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Runge-Kutta 系 7 段 7 次極限公式について

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1. はじめに

7段陽的Runge-Kutta 公式で達成し得る次数は高々 6 次の精度までであるが5段と6段の場合と同様に，分点を近づけた極限で7次公式が得られる．

2. Runge-Kutta 系公式

常微分方程式の初期値問題

(2-1) $$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0 \quad (y, f : \text{vector})$$

の7段陽的Runge-Kutta 公式：

$$y_{n+1} = y_n + h\sum_{i=1}^{7} \mu_i f_i$$

(2-2) $$\left| \begin{array}{l} f_1 = f(t_n, y_n) \\ f_i = f(t_n + \alpha_i h, \ y_n + h\sum_{j=1}^{i-1} \beta_{ij} f_j), \ (i = 2, \cdots, 7) \end{array} \right.$$

2.1　7段6次公式の条件方程式

(2.1-0) $\sum_{j=1}^{i-1} \beta_{ij} = \alpha_i$

(2.1-1) $\sum_{i=1}^{7} \mu_i = 1$

(2.1-2) $\sum_{i=2}^{7} \mu_i \alpha_i = 1/2$

(2.1-3)
$$\left|\begin{array}{l} \sum_{i=2}^{7} \mu_i \alpha_i^2 = 1/3 \\ \sum_{i=3}^{7} \mu_i c_i = 1/6 \end{array}\right.$$

(2.1-4)
$$\left|\begin{array}{l} \sum_{i=2}^{7} \mu_i \alpha_i^3 = 1/4 \\ \sum_{i=3}^{7} \mu_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 = 1/12 \\ \sum_{i=3}^{7} \mu_i \alpha_i c_i = 1/8 \\ \sum_{i=3}^{6} T_i c_i = 1/24 \end{array}\right.$$

(2.1-5)
$$\left|\begin{array}{l} \sum_{i=2}^{7} \mu_i \alpha_i^4 = 1/5 \\ \sum_{i=3}^{7} \mu_i \alpha_i^2 c_i = 1/10 \\ \sum_{i=3}^{7} \mu_i \alpha_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 = 1/15 \\ \sum_{i=3}^{6} T_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 = 1/60 \\ \sum_{i=3}^{7} \mu_i c_i^2 = 1/20 \\ \sum_{i=3}^{6} T_i \alpha_i c_i = 1/40 \\ \sum_{i=3}^{5} c_i \sum_{j=i+1}^{6} \mu_j \alpha_j \beta_{ji} = 1/30 \\ \sum_{i=3}^{7} \mu_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^3 = 1/20 \\ \sum_{i=3}^{5} c_i \sum_{j=i+1}^{6} T_j \beta_{ji} = 1/120 \end{array}\right.$$

$$
(2.1\text{-}6)\quad \left|\begin{array}{ll}
\sum_{i=2}^{7} \mu_i \alpha_i^5 = 1/6 \\
\sum_{i=3}^{7} \mu_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^4 = 1/30 \\
\sum_{i=3}^{6} T_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^3 = 1/120 \\
\sum_{i=3}^{5} \sigma_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 = 1/360 \\
\sum_{i=3}^{4} c_i = 1/720 \\
\sum_{i=3}^{7} \mu_i \alpha_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^3 = 1/24 \\
\sum_{i=3}^{6} T_i \alpha_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 = 1/90 \\
\sum_{i=3}^{6} \left(\sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2\right) \sum_{k=i+1}^{7} \mu_k \alpha_k \beta_{ki} = 1/72 \\
\sum_{i=3}^{5} \sigma_i \alpha_i c_i = 1/240 \\
\sum_{i=3}^{5} c_i \sum_{j=i+1}^{6} T_j \alpha_j \beta_{ji} = 1/180 \\
\sum_{i=3}^{5} c_i \sum_{j=i+1}^{6} \beta_{ji} \sum_{k=j+1}^{7} \mu_k \alpha_k \beta_{kj} = 1/144 \\
\sum_{i=3}^{7} \mu_i \alpha_i^3 c_i = 1/12 \\
\sum_{i=3}^{6} T_i \alpha_i^2 c_i = 1/60 \\
\sum_{i=3}^{6} c_i \sum_{j=i+1}^{7} \mu_j \alpha_j^2 \beta_{ji} = 1/36 \\
\sum_{i=3}^{7} \mu_i \alpha_i^2 \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 = 1/18 \\
\sum_{i=3}^{7} \mu_i c_i \sum_{j=2}^{i-1} \beta_{ij} \alpha_j^2 = 1/36 \\
\sum_{i=3}^{6} T_i c_i^2 = 1/120 \\
\sum_{i=3}^{6} c_i \sum_{j=i+1}^{7} \mu_j c_j \beta_{ji} = 1/72 \\
\sum_{i=3}^{7} \mu_i \alpha_i c_i^2 = 1/24 \\
\sum_{i=3}^{6} \alpha_i c_i \sum_{j=i+1}^{7} \mu_j \alpha_j \beta_{ji} = 1/48
\end{array}\right.
$$

ただし

$$T_i = \sum_{j=i+1}^{7} \mu_j \beta_{ji}\,, \qquad (i=2,3,\cdots,6)$$

$$c_i = \sum_{j=2}^{i-1} \beta_{ij} \alpha_j\,, \qquad (i=3,4,\cdots,7)$$

$$\sigma_i = \sum_{j=i+1}^{6} T_j \beta_{ji}\,, \qquad (i=2,3,4,5)$$

7段6次公式の場合分け

(I) $\alpha_7 = 1$

(II) $\alpha_7 \neq 1$, $p\alpha_7(\alpha_2-\alpha_7)+q\alpha_7^2(\alpha_2-\alpha_7)+r\alpha_7^3(\alpha_2-\alpha_7)+sc_7+t\alpha_7 c_7 = 0$

$\alpha_7 = 1$ の場合の条件式：

(2.1-7) $\sum_{i=1}^{7}\mu_i = 1$

(2.1-8)
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ 0 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \alpha_5^2 & \alpha_6^2 \\ 0 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 & \alpha_5^3 & \alpha_6^3 \\ 0 & \alpha_2^4 & \alpha_3^4 & \alpha_4^4 & \alpha_5^4 & \alpha_6^4 \end{bmatrix} \begin{bmatrix} \mu_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/6 \\ 1/12 \\ 1/20 \\ 1/30 \end{bmatrix}$$

(2.1-9)
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_3^2 & \alpha_4^2 & \alpha_5^2 & \alpha_6^2 \\ c_3 & c_4 & c_5 & c_6 \end{bmatrix} \begin{bmatrix} c_3T_3 \\ c_4T_4 \\ c_5T_5 \\ c_6T_6 \end{bmatrix} = \begin{bmatrix} 1/24 \\ 1/40 \\ 1/60 \\ 1/120 \end{bmatrix}$$

(2.1-10) $\mu_3 c_3^2 + \mu_4 c_4^2 + \mu_5 c_5^2 + \mu_6 c_6^2 + \mu_7 c_7^2 = 1/20$

(2.1-11)
$$\begin{bmatrix} \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \alpha_5^2 \\ \alpha_2^3 & \alpha_3^3 & \alpha_4^3 & \alpha_5^3 \\ 0 & c_3 & c_4 & c_5 \\ 0 & \alpha_3 c_3 & \alpha_4 c_4 & \alpha_5 c_5 \end{bmatrix} \begin{bmatrix} \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} = \begin{bmatrix} 1/60 \\ 1/120 \\ 1/120 \\ 1/240 \end{bmatrix}$$

$$(2.1\text{-}12)\quad \begin{bmatrix} \alpha_2^2 & \alpha_3^2 & \alpha_4^2 \\ 0 & c_3 & c_4 \end{bmatrix} \begin{bmatrix} \sigma_3\beta_{32}+\sigma_4\beta_{42}+\sigma_5\beta_{52} \\ \sigma_4\beta_{43}+\sigma_5\beta_{53} \\ \sigma_5\beta_{54} \end{bmatrix} = \begin{bmatrix} 1/360 \\ 1/720 \end{bmatrix}$$

$$(2.1\text{-}13)\quad \begin{bmatrix} \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \alpha_5^2 \\ 0 & c_3 & c_4 & c_5 \end{bmatrix} \begin{bmatrix} \tau_3\alpha_3\beta_{32}+\tau_4\alpha_4\beta_{42}+\tau_5\alpha_5\beta_{52}+\tau_6\alpha_6\beta_{62} \\ \tau_4\alpha_4\beta_{43}+\tau_5\alpha_5\beta_{53}+\tau_6\alpha_6\beta_{63} \\ \tau_5\alpha_5\beta_{54}+\tau_6\alpha_6\beta_{64} \\ \tau_6\alpha_6\beta_{65} \end{bmatrix} = \begin{bmatrix} 1/90 \\ 1/180 \end{bmatrix}$$

$$(2.1\text{-}14)\quad \begin{bmatrix} \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \alpha_5^2 & \alpha_6^2 \\ 0 & c_3 & c_4 & c_5 & c_6 \end{bmatrix} \begin{bmatrix} \mu_3c_3\beta_{32}+\mu_4c_4\beta_{42}+\mu_5c_5\beta_{52}+\mu_6c_6\beta_{62}+\mu_7c_7\beta_{72} \\ \mu_4c_4\beta_{43}+\mu_5c_5\beta_{53}+\mu_6c_6\beta_{63}+\mu_7c_7\beta_{73} \\ \mu_5c_5\beta_{54}+\mu_6c_6\beta_{64}+\mu_7c_7\beta_{74} \\ \mu_6c_6\beta_{65}+\mu_7c_7\beta_{75} \\ \mu_7c_7\beta_{76} \end{bmatrix} = \begin{bmatrix} 1/36 \\ 1/72 \end{bmatrix}$$

