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The Boltzmann Equation and Thirteen Moments

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1. Introduction

This is a summary of the author's recent paper [5].

We consider the initial value problem for the Boltzmann equation:

$$(1.1)\qquad F_t + v\cdot\nabla_x F = Q(F,F)\ ,$$

$$(1.2)\qquad F(0,x,v) = F_0(x,v)\ .$$

Here $F = F(t,x,v)$ denotes the mass density of gas molecules with velocity $v = (v_1,v_2,v_3) \in \mathbb{R}^3$ at time $t \geq 0$ and position $x = (x_1,x_2,x_3) \in \mathbb{R}^3$, ∇_x is the gradient with respect to x and $Q(F,F)$ is the term related to the binary collisions of molecules, which is given explicitly as follows:

$$(1.3)\qquad Q(F,G)(v) = \frac{1}{2}\iint_{S^2\times\mathbb{R}^3} q(\theta,|v_*-v|)\{F(v')G(v_*') + F(v_*')G(v') - F(v)G(v_*) - F(v_*)G(v)\}d\omega dv_*\ .$$

In (1.3) we use abbreviations such as $F(v) = F(t,x,v)$; v' and v_*' are molecular velocites which produce v and v_* after a collision, namely, $v' = v + ((v_*-v)\cdot\omega)\omega$, $v_*' = v_* - ((v_*-v)\cdot\omega)\omega$ for $\omega \in S^2$; $q(\theta,|v_*-v|)$ (where θ is defined by $(v_*-v)\cdot\omega = |v_*-v|\cos\theta$) is a function determined by the intermolecular potentials and is assumed to be of the cutoff hard type of Grad [2].

We study the problem concerning the existence of global solutions of (1.1),(1.2) in a neighborhood of a Maxwellian. The key of the problem is to get a suitable decay estimate for the linearized Boltzmann equation around the Maxwellian (see [6],[7]). In the previous works [6],[7], such a decay estimate was obtained by a method based on the spectral theory for the linearized Boltzmann operator investigated in [1]. Our aim is to show the same decay estimate by quite a different method. Our method is the so-called energy method and makes use of the matrix represetation of $v\cdot\xi$, the symbol of the streaming operator $v\cdot\nabla_x$, which maps the null space of the linearized collision operator into the subspace associated with the thirteen moments.

2. Preliminaries

We consider the problem (1.1),(1.2) in a neighborhood of the normalized Maxwellian $M = M(v)$:

(2.1) $$M(v) = (2\pi)^{-3/2}\exp(-|v|^2/2) .$$

M is an equilibrium of (1.1) since $Q(M,M) = 0$. Following Grad [2],[3], we introduce the new unknown function $f = f(t,x,v)$ by

(2.2) $$F = M + M^{1/2}f .$$

The problem (1.1),(1.2) is then transformed into

(2.3) $$f_t + v\cdot\nabla_x f + Lf = \Gamma(f,f) ,$$

(2.4) $$f(0,x,v) = f_0(x,v) .$$

Here $f_0(x,v) = M(v)^{-1/2}(F_0(x,v) - M(v))$ and

$$(2.5)\qquad \begin{aligned} Lf &= -\,2M^{-1/2}Q(M,M^{1/2}f)\ , \\ \Gamma(f,g) &= M^{-1/2}Q(M^{1/2}f,M^{1/2}g)\ . \end{aligned}$$

First we summarize some known properties of the linearized collision operator L. (For the details, see [2].) L is decomposed in the form

$$(2.6)\qquad Lf = \nu f - Kf\ ,$$

where $\nu = \nu(v)$ is the function satisfying $\nu_1 \le \nu(v) \le \nu_2(1+|v|)$ for positive constants ν_1, ν_2, and K is a compact selfadjoint operator on $L^2(v)$. Therefore L is a (unbounded) symmetric operator on $L^2(v)$. Also, L is nonnegative, namely, $(Lf,f) \ge 0$ for $f \in L^2(v)$ with $Lf \in L^2(v)$, where $(\ ,\ )$ is the standard inner product of $L^2(v)$. We denote the null space of L by $N(L)$. It is known that

$$(2.7)\qquad N(L) = \text{linear span of } \{\psi_1 M^{1/2},\cdots,\psi_5 M^{1/2}\}\ ,$$

where

$$(2.8)\qquad \psi_1 = 1\ ,\quad \psi_{j+1} = v_j\ ,\quad j=1,2,3,\quad \psi_5 = |v|^2\ .$$

(Recall that v_j is the j-th component of v.) Each ψ_k is called a summational invariant. The following five functions form an orthonormal basis of $N(L)$.

