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Reflection Groups, Combinatorics and Multi-Derivations

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0. Introduction.

Our object to study is a finite subset of non-zero elements of a vector space. Although it might look a too simple and too naive object, this object appears in various mathematics and seems interesting enough by itself.

First this object itself is a matroid which is extensively studied as a main theme in combinatorics. It is also studied under the different name, but the equivalent concept, as a geometric lattice[5][13][70][71].

Next if one considers the dual object in the dual vector space, one gets a finite family of hyperplanes, often called an arrangement of hyperplanes. When the coefficient field is a topological field, one can ask topological questions concerning the union of these hyperplanes or the complement of them. For example the homology and the homotopy groups of the complement space are studied in [4][8][48][53].

Thirdly assume that the vector space has a metric. Consider a group generated by all orthogonal reflections through a hyperplane of the arrangement in the dual vector space. (In some sense this is a group theoretic interpretation of our object.) In general this group is not finite. In case that it is finite and the coefficient field is the real number field, the group is called a Coxeter group, which plays an important role in the theory of Lie groups. If the coefficient field is the complex number field, it is called unitary reflection groups, which are classified in [61] for finite groups.

An algebro-geometric study of our object is the study of the product of all the elements in the symmetric algebra of the vector space. It can also be interpreted as a defining equation for the arrangement in the dual vector space. It is thus the study of a product of linear forms or of a divisor consisting of hyperplanes.

This note is a general survey on these four aspects of the study of a finite set of non-zero elements of a vector space from a specific standpoint: we study a certain class of derivations of the symmetric algebra of the vector space and show that it relates the four (combinatorial, topological, group theoretic and algebro-geometric) aspects. Our standpoint is the theory of logarithmic multi-derivations or multi-derivations along the divisor consisting of the hypereplanes of the arrangement. We will explain how the theory of logarithmic multi-derivations connect the different aspects of our object by using examples. One of the central examples is the polynomial called the characteristic polynomial studied in the matroid theory. It permits other interpretations [66][48][82][68] which we will explain.

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