$$(2.1\text{-}15)\quad \sum_{j=1}^{i-1}\beta_{ij} = \alpha_i\,, \quad (i=3,4,\cdots,7)$$

$$(2.1\text{-}16)\quad \mu_i(1-\alpha_i) = \tau_i\,, \quad (i=2,3,\cdots,6)$$

2.2 誤差の主要項

$$(2.2\text{-}1)\quad E = y_{n+1} - y(t_n+h) = h^7\sum_{j,k}\delta_{7,j}^{(k)}D_{7,j}^{(k)} + h^8\sum_{j,k}\delta_{8,j}^{(k)}D_{8,j}^{(k)} + O(h^9)$$

$$\left(D_{7,j}^{(k)},\ D_{8,j}^{(k)} : \text{derivatives}\right)$$

$$\delta_{7,1} = \frac{1}{6!}\left[\sum_{i=2}^{7} \mu_i \alpha_i^6 - \frac{1}{7}\right]$$

$$= \frac{-1}{6!}\Big[T_6 \alpha_6(\alpha_6-\alpha_2)(\alpha_6-\alpha_3)(\alpha_6-\alpha_4)(\alpha_6-\alpha_5)$$

$$-\frac{1}{420}\left(70 q_4(2,3,4,5) - 35 q_3(2,3,4,5) + 21 q_2(2,3,4,5) - 14 q_1(2,3,4,5) + 10\right)\Big] = -\frac{1}{6}\delta_{7,2}$$

$$\delta_{7,3} = \frac{1}{24}\left[\sum_{i=4}^{7} \mu_i \sum_{j=3}^{i-1} \beta_{ij} \sum_{k=2}^{j-1} \beta_{jk}\alpha_k^4 - \frac{1}{210}\right]$$

$$= \frac{1}{24}\left[\sigma_5 \alpha_5(\alpha_5-\alpha_2)(\alpha_5-\alpha_3)(\alpha_5-\alpha_4) - \frac{1}{840}\left(-35\alpha_2\alpha_3\alpha_4 + 14(\alpha_2\alpha_3+\alpha_3\alpha_4+\alpha_4\alpha_2) - 7(\alpha_2+\alpha_3+\alpha_4)+4\right)\right] = -\delta_{7,7}$$

$$\delta_{7,4} = \frac{1}{6}\left[\sum_{i=5}^{7} \mu_i \sum_{j=4}^{i-1} \beta_{ij} \sum_{k=3}^{j-1} \beta_{jk} \sum_{l=2}^{k-1} \beta_{kl}\alpha_l^3 - \frac{1}{840}\right]$$

$$= \frac{1}{6}\left[\sigma_5 \beta_{54}\alpha_4(\alpha_4-\alpha_2)(\alpha_4-\alpha_3) - \frac{1}{2520}\left(21\alpha_2\alpha_3 - 7(\alpha_2+\alpha_3)+3\right)\right] = -\delta_{7,8}^{(2)}$$

$$\delta_{7,5} = \frac{1}{2}\left[\sum_{i=6}^{7} \mu_i \sum_{j=5}^{i-1} \beta_{ij} \sum_{k=4}^{j-1} \beta_{jk} \sum_{l=3}^{k-1} \beta_{kl} \sum_{m=2}^{l-1} \beta_{lm}\alpha_m^2 - \frac{1}{2520}\right]$$

$$= \frac{1}{2}\left[\sigma_5 \beta_{54}\beta_{43}\alpha_3(\alpha_3-\alpha_2) - \frac{1}{5040}(-7\alpha_2+2)\right] = -\delta_{7,9}^{(3)}$$

$$\delta_{7,6} = \mu_7\beta_{76}\beta_{65}\beta_{54}\beta_{43}\beta_{32}\alpha_2 - \frac{1}{5040} = \sigma_5\beta_{54}\beta_{43}c_3 - \frac{1}{5040} = -\delta_{7,10}^{(4)}$$

$$\delta_{7,8}^{(1)} = \frac{1}{6}\left[\sum_{i=4}^{7} \mu_i \sum_{j=3}^{i-1} \beta_{ij}\alpha_j \sum_{k=2}^{j-1} \beta_{jk}\alpha_k^3 - \frac{1}{168}\right]$$

$$= \frac{1}{6}\Big[T_5 \beta_{54}\alpha_4(\alpha_4-\alpha_2)(\alpha_4-\alpha_3)(\alpha_5-\alpha_6)$$

$$-\frac{1}{2520}\left(-105\alpha_2\alpha_3\alpha_6 + 63\alpha_2\alpha_3 + 42(\alpha_2\alpha_6+\alpha_3\alpha_6) - 21\alpha_6 - 28(\alpha_2+\alpha_3)+15\right)\Big] = -2\,\delta_{7,16}$$

$$\delta_{7,9}^{(1)} = \frac{1}{2}\left[\sum_{i=5}^{7} \mu_i \sum_{j=4}^{i-1} \beta_{ij} \sum_{k=3}^{j-1} \beta_{jk}\alpha_k \sum_{l=2}^{k-1} \beta_{kl}\alpha_l^2 - \frac{1}{630}\right]$$

$$= \frac{-1}{2}\left[\sigma_4\beta_{43}\alpha_3(\alpha_3-\alpha_2)(\alpha_4-\alpha_5) - \frac{1}{5040}\left(42\alpha_2\alpha_5 - 14\alpha_5 - 21\alpha_2 + 8\right)\right] = -\delta_{7,21}$$

$$\delta_{7,9}^{(2)} = \frac{1}{2}\left[\sum_{i=5}^{7}\mu_i\sum_{j=4}^{i-1}\beta_{ij}\alpha_j\sum_{k=3}^{j-1}\beta_{jk}\sum_{l=2}^{k-1}\beta_{kl}\alpha_l^2 - \frac{1}{504}\right]$$

$$= \frac{1}{2}\left[T_5\beta_{54}\beta_{43}\alpha_3(\alpha_3-\alpha_2)(\alpha_5-\alpha_6) - \frac{1}{2520}(21\alpha_2\alpha_6 - 7\alpha_6 - 14\alpha_2 + 5)\right] = -2\,\delta_{7,15}^{(2)}$$

$$\delta_{7,10}^{(1)} = \sum_{i=6}^{7}\mu_i\sum_{j=5}^{i-1}\beta_{ij}\sum_{k=4}^{j-1}\beta_{jk}\sum_{l=3}^{k-1}\beta_{kl}\alpha_l\sum_{m=2}^{l-1}\beta_{lm}\alpha_m - \frac{1}{1680}$$

$$= \sigma_5\beta_{54}c_4(\alpha_4-\alpha_3) - \frac{1}{5040}(-7\alpha_3+3) = -\delta_{7,22}^{(3)}$$

$$\delta_{7,10}^{(2)} = \sum_{i=6}^{7}\mu_i\sum_{j=5}^{i-1}\beta_{ij}\sum_{k=4}^{j-1}\beta_{jk}\alpha_k\sum_{l=3}^{k-1}\beta_{kl}\sum_{m=2}^{l-1}\beta_{lm}\alpha_m - \frac{1}{1260}$$

$$= \sigma_4\beta_{43}c_3(\alpha_4-\alpha_5) - \frac{1}{5040}(-7\alpha_5+4) = -\delta_{7,22}^{(2)}$$

$$\delta_{7,10}^{(3)} = \sum_{i=6}^{7}\mu_i\sum_{j=5}^{i-1}\beta_{ij}\alpha_j\sum_{k=4}^{j-1}\beta_{jk}\sum_{l=3}^{k-1}\beta_{kl}\sum_{m=2}^{l-1}\beta_{lm}\alpha_m - \frac{1}{1008}$$

$$= T_5\beta_{54}\beta_{43}c_3(\alpha_5-\alpha_6) - \frac{1}{5040}(-7\alpha_6+5) = -2\,\delta_{7,13}^{(3)}$$

$$\delta_{7,11} = \frac{1}{24}\left[\sum_{i=3}^{7}\mu_i\alpha_i^4\sum_{j=2}^{i-1}\beta_{ij}\alpha_j - \frac{1}{14}\right]$$

$$= \frac{-1}{24}\left[T_6c_6(\alpha_6-\alpha_3)(\alpha_6-\alpha_4)(\alpha_6-\alpha_5) - \frac{1}{840}\left(-35\alpha_3\alpha_4\alpha_5 + 21(\alpha_3\alpha_4+\alpha_4\alpha_5+\alpha_5\alpha_3) - 14(\alpha_3+\alpha_4+\alpha_5) + 10\right)\right]$$

$$= -\frac{1}{4}\delta_{7,12}^{(1)}$$

$$\delta_{7,12}^{(2)} = \frac{1}{6}\left[\sum_{i=4}^{7}\mu_i\alpha_i^3\sum_{j=3}^{i-1}\beta_{ij}\sum_{k=2}^{j-1}\beta_{jk}\alpha_k - \frac{1}{42}\right]$$

