$ and put $U_1' = \mathrm{Im}\, \bar{f}_1'$. We take a closed tube T_1' connecting $\bar{D}$ with U_1' along $\partial T_1 - \bar{T}$ in T_1, and we set

$$\bar{U}_1' = U_1' \cup gU_1' ,$$

$$\bar{T}_1' = T_1' \cup gT_1' ,$$

$$\bar{U}' = \bar{D} \cup (\bar{T}_1' \cup \bar{U}_1') \cup \bigcup_{i=2}^{n} (\bar{T}_i \cup \bar{U}_i) \quad \text{and}$$

$$X_0' = X - \mathrm{Int}\, \bar{U}.$$

We put $e_1' = \beta^{-1} \circ e_1''$ and $f_1' = \beta^{-1} \circ f_1''$. $K_1(\partial\bar{U}')$ is regarded as a $\mathbb{Z}[G]$-free module with basis $\{e_1', e_2, \ldots, e_n, f_1', f_2, \ldots, f_n\}$. By the construction there is a $\mathbb{Z}[G]$-isomorphism $\gamma : K_1(\partial\bar{U}') \longrightarrow K_1(\partial\bar{U}')$ such that (1) $\gamma(K_2(\bar{U}', \partial\bar{U}')) = K_2(X_0', \partial\bar{U}')$, (2) γ preserves the (equivariant) intersection form λ' on $K_1(\partial\bar{U}')$ and (3) γ has the matrix form $\Phi(\alpha)\varepsilon_1(g)$, that is,

$$\gamma(e_1', e_2, \ldots, e_n, f_1', f_2, \ldots, f_n)$$

$$= \Phi(\alpha)\varepsilon_1(g)\ {}^t(e_1', e_2, \ldots, e_n, f_1', f_2, \ldots, f_n).$$

This completes the proof under the assumption of existence of β.

Now we show the way to obtain the G-diffeomorphism $\beta : W \longrightarrow W$ ($= W''$).

We regard W as the union of 3-dimensional disks $H(1)$, $H(0)$ and $H(g)$ with the picture:

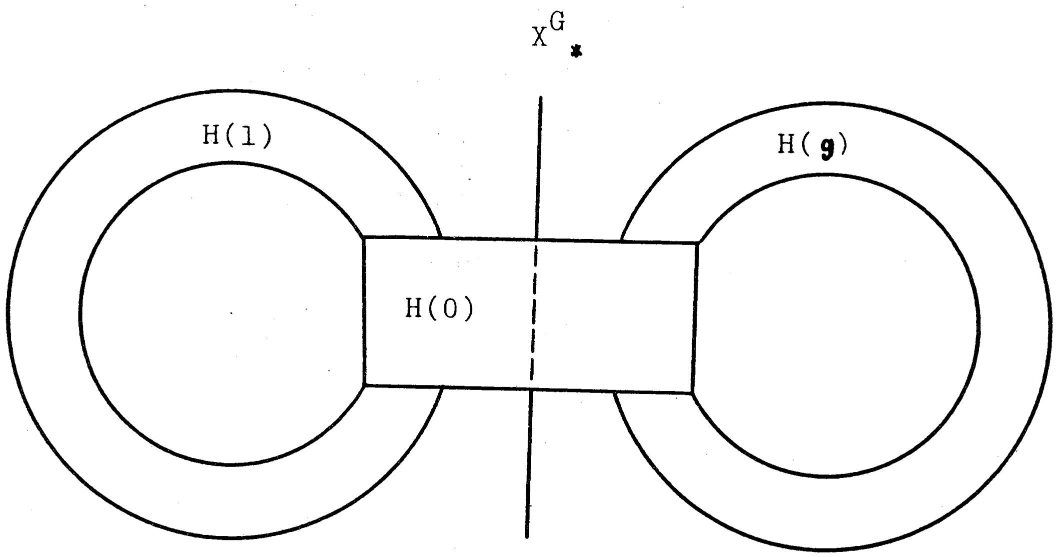


Figure 4.

Here the action of g on W is regarded as the rotation with angle π around the axis X^G_*. We assume that $H(g) = gH(1)$ and $H(0)$ is G-invariant. We identify $H(1)$ with $I \times D^2$, $I = [0, 1]$, and

$H(0)$ with $D^2 \times I$, where the action of g on $H(0)$ is the rotation on $D^2 \times I$ with angle π around $\{0\} \times I$. We suppose that $\{0\} \times D^2$ of $H(1)$ lies in $D^2 \times \{1\}$ of $H(0)$, and $\{1\} \times D^2$ of $H(1)$ lies in $D^2 \times \{0\}$ of $H(0)$. Let $\beta_1 : W \longrightarrow W$ be the G-diffeomorphism given by twisting $H(0)$ once around $X^G{}_*$. That is, $\beta_1(x, t) = (x \exp(2\pi t \sqrt{-1}), t)$ for $(x, t) \in D^2 \times I = H(0)$ and $\beta_1(y) = y$ for $y \in H(1) \cup H(g)$. Then f_1 is mapped by β_1 to $f_1 + e_1 + ge_1$ (caution: not $f_1 + ge_1$) as elements of the homology group $H_1(\partial W)$. Let $\beta_2 : W \longrightarrow W$ be the G-diffeomorphism given by twisting $H(1)$ and $H(g)$ equivariantly once in the opposite direction around $I \times \{0\}$. That is $\beta_2(t, x) = (t, x \exp(-2\pi t \sqrt{-1}))$ for $(t, x) \in I \times D^2 = H(1)$, $\beta_2(y) = y$ for $y \in H(0)$ and $\beta_2(gz) = g\beta_2(z)$ for $gz \in H(g)$. Then f_1 is mapped by β_2 to $f_1 - e_1$ as elements of $H_1(\partial W)$. The required map β is the composition $\beta_1 \circ \beta_2$ of β_1 and β_2.