$$= \frac{-1}{6}\left[T_4\beta_{43}c_3(\alpha_4-\alpha_6)(\alpha_4-\alpha_5) - \frac{1}{2520}(21\alpha_5\alpha_6 - 14(\alpha_5+\alpha_6) + 10)\right] = -\frac{1}{3}\delta_{7,13}^{(2)}$$

$$\delta_{7,13}^{(1)} = \frac{1}{2}\left[\sum_{i=5}^{7}\mu_i\sum_{j=4}^{i-1}\beta_{ij}\sum_{k=3}^{j-1}\beta_{jk}\alpha_k^2\sum_{l=2}^{k-1}\beta_{kl}\alpha_l - \frac{1}{420}\right]$$

$$= \frac{1}{2}\left[\sigma_5c_5(\alpha_5-\alpha_3)(\alpha_5-\alpha_4) - \frac{1}{1680}(14\alpha_3\alpha_4 - 7(\alpha_3+\alpha_4) + 4)\right] = -\delta_{7,17}^{(1)}$$

$$\delta_{7,16} = \frac{1}{12}\left[\sum_{i=3}^{7} \mu_i \alpha_i^3 \sum_{j=2}^{i-1} \beta_{ij}\alpha_j^2 - \frac{1}{21}\right]$$

$$= \frac{-1}{12}\left[T_4\beta_{43}\alpha_3(\alpha_3-\alpha_2)(\alpha_4-\alpha_6)(\alpha_4-\alpha_5) - \frac{1}{2520}\left(-105\alpha_2\alpha_5\alpha_6 + 63(\alpha_2\alpha_6+\alpha_2\alpha_5)+42\alpha_5\alpha_6-28(\alpha_5+\alpha_6)\right.\right.$$
$$\left.\left. - 42\alpha_2+20\right)\right] = -\frac{1}{3}\,\delta_{7,15}^{(1)}$$

$$\delta_{7,17}^{(2)} = \frac{1}{2}\left[\sum_{i=4}^{7} \mu_i \alpha_i^2 \sum_{j=3}^{i-1} \beta_{ij}\alpha_j \sum_{k=2}^{j-1} \beta_{jk}\alpha_k - \frac{1}{56}\right]$$

$$= \frac{-1}{2}\left[T_5\beta_{54}c_4(\alpha_4-\alpha_3)(\alpha_5-\alpha_6) - \frac{1}{5040}(42\alpha_3\alpha_6-21\alpha_6-28\alpha_3+15)\right] = -\frac{1}{2}\,\delta_{7,22}^{(1)}$$

$$\delta_{7,18} = \frac{1}{6}\left[\sum_{i=3}^{7} \mu_i \left(\sum_{j=2}^{i-1} \beta_{ij}\alpha_j\right)\left(\sum_{j=2}^{i-1} \beta_{ij}\alpha_j^3\right) - \frac{1}{56}\right]$$

$$= \frac{1}{6}\left[(\mu_3c_3\beta_{32}+\mu_4c_4\beta_{42}+\mu_5c_5\beta_{52}+\mu_6c_6\beta_{62}+\mu_7c_7\beta_{72})\alpha_2^3+(\mu_4c_4\beta_{43}+\mu_5c_5\beta_{53}+\mu_6c_6\beta_{63}+\mu_7c_7\beta_{73})\alpha_3^3\right.$$
$$\left. + (\mu_5c_5\beta_{54}+\mu_6c_6\beta_{64}+\mu_7c_7\beta_{74})\alpha_4^3 + (\mu_6c_6\beta_{65}+\mu_7c_7\beta_{75})\alpha_5^3 + \mu_7c_7\beta_{76}\alpha_6^3 - \frac{1}{56}\right]$$

$$\delta_{7,19} = \frac{1}{2}\left[\sum_{i=3}^{7} \mu_i\alpha_i \left(\sum_{j=2}^{i-1} \beta_{ij}\alpha_j\right)\left(\sum_{j=2}^{i-1} \beta_{ij}\alpha_j^2\right) - \frac{1}{42}\right]$$

$$= \frac{-1}{2}\left[T_3c_3d_3+T_4c_4d_4+T_5c_5d_5+T_6c_6d_6 - \frac{1}{252}\right] = -\,\delta_{7,20}^{(1)}$$

$$\delta_{7,20}^{(2)} = \frac{1}{2}\left[\sum_{i=4}^{7} \mu_i \left(\sum_{j=2}^{i-1} \beta_{ij}\alpha_j\right)\sum_{j=3}^{i-1}\beta_{ij}\sum_{k=2}^{j-1}\beta_{jk}\alpha_k^2 - \frac{1}{168}\right]$$

$$= \frac{1}{2}\left[(\mu_4c_4\beta_{43}+\mu_5c_5\beta_{53}+\mu_6c_6\beta_{63}+\mu_7c_7\beta_{73})d_3 + (\mu_5c_5\beta_{54}+\mu_6c_6\beta_{64}+\mu_7c_7\beta_{74})d_4\right.$$
$$\left. + (\mu_6c_6\beta_{65}+\mu_7c_7\beta_{75})d_5 + \mu_7c_7\beta_{76}d_6 - \frac{1}{168}\right]$$

$$\delta_{7,20}^{(3)} = \frac{1}{2}\left[\sum_{i=4}^{7} \mu_i \left(\sum_{j=2}^{i-1} \beta_{ij}\alpha_j^2\right)\sum_{j=3}^{i-1}\beta_{ij}\sum_{k=2}^{j-1}\beta_{jk}\alpha_k - \frac{1}{126}\right]$$

$$= \frac{1}{2}\left[(\mu_4d_4\beta_{43}+\mu_5d_5\beta_{53}+\mu_6d_6\beta_{63}+\mu_7d_7\beta_{73})c_3 + (\mu_5d_5\beta_{54}+\mu_6d_6\beta_{64}+\mu_7d_7\beta_{74})c_4\right.$$
$$\left. + (\mu_6d_6\beta_{65}+\mu_7d_7\beta_{75})c_5 + \mu_7d_7\beta_{76}c_6 - \frac{1}{126}\right]$$

$$\delta_{7,23} = \frac{1}{4}\left[\sum_{i=3}^{7} \mu_i \alpha_i^2 \left(\sum_{j=2}^{i-1} \beta_{ij}\alpha_j\right)^2 - \frac{1}{28}\right] = \frac{-1}{4}\left[T_3\alpha_3 c_3^2 + T_4\alpha_4 c_4^2 + T_5\alpha_5 c_5^2 + T_6\alpha_6 c_6^2 - \frac{1}{168}\right] = -\frac{1}{2}\delta_{7,24}^{(1)}$$

$$\delta_{7,24}^{(2)} = \sum_{i=4}^{7} \mu_i\alpha_i \left(\sum_{j=2}^{i-1} \beta_{ij}\alpha_j\right)\left(\sum_{j=3}^{i-1}\beta_{ij}\sum_{k=2}^{i-1}\beta_{jk}\alpha_k\right) - \frac{1}{84}$$

$$= -\left[(T_4 c_4\beta_{43} + T_5 c_5\beta_{53} + T_6 c_6 \beta_{63})c_3 + (T_5 c_5\beta_{54} + T_6 c_6\beta_{64})c_4 + T_6 c_6 \beta_{65} c_5 - \frac{1}{504}\right] = -\delta_{7,26}^{(4)}$$

$$\delta_{7,25}^{(1)} = \frac{1}{2}\left[\sum_{i=4}^{7} \mu_i\alpha_i \sum_{j=3}^{i-1}\beta_{ij}\left(\sum_{k=2}^{i-1}\beta_{jk}\alpha_k\right)^2 - \frac{1}{140}\right] = \frac{-1}{2}\left[\sigma_3 c_3^2 + \sigma_4 c_4^2 + \sigma_5 c_5^2 - \frac{1}{840}\right] = -\delta_{7,26}^{(3)}$$

$$\delta_{7,25}^{(2)} = \sum_{i=4}^{7}\mu_i\left(\sum_{j=2}^{i-1}\beta_{ij}\alpha_j\right)\left(\sum_{j=3}^{i-1}\beta_{ij}\alpha_j\sum_{k=2}^{i-1}\beta_{jk}\alpha_k\right) - \frac{1}{112}$$

$$= (\mu_4 c_4\beta_{43} + \mu_5 c_5\beta_{53} + \mu_6 c_6\beta_{63} + \mu_7 c_7\beta_{73})\alpha_3 c_3 + (\mu_5 c_5\beta_{54} + \mu_6 c_6\beta_{64} + \mu_7 c_7\beta_{74})\alpha_4 c_4$$

$$+ (\mu_6 c_6\beta_{65} + \mu_7 c_7\beta_{75})\alpha_5 c_5 + \mu_7 c_7\beta_{76}\alpha_6 c_6 - \frac{1}{112}$$

$$\delta_{7,26}^{(1)} = \frac{1}{2}\left[\sum_{i=4}^{7}\mu_i\left(\sum_{j=3}^{i-1}\beta_{ij}\sum_{k=2}^{i-1}\beta_{jk}\alpha_k\right)^2 - \frac{1}{252}\right]$$

$$= \frac{1}{2}\left[\mu_4(\beta_{43}c_3)^2 + \mu_5(\beta_{53}c_3 + \beta_{54}c_4)^2 + \mu_6(\beta_{63}c_3 + \beta_{64}c_4 + \beta_{65}c_5)^2\right.$$

$$\left. + \mu_7(\beta_{73}c_3 + \beta_{74}c_4 + \beta_{75}c_5 + \beta_{76}c_6)^2 - \frac{1}{252}\right]$$

$$\delta_{7,26}^{(2)} = \sum_{i=5}^{7}\mu_i\left(\sum_{j=2}^{i-1}\beta_{ij}\alpha_j\right)\sum_{j=4}^{i-1}\beta_{ij}\sum_{k=3}^{i-1}\beta_{jk}\sum_{l=2}^{k-1}\beta_{kl}\alpha_l - \frac{1}{336}$$

$$= \left((\mu_5 c_5\beta_{54} + \mu_6 c_6\beta_{64} + \mu_7 c_7\beta_{74})\beta_{43} + (\mu_6 c_6\beta_{65} + \mu_7 c_7\beta_{75})\beta_{53} + \mu_7 c_7\beta_{76}\beta_{63}\right)c_3$$

$$+\left((\mu_6 c_6\beta_{65} + \mu_7 c_7\beta_{75})\beta_{54} + \mu_7 c_7\beta_{76}\beta_{64}\right)c_4 + \mu_7 c_7\beta_{76}\beta_{65}c_5 - \frac{1}{336}$$

$$\delta_{7,27} = \frac{1}{6}\left[\sum_{i=3}^{7}\mu_i\left(\sum_{j=2}^{i-1}\beta_{ij}\alpha_j\right)^3 - \frac{1}{56}\right] = \frac{1}{6}\left[\mu_3 c_3^3 + \mu_4 c_4^3 + \mu_5 c_5^3 + \mu_6 c_6^3 + \mu_7 c_7^3 - \frac{1}{56}\right]$$

$$\delta_{7,28} = \frac{1}{8}\left[\sum_{i=3}^{7}\mu_i\left(\sum_{j=2}^{i-1}\beta_{ij}\alpha_j^2\right)^2 - \frac{1}{63}\right] = \frac{1}{8}\left[\mu_3 d_3^2 + \mu_4 d_4^2 + \mu_5 d_5^2 + \mu_6 d_6^2 + \mu_7 d_7^2 - \frac{1}{63}\right]$$

ただし

$$g_1(ijkl)=\alpha_i+\alpha_j+\alpha_k+\alpha_l,$$

$$g_2(ijkl)=\alpha_i\alpha_j+\alpha_i\alpha_k+\alpha_i\alpha_l+\alpha_j\alpha_k+\alpha_j\alpha_l+\alpha_k\alpha_l,$$

$$g_3(ijkl)=\alpha_j\alpha_k\alpha_l+\alpha_i\alpha_k\alpha_l+\alpha_i\alpha_j\alpha_l+\alpha_i\alpha_j\alpha_k,$$

$$g_4(ijkl)=\alpha_i\alpha_j\alpha_k\alpha_l,$$

$$d_i=\sum_{j=2}^{i-1}\beta_{ij}\alpha_j^2, \qquad (i=3,4,\cdots,7).$$

2.3 7次極限公式の条件

2.3.1 条件 $\alpha_2\to 0$ の誘導

α_i のほか T_6 だけを含む $\delta_{7,1}$ を 0 とおくと (2.1−8) から

$$T_i=\frac{70g_4(jklm)-35g_3(jklm)+21g_2(jklm)-14g_1(jklm)+10}{420\alpha_i(\alpha_i-\alpha_j)(\alpha_i-\alpha_k)(\alpha_i-\alpha_l)(\alpha_i-\alpha_m)} \tag{2.3−1}$$

$$(i,j,k,l,m=2,\cdots,6;\ \ j,k,l,m\neq i)$$

(2.1−7), (2.1−16) から μ_i が決る.

(2.1−9)のうち線形のはじめの3式と $\delta_{7,11}=0$ から

$$c_i=\frac{-35\alpha_j\alpha_k\alpha_l+21(\alpha_j\alpha_k+\alpha_k\alpha_l+\alpha_l\alpha_j)-14(\alpha_j+\alpha_k+\alpha_l)+10}{840T_i(\alpha_i-\alpha_j)(\alpha_i-\alpha_k)(\alpha_i-\alpha_l)} \tag{2.3−2}$$

$$(i,j,k,l=3,4,5,6;\ \ j,k,l\neq i)$$

極限公式の可能性を調べる.

$\delta_{7,3}=0$ と $\delta_{7,13}^{(1)}=0$ から

$$\sigma_5=\frac{-35\alpha_2\alpha_3\alpha_4+14(\alpha_2\alpha_3+\alpha_3\alpha_4+\alpha_4\alpha_2)-7(\alpha_2+\alpha_3+\alpha_4)+4}{840\,\alpha_5(\alpha_5-\alpha_2)(\alpha_5-\alpha_3)(\alpha_5-\alpha_4)}$$

$$\sigma_5=\frac{-70g_4(2,3,4,6)+35g_3(2,3,4,6)-21g_2(2,3,4,6)+14g_1(2,3,4,6)-10}{35\alpha_3\alpha_4\alpha_6-21(\alpha_3\alpha_4+\alpha_4\alpha_6+\alpha_6\alpha_3)+14(\alpha_3+\alpha_4+\alpha_6)-10}$$

$$\times\frac{14\alpha_3\alpha_4-7(\alpha_3+\alpha_4)+4}{840\,\alpha_5(\alpha_5-\alpha_2)(\alpha_5-\alpha_3)(\alpha_5-\alpha_4)}$$

この二つの σ_5 が等しい条件：

$$(2.3-3)\quad \alpha_2(\alpha_6-1)\{35\alpha_3^2\alpha_4^2-35\alpha_3\alpha_4(\alpha_3+\alpha_4)-23\alpha_3\alpha_4+7(\alpha_3+\alpha_4)^2$$
$$-8(\alpha_3+\alpha_4)+2\}=0$$

$\delta_{7,4}=0$ と $\delta_{7,10}^{(1)}=0$ から

$$\sigma_5\beta_{54}=\frac{21\alpha_2\alpha_3-7(\alpha_2+\alpha_3)+3}{2520\,\alpha_4(\alpha_4-\alpha_2)(\alpha_4-\alpha_3)}$$

$$\sigma_5\beta_{54}=\frac{70g_4(2,3,5,6)-35g_3(2,3,5,6)+21g_2(2,3,5,6)-14g_1(2,3,5,6)+10}{-35\alpha_3\alpha_5\alpha_6+21(\alpha_3\alpha_5+\alpha_5\alpha_6+\alpha_6\alpha_3)-14(\alpha_3+\alpha_5+\alpha_6)+10}$$

$$\times\frac{-7\alpha_3+3}{2520\,\alpha_4(\alpha_4-\alpha_2)(\alpha_4-\alpha_3)}$$

$$(2.3-4)\quad \alpha_2\{7\alpha_3^2(5\alpha_5\alpha_6-4(\alpha_5+\alpha_6)+3)-\alpha_3(33\alpha_5\alpha_6-27(\alpha_5+\alpha_6)+21)$$
$$+6\alpha_5\alpha_6-5(\alpha_5+\alpha_6)+4\}=0$$

$\delta_{7,8}^{(1)} = 0$ と $\delta_{7,11}^{(2)} = 0$ から

$$\tau_5\beta_{54} = \frac{105\alpha_2\alpha_3\alpha_6 - 63\alpha_2\alpha_3 - 42(\alpha_3\alpha_6+\alpha_6\alpha_2) + 28(\alpha_2+\alpha_3) + 21\alpha_6 - 15}{2520\,\alpha_4(\alpha_4-\alpha_2)(\alpha_4-\alpha_3)(\alpha_6-\alpha_5)}$$

$$\tau_5\beta_{54} = \frac{70g_4(2,3,5,6) - 35g_3(2,3,5,6) + 21g_2(2,3,5,6) - 14g_1(2,3,5,6) + 10}{-35\alpha_3\alpha_5\alpha_6 + 21(\alpha_3\alpha_5+\alpha_5\alpha_6+\alpha_6\alpha_3) - 14(\alpha_3+\alpha_5+\alpha_6) + 10} \times \frac{-42\alpha_3\alpha_6 + 28\alpha_3 + 21\alpha_6 - 15}{2520\alpha_4(\alpha_4-\alpha_2)(\alpha_4-\alpha_3)(\alpha_6-\alpha_5)}$$

$$(2.3\text{-}5)\quad \alpha_2\Big[\alpha_5\big\{(15\alpha_3\alpha_6 - 9\alpha_3 - 6\alpha_6 + 4)(35\alpha_3\alpha_6 - 21(\alpha_3+\alpha_6) + 14) - (42\alpha_3\alpha_6 - 28\alpha_3 - 21\alpha_6 + 15)(10\alpha_3\alpha_6 - 5(\alpha_3+\alpha_6) + 3)\big\} - \big\{(15\alpha_3\alpha_6 - 9\alpha_3 - 6\alpha_6 + 4)(21\alpha_3\alpha_6 - 14(\alpha_3+\alpha_6) + 10) - (42\alpha_3\alpha_6 - 28\alpha_3 - 21\alpha_6 + 15)(5\alpha_3\alpha_6 - 3(\alpha_3+\alpha_6) + 2\big\}\Big] = 0$$