Remark 5.6. It will turn out to be a key to our G-surgery of the framed case, that $\bar{f}_1$ and $\bar{f}_1{}' : S^1 \times D^2 \longrightarrow W$ are isotopic to each other.

Assertion 5.7. <u>By some replacement of</u> $\bar{f}_i$, $\Phi(\alpha)$ <u>is converted to</u> $\Phi(\alpha)\varepsilon_{ij}(y)$ <u>for arbitrarily given</u> $y \in \{1, -1, g, -g\}$ <u>and</u> $i \neq j$.

Proof. The reader may have the strategy for the proof on the analogy of ordinaly surgery theory as follows. Fix a G-invariant

Riemannian metric on X. Consider the effect of varying h_i by regular homotopies. Here $\bar{f}_i$ is also varied by parallel translation along the regular homotopies. We should, of course, do it by using the covering : $X - X^G \longrightarrow (X - X^G)/G$. As in ordinary surgery theory, it seems possible to convert the matrix $\Phi(\alpha)$ to $\Phi(\alpha)\varepsilon_{ij}(y)$ by some choice of an element in the regular homotopy class of h_i .

There is, however, a more elementary way of proof. Since the proof is quite similar to the proof of Assertion 5.5, we give only a rough sketch.

We suppose that $\varepsilon_{ij}(y) = \varepsilon_{12}(1)$.

Take a connecting tube T from T_1 to T_2 which does not meet with gT. We have two disjoint handle bodies of genus two:

$$H(1) = U_1 \cup T_1 \cup T \cup T_2 \cup U_2 \quad \text{and} \quad H(g) = gH(1).$$

If we forget the G-action of W in the proof of Assertion 5.5, then W is homeomorphic to $H(1)$ and $H(g)$. Thus we consider the effect of twisting $H(1)$ (simultaneously $H(g)$) as W was done by β. By the twisting, $\bar{f}_2$ is also replaced by a new emmbedding $\bar{f}_2'$. This $\bar{f}_2'$ is isotopic to the original $\bar{f}_2$. We can suppose $\bar{f}_2' = \bar{f}_2$. By the twisting $H(1)$ and $H(g)$, we can convert the matrix $\Phi(\alpha)$ to $\Phi(\alpha)\varepsilon_{12}(1)$.

The other cases are quite similarly shown.

We complete the proof of Theorem 3.3.

References

[1] Anderson, G. A.: Surgery with coefficients. Lecture Notes in Math. 591. Berlin-Heidelberg-New York: Springer 1977

[2] Bak, A.: K-theory of forms. Princeton: Princeton University Press 1981

[3] Bak, A. - Sharlou, W.: Grothendieck and Witt group of orders and finite groups. Inv. Math. 23, 207-240 (1974)

[4] Dieck, tom T.: Transformation groups and representation theory. Lecture Notes in Math. 766. Berlin-Heidelberg-New York: Springer 1979

[5] Dovermann, K. H. - Petrie, T.: G surgery II. Memoirs of Amer. Math. Soc. vol. 37 no. 260 (1982)

[6] Dovermann, K. H. - Petrie, T.: An induction theorem for equivariant surgery (G-surgery III). Amer. J. Math. 105, 1369-1403 (1983)

[7] Dress, A.: Induction and structure theorems for orthogonal representations of finite groups. Ann. of Math. 102, 291-325 (1975)

[8] Laitinen, E. - Traczyk, P.: Pseudofree representations and 2-pseudofree actions on spheres. Proc. Amer. Math. Soc. 97, 151-157 (1986)

[9] Petrie, T.: One fixed point actions on spheres, I. Adv. in Math. 46, 3-14 (1982)

[10] Petrie, T.: One fixed point actions on spheres, II. Adv. in Math. 46, 15-70 (1982)

[11] Reiner, I. - Ullom, S.: Remarks on class groups of integral group rings. Symposia Mathematica vol. XIII, 501-516. London-New York-San Francisco: Academic Press 1974

[12] Rolfsen, D.: Knots and links. Math. Lecture Series 7. Berkeley: Publish or Perish Inc. 1976

[13] Steenrod, N.: The topology of fiber bundles. Princeton: Princeton Univ. Press 1951

[14] Stein, E.: Surgery on products with finite fundamental group. Topology 16, 473-493 (1977)

[15] Thurston, W. P.: A norm for the homology of 3-manifolds. Memoirs of Amer. Math. Soc. vol. 59 no. 339 (1986)

[16] Wall, C. T. C.: Surgery on compact manifolds. London-New York-San Francisco: Academic Press 1970

[17] Wall, C. T. C.: On the classification of Hermitian forms. III Complete semilocal rings. Invent. Math. 19, 59-71 (1973)

[18] Wall, C. T. C.: Foundations of algebraic L-theory, in Algebraic K-theory III. Lecture Notes in Math. 343, 266-300. Berlin-Heidelberg-New York: Springer 1973