$\delta_{7,5} = 0$ と $\delta_{7,6} = 0$ から

$$\sigma_5\beta_{54}\beta_{43} = \frac{-7\alpha_2 + 2}{5040\,\alpha_3(\alpha_3-\alpha_2)}$$

$$\sigma_5\beta_{54}\beta_{43} = \frac{-70g_4(2,4,5,6) + 35g_3(2,4,5,6) - 21g_2(2,4,5,6) + 14g_1(2,4,5,6) - 10}{2520\alpha_3(\alpha_3-\alpha_2)(35\alpha_4\alpha_5\alpha_6 - 21(\alpha_4\alpha_5+\alpha_5\alpha_6+\alpha_6\alpha_4) + 14(\alpha_4+\alpha_5+\alpha_6) - 10)}$$

$$(2.3\text{-}6)\quad \alpha_2(15\alpha_4\alpha_5\alpha_6 - 11(\alpha_4\alpha_5+\alpha_5\alpha_6+\alpha_6\alpha_4) + 8(\alpha_4+\alpha_5+\alpha_6) - 6) = 0$$

$\delta_{7,9}^{(1)} = 0$ と $\delta_{7,10}^{(2)} = 0$ から

$$\sigma_4\beta_{43} = \frac{-42\alpha_2\alpha_5 + 21\alpha_2 + 14\alpha_5 - 8}{5040\,\alpha_3(\alpha_3-\alpha_2)(\alpha_5-\alpha_4)}$$

$$\sigma_4\beta_{43} = \frac{-70g_4(2,4,5,6) + 35g_3(2,4,5,6) - 21g_2(2,4,5,6) + 14g_1(2,4,5,6) - 10}{35\alpha_4\alpha_5\alpha_6 - 21(\alpha_4\alpha_5+\alpha_5\alpha_6+\alpha_6\alpha_4) + 14(\alpha_4+\alpha_5+\alpha_6) - 10} \times \frac{7\alpha_5 - 4}{2520\alpha_3(\alpha_3-\alpha_2)(\alpha_5-\alpha_4)}$$

(2.3-7) $\alpha_2\Big[2\alpha_5\{35\alpha_3\alpha_5\alpha_6-21(\alpha_3\alpha_5+\alpha_5\alpha_6+\alpha_6\alpha_3)+21(\alpha_3+\alpha_5+\alpha_6)-16\}$

$-\{25\alpha_3\alpha_5\alpha_6-23(\alpha_3\alpha_5+\alpha_5\alpha_6+\alpha_6\alpha_3)+18(\alpha_3+\alpha_5+\alpha_6)-14\}\Big]=0$

$\delta_{7,9}^{(2)}=0$ と $\delta_{7,10}^{(3)}=0$ から

$$T_5\beta_{54}\beta_{43}=\frac{-21\alpha_2\alpha_6+14\alpha_2+7\alpha_6-5}{2520\alpha_3(\alpha_3-\alpha_2)(\alpha_6-\alpha_5)}$$

$$T_5\beta_{54}\beta_{43}=\frac{-70g_4(2,4,5,6)+35g_3(2,4,5,6)-21g_2(2,4,5,6)+14g_1(2,4,5,6)-10}{35\alpha_4\alpha_5\alpha_6-21(\alpha_4\alpha_5+\alpha_5\alpha_6+\alpha_6\alpha_4)+14(\alpha_4+\alpha_5+\alpha_6)-10}\times\frac{7\alpha_6-5}{2520\alpha_3(\alpha_3-\alpha_2)(\alpha_6-\alpha_5)}$$

(2.3-8) $\alpha_2\Big[\alpha_6\{35\alpha_4\alpha_5\alpha_6-28(\alpha_4\alpha_5+\alpha_5\alpha_6+\alpha_6\alpha_4)+21(\alpha_4+\alpha_5+\alpha_6)-16\}$

$-\{20\alpha_4\alpha_5\alpha_6-17(\alpha_4\alpha_5+\alpha_5\alpha_6+\alpha_6\alpha_4)+13(\alpha_4+\alpha_5+\alpha_6)-10\}\Big]=0$

$\delta_{7,14}=0$ と $\delta_{7,12}^{(2)}=0$ から

$$T_4\beta_{43}=\frac{-105\alpha_2\alpha_5\alpha_6+63(\alpha_2\alpha_5+\alpha_6\alpha_2)+42\alpha_5\alpha_6-42\alpha_2-28(\alpha_5+\alpha_6)+20}{2520\alpha_3(\alpha_3-\alpha_2)(\alpha_5-\alpha_4)(\alpha_6-\alpha_4)}$$

$$T_4\beta_{43}=\frac{-70g_4(2,4,5,6)+35g_3(2,4,5,6)-21g_2(2,4,5,6)+14g_1(2,4,5,6)-10}{35\alpha_4\alpha_5\alpha_6-21(\alpha_4\alpha_5+\alpha_5\alpha_6+\alpha_6\alpha_4)+14(\alpha_4+\alpha_5+\alpha_6)-10}\times\frac{21\alpha_5\alpha_6-14(\alpha_5+\alpha_6)+10}{1260\alpha_3(\alpha_3-\alpha_2)(\alpha_5-\alpha_4)(\alpha_6-\alpha_4)}$$

(2.3-9) $\alpha_2\Big[(15\alpha_5\alpha_6-9(\alpha_5+\alpha_6)+6)(35\alpha_4\alpha_5\alpha_6-21(\alpha_4\alpha_5+\alpha_5\alpha_6+\alpha_6\alpha_4)+14(\alpha_4+\alpha_5+\alpha_6)-10)$

$-(21\alpha_5\alpha_6-14(\alpha_5+\alpha_6)+10)(20\alpha_4\alpha_5\alpha_6-10(\alpha_4\alpha_5+\alpha_5\alpha_6+\alpha_6\alpha_4)+6(\alpha_4+\alpha_5+\alpha_6)-4)\Big]$

$=0$

(2.3-3) から (2.3-9) は $\alpha_2=0$ ならばすべて成り立つ.

2.3.2 $\alpha_2 \to 0$ のときの残りの条件方程式と $\delta_{7,j}^{(k)}$

(2.3-1) から

(2.3-10)
$$\left|\begin{aligned} T_2\alpha_2 &= \frac{70g_4(3,4,5,6)-35g_3(3,4,5,6)+21g_2(3,4,5,6)-14g_1(3,4,5,6)+10}{420\,\alpha_3\alpha_4\alpha_5\alpha_6} \\ T_i\alpha_i &= \frac{-35\alpha_j\alpha_k\alpha_\ell+21(\alpha_j\alpha_k+\alpha_k\alpha_\ell+\alpha_\ell\alpha_j)-14(\alpha_j+\alpha_k+\alpha_\ell)+10}{420\,\alpha_i(\alpha_i-\alpha_j)(\alpha_i-\alpha_k)(\alpha_i-\alpha_\ell)} \end{aligned}\right.$$

$$(i, j, k, \ell = 3, 4, 5, 6;\ j, k, \ell \neq i)$$

(2.3-2), (2.1-3) から

(2.3-11) $\quad c_i \longrightarrow \dfrac{\alpha_i^2}{2}, \qquad (i = 3, 4, 5, 6)$

(2.3-12) $\quad c_7 \longrightarrow \dfrac{1}{2}\left(= \dfrac{\alpha_7^2}{2}\right)$

<u>残りの条件式</u>:

$$\sigma_2\alpha_2^2 \longrightarrow 0$$

$$(\sigma_3\beta_{32}+\sigma_4\beta_{42}+\sigma_5\beta_{52})\alpha_2^2 \longrightarrow 0$$

$$(T_3\alpha_3\beta_{32}+T_4\alpha_4\beta_{42}+T_5\alpha_5\beta_{52})\alpha_2^2 \longrightarrow 0$$

となり，残りの条件式は次の4式になる:

(2.3-13) $\quad \sigma_3\alpha_3^2+\sigma_4\alpha_4^2+\sigma_5\alpha_5^2 = \dfrac{1}{60}$

(2.3-14) $\quad \sigma_3\alpha_3^3+\sigma_4\alpha_4^3+\sigma_5\alpha_5^3 = \dfrac{1}{120}$

(2.3-15) $\quad (\sigma_4\beta_{43}+\sigma_5\beta_{53})\alpha_3^2+\sigma_5\beta_{54}\alpha_4^2 = \dfrac{1}{360}$

(2.3-16) $\quad (T_4\alpha_4\beta_{43}+T_5\alpha_5\beta_{53}+T_6\alpha_6\beta_{63})\alpha_3^2+(T_5\alpha_5\beta_{54}+T_6\alpha_6\beta_{64})\alpha_4^2$
$$+\,T_6\alpha_6\beta_{65}\alpha_5^2 = \frac{1}{90}$$

7 次の誤差項の係数：

$$-720\delta_{7,1},\ 120\delta_{7,2},\ -48\delta_{7,11},\ 12\delta_{7,12}^{(1)},\ -16\delta_{7,23},\ 8\delta_{7,24}^{(1)},\ -48\delta_{7,27}$$
$$\longrightarrow T_3\alpha_3^5+T_4\alpha_4^5+T_5\alpha_5^5+T_6\alpha_6^5 - 1/42$$

$$24\delta_{7,3},\ -24\delta_{7,7},\ 4\delta_{7,13}^{(1)},\ -4\delta_{7,17}^{(1)},\ -8\delta_{7,25}^{(1)},\ 8\delta_{7,26}^{(3)}$$
$$\longrightarrow \sigma_5\alpha_5^2(\alpha_5-\alpha_3)(\alpha_5-\alpha_4) - (14\alpha_3\alpha_4-7(\alpha_3+\alpha_4)+4)/840$$

$$6\delta_{7,4},\ -6\delta_{7,8}^{(2)},\ 2\delta_{7,10}^{(1)},\ -2\delta_{7,22}^{(3)} \longrightarrow \sigma_5\beta_{54}\alpha_4^2(\alpha_4-\alpha_3) - (-7\alpha_3+3)/2520$$

$$2\delta_{7,5},\ -2\delta_{7,9}^{(3)},\ 2\delta_{7,6},\ -2\delta_{7,10}^{(4)} \longrightarrow \sigma_5\beta_{54}\beta_{43}\alpha_3^2 - 1/2520$$

$$-6\delta_{7,8}^{(1)},\ 12\delta_{7,16},\ 4\delta_{7,17}^{(2)},\ -2\delta_{7,22}^{(1)},\ 12\delta_{7,18},\ 4\delta_{7,25}^{(2)}$$
$$\longrightarrow T_5\beta_{54}\alpha_4^2(\alpha_4-\alpha_3)(\alpha_6-\alpha_5) - (-42\alpha_3\alpha_6+21\alpha_6+28\alpha_3-15)/2520$$

$$2\delta_{7,9}^{(1)},\ -2\delta_{7,21},\ -2\delta_{7,10}^{(2)},\ 2\delta_{7,22}^{(2)} \longrightarrow \sigma_4\beta_{43}\alpha_3^2(\alpha_5-\alpha_4) - (7\alpha_5-4)/2520$$

$$-2\delta_{7,9}^{(2)},\ 4\delta_{7,15}^{(2)},\ -2\delta_{7,10}^{(3)},\ 4\delta_{7,13}^{(3)},\ 4\delta_{7,20}^{(2)},\ 4\delta_{7,26}^{(2)}$$
$$\longrightarrow T_5\beta_{54}\beta_{43}\alpha_3^2(\alpha_6-\alpha_5) - (7\alpha_6-5)/2520$$

$$-12\delta_{7,12}^{(2)},\ 4\delta_{7,13}^{(2)},\ -12\delta_{7,14},\ 4\delta_{7,15}^{(1)},\ -4\delta_{7,19},\ 4\delta_{7,20}^{(1)},\ -4\delta_{7,24}^{(2)},\ 4\delta_{7,26}^{(4)}$$
$$\longrightarrow T_4\beta_{43}\alpha_3^2(\alpha_6-\alpha_4)(\alpha_5-\alpha_4) - (21\alpha_5\alpha_6-14(\alpha_5+\alpha_6)+10)/1260$$

$$4\delta_{7,20}^{(3)} \longrightarrow (\mu_4 d_4\beta_{43}+\mu_5 d_5\beta_{53}+\mu_6 d_6\beta_{63}+\mu_7 d_7\beta_{73})\alpha_3^2+(\mu_5 d_5\beta_{54}+\mu_6 d_6\beta_{64}+\mu_7 d_7\beta_{74})\alpha_4^2$$
$$+(\mu_6 d_6\beta_{65}+\mu_7 d_7\beta_{75})\alpha_5^2+\mu_7 d_7\beta_{76}\alpha_6^2 - 1/63$$

$$8\delta_{7,26}^{(1)} \longrightarrow \mu_4(\beta_{43}\alpha_3^2)^2+\mu_5(\beta_{53}\alpha_3^2+\beta_{63}\alpha_3^2)^2+\mu_6(\beta_{63}\alpha_3^2+\beta_{64}\alpha_4^2+\beta_{65}\alpha_5^2)^2$$
$$+\mu_7(\beta_{73}\alpha_3^2+\beta_{74}\alpha_4^2+\beta_{75}\alpha_5^2+\beta_{76}\alpha_6^2)^2 - 1/63$$

$$8\delta_{7,28} = \mu_3 d_3^2+\mu_4 d_4^2+\mu_5 d_5^2+\mu_6 d_6^2+\mu_7 d_7^2 - 1/63$$

2.3.3 7次極限公式の条件

すべての項が0になるためには，はじめの7式から

$$(2.3-17)\qquad \sigma_5 = \frac{14\alpha_3\alpha_4 - 7(\alpha_3+\alpha_4) + 4}{840\alpha_5^2(\alpha_5-\alpha_3)(\alpha_5-\alpha_4)}$$

$$(2.3-18)\qquad \sigma_5\beta_{54} = \frac{-7\alpha_3 + 3}{2520\alpha_4^2(\alpha_4-\alpha_3)}$$

$$(2.3-19)\qquad \sigma_5\beta_{54}\beta_{43} = \frac{1}{2520\alpha_3^2}$$

$$(2.3-20)\qquad T_5\beta_{54} = \frac{-42\alpha_3\alpha_6 + 28\alpha_3 + 21\alpha_6 - 15}{2520\alpha_4^2(\alpha_4-\alpha_3)(\alpha_6-\alpha_5)}$$

$$(2.3-21)\qquad \sigma_4\beta_{43} = \frac{7\alpha_5 - 4}{2520\alpha_3^2(\alpha_5-\alpha_4)}$$

$$(2.3-22)\qquad T_5\beta_{54}\beta_{43} = \frac{7\alpha_6 - 5}{2520\alpha_3^2(\alpha_6-\alpha_5)}$$

$$(2.3-23)\qquad T_4\beta_{43} = \frac{21\alpha_5\alpha_6 - 14(\alpha_5+\alpha_6) + 10}{1260\alpha_3^2(\alpha_6-\alpha_4)(\alpha_5-\alpha_4)}$$

が得られる．

(2.3-17)，(2.3-13)，(2.3-14) から

$$(2.3-24)\qquad \sigma_i = \frac{14\alpha_j\alpha_k - 7(\alpha_j+\alpha_k) + 4}{840\alpha_i^2(\alpha_i-\alpha_j)(\alpha_i-\alpha_k)},\qquad (i,j,k=3,4,5;\ j,k\neq i)$$

(2.3-18)と(2.3-20)から得られる β_{54} が等しいための条件：

$$(2.3-25)\qquad (\alpha_6-1)((14\alpha_3^2 - 12\alpha_3 + 3)\alpha_4 - \alpha_3) = 0$$

残りの4式から得られる β_{43} を二つづつ等しいとおいて得られる条件：

$$
(2.3\text{-}26)\quad
\left|\begin{array}{l}
\alpha_3(\alpha_6-1)(7\alpha_5^2-8\alpha_5+2)=0 \\
\alpha_3(\alpha_5-1)=0 \\
\alpha_3(3\alpha_6-2\alpha_5-1)=0 \\
\alpha_3(6\alpha_5\alpha_6-5(\alpha_5+\alpha_6)+4)=0 \\
\alpha_3\{(21\alpha_6^2-30\alpha_6+11)\alpha_5-(21\alpha_6^2-29\alpha_6+10)\}=0 \\
\alpha_3(\alpha_6-1)=0
\end{array}\right.
$$

(2.3-25) と (2.3-26) を共に満足する場合分け

i) $\alpha_6=1$ かつ $\alpha_3=0$ （α_4, α_5：任意）

ii) $\alpha_6=1$ かつ $\alpha_5=1$ （α_3, α_4：任意）

iii) $\alpha_3=0$ かつ $\alpha_4=0$ （α_5, α_6：任意）

i)の場合に限って考察する．

$$T_i(1-\alpha_i)\alpha_i^2 \longrightarrow 2\sigma_i\alpha_i^2, \quad (i=3,4,5)$$

<u>残りの条件式</u>：

$$
(2.3\text{-}16)'\quad (T_4\alpha_4\beta_{43}+T_5\alpha_5\beta_{53}+T_6\alpha_6\beta_{63})\alpha_3^2+(T_5\alpha_5\beta_{54}+T_6\alpha_6\beta_{64})\alpha_4^2+T_6\alpha_6\beta_{65}\alpha_5^2
$$

$$
=T_3\alpha_3 d_3+T_4\alpha_4 d_4+T_5\alpha_5 d_5+T_6\alpha_6 d_6=\frac{1}{90}
$$

残りの7次の誤差項の係数：

$(2.3-13)$, $(2.3-16)'$, $\delta_{7,14}=0$ と $(2.1-6)$ の15番目の式から

$$(2.3-28)\qquad d_i \rightarrow \frac{\alpha_i^3}{3},\quad (i=4,5,6)$$

$$(2.3-29)\qquad d_7 \rightarrow \frac{1}{3}\left(=\frac{\alpha_7^3}{3}\right)$$

$$d_4=\beta_{42}\alpha_2^2+\beta_{43}\alpha_3^2 \rightarrow \frac{\alpha_4^3}{3},\quad \beta_{43}\alpha_3^2 \rightarrow \frac{\alpha_4^3}{3},\quad \sigma_2\alpha_2^2 \rightarrow 0,\quad \sum_{i=3}^{5}\sigma_i\beta_{i2}\alpha_2^2 \rightarrow 0,\quad \sum_{i=3}^{5}T_i\alpha_i\beta_{i2}\alpha_2^2 \rightarrow 0$$

から

$$\beta_{i2}\alpha_2^2 \rightarrow 0,\quad (i=4,5,6)$$

従って

$$(2.3-30)\qquad \sum_{j=3}^{i-1}\beta_{ij}\alpha_j^2 \longrightarrow \sum_{j=2}^{i-1}\beta_{ij}\alpha_j^2 = d_i,\quad (i=4,5,6,7)$$

$(2.3-11)$, $(2.3-12)$, $(2.3-28)$, $(2.3-29)$, $(2.3-30)$ から,

$$\begin{aligned}
\delta_{7,20}^{(3)} \rightarrow\ & (\mu_4 d_4\beta_{43}+\mu_5 d_5\beta_{53}+\mu_6 d_6\beta_{63}+\mu_7 d_7\beta_{73})\alpha_3^2 \\
& +(\mu_5 d_5\beta_{54}+\mu_6 d_6\beta_{64}+\mu_7 d_7\beta_{74})\alpha_4^2 \\
& +(\mu_6 d_6\beta_{65}+\mu_7 d_7\beta_{75})\alpha_5^2 \\
& +\mu_7 d_7\beta_{76}\alpha_6^2-\frac{1}{63} \\
\rightarrow\ & \frac{1}{3}\left[\mu_4\alpha_4^3 d_4+\mu_5\alpha_5^3 d_5+\mu_6\alpha_6^3 d_6+\mu_7 d_7-\frac{1}{21}\right] \longrightarrow 4\delta_{7,14} \rightarrow 0
\end{aligned}$$

$$\begin{aligned}
\delta_{7,26}^{(1)} \rightarrow\ & \frac{1}{8}\left[\mu_4(\beta_{43}\alpha_3^2)^2+\mu_5\Big(\sum_{j=3}^{4}\beta_{5j}\alpha_j^2\Big)^2+\mu_6\Big(\sum_{j=3}^{5}\beta_{6j}\alpha_j^2\Big)^2+\mu_7\Big(\sum_{j=3}^{6}\beta_{7j}\alpha_j^2\Big)^2-\frac{1}{63}\right] \\
\rightarrow\ & \frac{1}{24}\left[\mu_4\alpha_4^3 d_4+\mu_5\alpha_5^3 d_5+\mu_6\alpha_6^3 d_6+\mu_7 d_7-\frac{1}{21}\right] \rightarrow \frac{1}{2}\delta_{7,14} \rightarrow 0
\end{aligned}$$

$$\begin{aligned}
\delta_{7,28} =\ & \frac{1}{8}\left[\mu_3 d_3^2+\mu_4 d_4^2+\mu_5 d_5^2+\mu_6 d_6^2+\mu_7 d_7^2-\frac{1}{63}\right] \\
\rightarrow\ & \frac{1}{72}\left[\mu_3\alpha_3^6+\mu_4\alpha_4^6+\mu_5\alpha_5^6+\mu_7-\frac{1}{7}\right] \longrightarrow 10\delta_{7,1}=0
\end{aligned}$$

3. 7次極限公式

3.1 極限公式

$$
(3.1\text{-}1)\quad
\left|\begin{array}{l}
f_1 = f(t_n, y_n) \\
\qquad F_2 = \dfrac{\partial}{\partial t} f(t_n, y_n) + f_1 \dfrac{\partial}{\partial y} f(t_n, y_n) \\
\qquad F_3 = \dfrac{\partial^2}{\partial t^2} f(t_n, y_n) + 2 f_1 \dfrac{\partial^2}{\partial t \partial y} f(t_n, y_n) + f_1^2 \dfrac{\partial^2}{\partial y^2} f(t_n, y_n) \quad + F_2 \dfrac{\partial}{\partial y} f(t_n, y_n) \\
f_4 = f\left(t_n + \alpha_4 h,\ y_n + h\left(b_{41} f_1 + h\, b_{42} F_2 + \dfrac{h^2}{2} b_{43} F_3\right)\right) \\
f_5 = f\left(t_n + \alpha_5 h,\ y_n + h\left(b_{51} f_1 + h\, b_{52} F_2 + \dfrac{h^2}{2} b_{53} F_3 + b_{54} f_4\right)\right) \\
\qquad y_p = y_n + h\left(b_{71} f_1 + h\, b_{72} F_2 + \dfrac{h^2}{2} b_{73} F_3 + b_{74} f_4 + b_{75} f_5\right) \\
f_7 = f(t_n + h,\ y_p) \\
\qquad F_6 = \dfrac{\partial}{\partial t} f(t_n + h, y_p) \\
\qquad\qquad + \dfrac{\partial}{\partial y} f(t_n + h, y_p) \cdot \left\{ b_{671} f_1 + h\, b_{672} F_2 + \dfrac{h^2}{2} b_{673} F_3 + b_{674} f_4 + b_{675} f_5 + b_{676} f_7 \right\} \\
y_{n+1} = y_n + h\left(m_1 f_1 + h\, m_2 F_2 + \dfrac{h^2}{2} m_3 F_3 + m_4 f_4 + m_5 f_5 + h\, m_6 F_6 + m_7 f_7\right)
\end{array}\right.
$$

ただし

$b_{i1} = \lim(\beta_{i1} + \beta_{i2} + \beta_{i3})$, $\quad b_{i2} = \lim(\beta_{i2}\alpha_2 + \beta_{i3}\alpha_3)$, $\quad b_{i3} = \lim \beta_{i3}\alpha_3^2$, $\ (i = 4, 5, 6, 7)$;

$b_{ij} = \lim \beta_{ij}$, $\ (i=5,\ j=4;\ \ i=7,\ j=4,5)$;

$b_{67j} = \lim \dfrac{b_{7j} - b_{6j}}{1 - \alpha_6}$, $\ (j = 1, 2, \cdots, 5)$; $\qquad b_{676} = \lim \dfrac{\beta_{76}}{1 - \alpha_6}$;

$m_1 = \lim(\mu_1 + \mu_2 + \mu_3)$, $\quad m_2 = \lim(\mu_2\alpha_2 + \mu_3\alpha_3)$, $\quad m_3 = \lim \mu_3\alpha_3^2$,

$m_i = \lim \mu_i$, $\ (i = 4, 5)$; $\quad m_6 = -\lim \mu_6(1 - \alpha_6)$, $\quad m_7 = \lim(\mu_6 + \mu_7)$

係数は α_4, α_5 を自由パラメタとして次のように表わされる：

$$m_2 = \frac{\alpha_4\alpha_5(63\alpha_4\alpha_5 - 28(\alpha_4+\alpha_5) + 15) + (\alpha_4+\alpha_5)(14\alpha_4\alpha_5 - 7(\alpha_4+\alpha_5) + 4)}{420\,\alpha_4^2\alpha_5^2}$$

$$m_3 = \frac{14\alpha_4\alpha_5 - 7(\alpha_4+\alpha_5) + 4}{420\alpha_4^2\alpha_5^2}, \quad m_4 = \frac{7\alpha_5 - 4}{420\alpha_4^3(\alpha_5-\alpha_4)(1-\alpha_4)^2}, \quad m_5 = \frac{-7\alpha_4 + 4}{420\alpha_5^3(\alpha_5-\alpha_4)(1-\alpha_5)^2}$$

$$m_6 = \frac{-21\alpha_4\alpha_5 + 14(\alpha_4+\alpha_5) - 10}{420(1-\alpha_4)(1-\alpha_5)}$$

$$m_7 = \frac{2\alpha_4\alpha_5(84\alpha_4\alpha_5 - 63(\alpha_4+\alpha_5) + 50) - (\alpha_4+\alpha_5)(189\alpha_4\alpha_5 - 140(\alpha_4+\alpha_5) + 110) + 210\alpha_4\alpha_5 - 154(\alpha_4+\alpha_5) + 120}{420(1-\alpha_4)^2(1-\alpha_5)^2}$$

$$m_1 = 1 - m_4 - m_5 - m_7$$

$$b_{41} = \alpha_4, \qquad b_{42} = \frac{\alpha_4^2}{2}, \qquad b_{43} = \frac{\alpha_4^3}{3}$$

$$b_{54} = \frac{\alpha_5^3(\alpha_5-\alpha_4)}{\alpha_4^3(4-7\alpha_4)}, \quad b_{51} = \alpha_5 - b_{54}, \quad b_{52} = \frac{\alpha_5^2}{2} - b_{54}\alpha_4, \quad b_{53} = \frac{\alpha_5^3}{3} - b_{54}\alpha_4^2$$

$$b_{75} = \frac{-m_5(1-\alpha_5)^2}{2m_6}, \quad b_{74} = \frac{1}{m_6}\left(m_5(1-\alpha_5)b_{54} - \frac{1}{2}m_4(1-\alpha_4)^2\right)$$

$$b_{71} = 1 - b_{74} - b_{75}, \quad b_{72} = \frac{1}{2} - b_{74}\alpha_4 - b_{75}\alpha_5, \quad b_{73} = \frac{1}{3} - b_{74}\alpha_4^2 - b_{75}\alpha_5^2$$

$$b_{675} = \frac{1}{m_6}\left(m_5(1-\alpha_5) - m_7 b_{75}\right)$$

$$b_{674} = \frac{1}{m_6}\left(m_4(1-\alpha_4) - m_5 b_{54} - m_7 b_{74}\right)$$

$$b_{673} = 2 - b_{674}\alpha_4^2 - b_{675}\alpha_5^2, \qquad b_{672} = 2 - b_{674}\alpha_4 - b_{675}\alpha_5$$

$$b_{671} = 2 - b_{674} - b_{675}, \qquad b_{676} = -1$$

3.2 自由パラメタ α_4, α_5 の決め方

$O(h^8)$ の誤差項の係数 $\delta_{8j}^{(k)}$ 115項の分類:

$$K=\frac{1}{6720},\quad \xi=\frac{14\alpha_4\alpha_5-8(\alpha_4+\alpha_5)+5}{3360},\quad \eta=\frac{8\alpha_5-5}{6720},\quad \zeta=\frac{2\alpha_4-1}{1680}$$ とおく

K の倍数 ------ 20項

ξ の倍数 ------ 38項

η の倍数 ------ 32項

ζ の倍数 ------ 25項

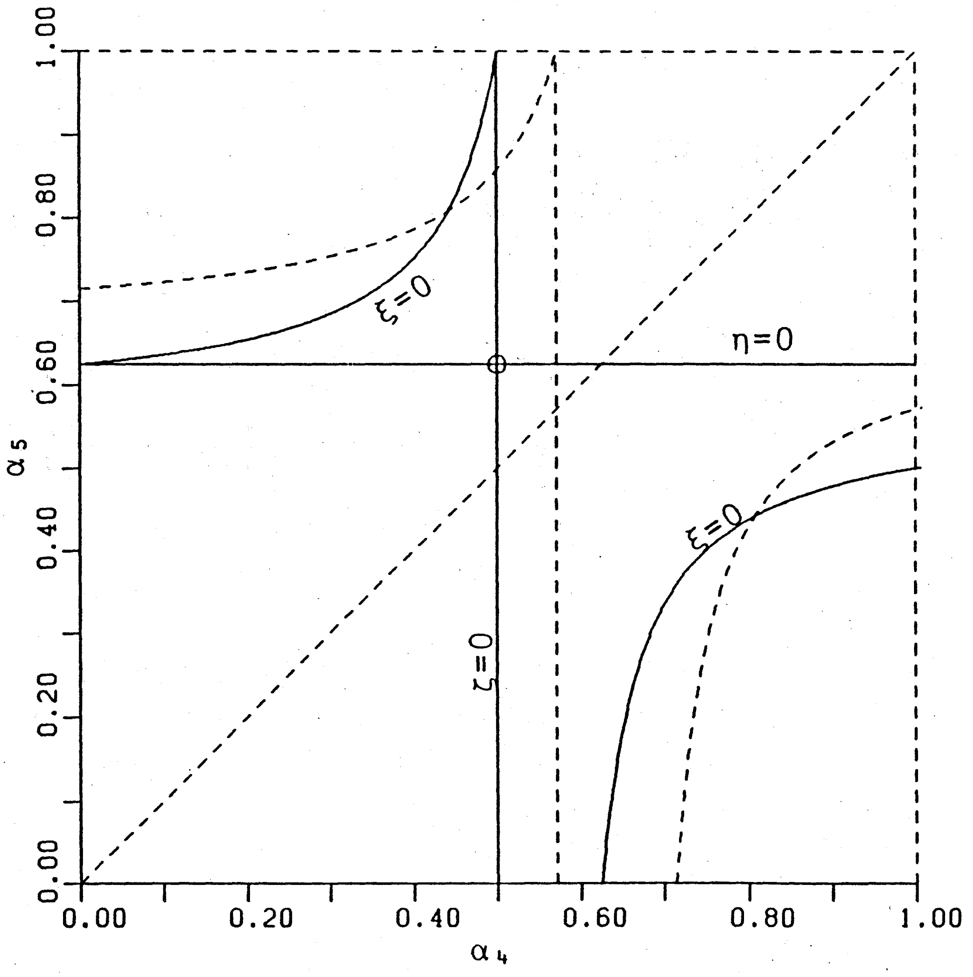

図1 $\xi, \eta, \zeta=0$ の曲線 　　----は禁止線, ○…$(\frac{1}{2}, \frac{5}{8})$

3.3 7次極限公式 [RKD7]

[RKD7]

$$f_1 = f(t_n, y_n)$$

$$F_2 = \frac{\partial}{\partial t} f(t_n, y_n) + f_1 \frac{\partial}{\partial y} f(t_n, y_n)$$

$$F_3 = \frac{\partial^2}{\partial t^2} f(t_n, y_n) + 2 f_1 \frac{\partial^2}{\partial t \partial y} f(t_n, y_n) + f_1^2 \frac{\partial^2}{\partial y^2} f(t_n, y_n) + F_2 \frac{\partial}{\partial y} f(t_n, y_n)$$

$$f_4 = f(t_n + \tfrac{1}{2} h,\ y_n + \tfrac{1}{2} h f_1 + \tfrac{1}{8} h^2 F_2 + \tfrac{1}{48} h^3 F_3)$$

$$f_5 = f(t_n + \tfrac{5}{8} h,\ y_n + \tfrac{35}{256} h f_1 - \tfrac{25}{512} h^2 F_2 - \tfrac{125}{6144} h^3 F_3 + \tfrac{125}{256} h f_4)$$

$$y_p = y_n + \frac{2053}{1625} h f_1 + \frac{257}{650} h^2 F_2 + \frac{1}{15} h^3 F_3 - \frac{28}{13} h f_4 + \frac{3072}{1625} h f_5$$

$$f_7 = f(t_n + h, y_p)$$

$$\begin{aligned} F_6 = & \frac{\partial}{\partial t} f(t_n + h, y_p) \\ & + \frac{\partial}{\partial y} f(t_n + h, y_p) \\ & \times \left(\frac{62298}{4225} f_1 + \frac{4566}{845} h F_2 + \frac{12}{13} h^2 F_3 - \frac{6168}{169} f_4 + \frac{100352}{4225} f_5 - f_7 \right) \end{aligned}$$

$$\begin{aligned} y_{n+1} = y_n & + \frac{2707}{8750} h f_1 + \frac{19}{500} h^2 F_2 + \frac{1}{525} h^3 F_3 + \frac{8}{35} h f_4 + \frac{32768}{118125} h f_5 \\ & - \frac{13}{1260} h^2 F_6 + \frac{349}{1890} h f_7 \end{aligned}$$

4. 数値例

$$\frac{dy}{dt} = \frac{e^t(y^3(t+1)+1)}{3y^2(6-te^t)}, \qquad y(0)=1$$

$t=1$ における累積誤差。16進14桁15切捨て計算

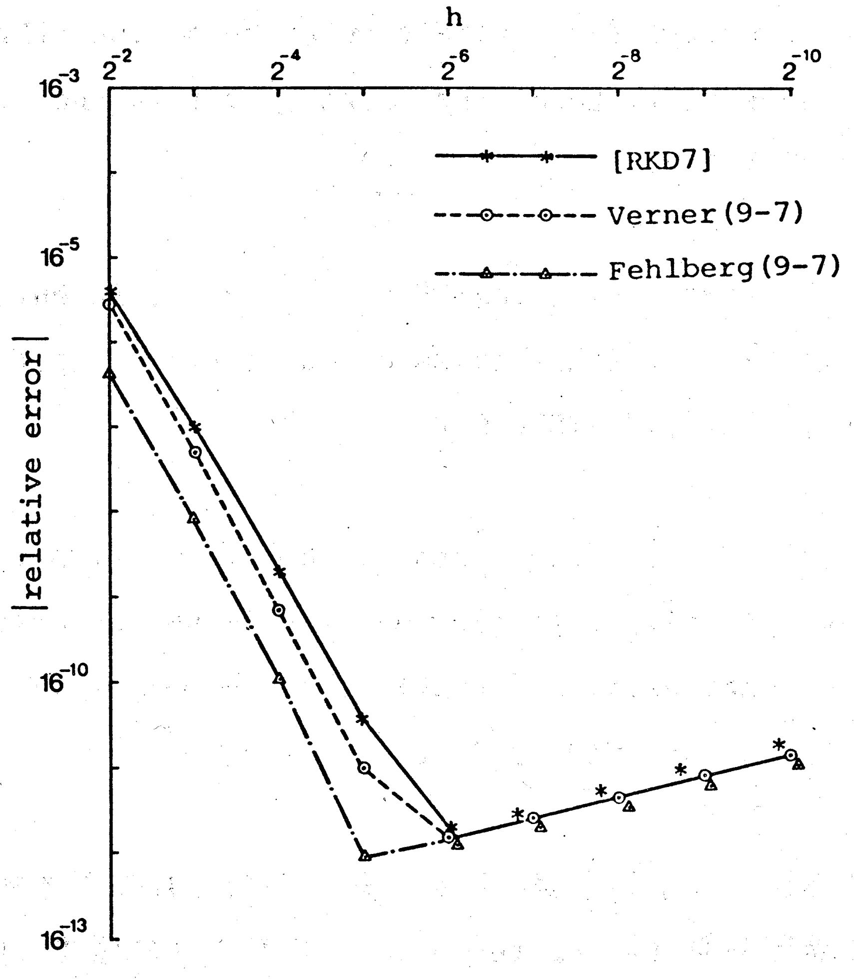

図2　$t=1$ における累積誤差の比較

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