$$(2.9)\qquad e_1 = M^{1/2}\ ,\quad e_{j+1} = v_j M^{1/2}\ ,\quad j=1,2,3,\quad e_5 = \frac{1}{\sqrt{6}}\,(|v|^2 - 3)M^{1/2}\ .$$

From the properties of L stated above we deduce that for $f \in L^2(v)$ with $Lf \in L^2(v)$,

$$(2.10)\qquad (Lf,f) \ge \delta_1 |(I - P_0)f|_2^2\ ,$$

where δ_1 is a positive constant, $|\cdot|_2$ denotes the norm of $L^2(v)$, and P_0 is the orthogonal projection from $L^2(v)$ onto $N(L)$:

$$P_0 f = \sum_{k=1}^{5} (f, e_k) e_k \ .$$

We remark that (2.10) holds true also for $f \in L_1^2(v)$, where $L_1^2(v)$ is the space of functions $f \in L^2(v)$ such that $(1+|v|)f \in L^2(v)$.

Next we introduce the following thirteen functions.

$$\phi_1 = 1 \ , \quad \phi_{j+1} = v_j \ , \quad j=1,2,3, \quad \phi_{j+4} = v_j^2 \ , \quad j=1,2,3,$$

$$\text{(2.11)} \qquad \phi_8 = v_1 v_2 \ , \quad \phi_9 = v_2 v_3 \ , \quad \phi_{10} = v_3 v_1 \ ,$$

$$\phi_{j+10} = |v|^2 v_j \ , \quad j=1,2,3.$$

Notice that each summational invariant in (2.8) is a linear combination of the above functions: $\psi_k = \phi_k$, $k=1,\cdots,4$, and $\psi_5 = \phi_5 + \phi_6 + \phi_7$. We denote by W the subspace of $L^2(v)$ spanned by the thirteen functions $\phi_k M^{1/2}$, $k=1,\cdots,13$, namely,

$$\text{(2.12)} \qquad W = \text{linear span of } \{\phi_1 M^{1/2}, \cdots, \phi_{13} M^{1/2}\} \ .$$

W is the subspace associated with the thirteen moments, since the quantities $\int \phi_k F \, dv = (M^{1/2} + f, \phi_k M^{1/2})$ are called moments of the distribution function F. We shall introduce an orthonormal basis of W. Since $N(L) \subset W$ and the five functions $e_1, \cdots, e_5$ given by (2.9) form an orthonormal basis of $N(L)$, we choose additional eight functions $e_6, \cdots, e_{13}$ such that $\{e_1, \cdots, e_{13}\}$ becomes an orthonormal basis of W. They are given as follows:

$$e_{k+4} = \sum_{j=1}^{3} c_{kj} \tilde{e}_{j+4} \ , \quad k=2,3,$$

$$\text{(2.13)} \qquad e_8 = v_1 v_2 M^{1/2} \ , \quad e_9 = v_2 v_3 M^{1/2} \ , \quad e_{10} = v_3 v_1 M^{1/2} \ ,$$

$$e_{j+10} = \frac{1}{\sqrt{10}} (|v|^2 - 5)v_j M^{1/2}, \quad j = 1,2,3.$$

Here

(2.14) $$\tilde{e}_{j+4} = \frac{1}{\sqrt{2}} (v_j^2 - 1)M^{1/2}, \quad j = 1,2,3,$$

and the coefficients c_{kj} are chosen such that the three vectors $c_1 = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $c_2 = (c_{21}, c_{22}, c_{23})$ and $c_3 = (c_{31}, c_{32}, c_{33})$ form an orthonormal basis of $\mathbb{R}^3$. This choice of c_{kj} is based on the following observation: The three functions in (2.14) form an orthonormal system of $L^2(v)$ and $e_5 = (\tilde{e}_5 + \tilde{e}_6 + \tilde{e}_7)/\sqrt{3}$.

Now we consider $v \cdot \xi$ ($\xi \in \mathbf{R}^3$), the symbol of the streaming operator $v \cdot \nabla_x$, on the null space $N(L)$. For each $\xi \in \mathbf{R}^3$, $v \cdot \xi$ is regarded as a linear operator from $N(L)$ into W, and therefore can be represented by the 13×5 matrix with the entries $((v \cdot \xi)e_k, e_\ell)$, $1 \le k \le 13$, $1 \le \ell \le 5$. Hence we introduce for $\xi \in \mathbf{R}^3$,

(2.15) $$V(\xi) = (((v \cdot \xi)e_k, e_\ell))_{1 \le k, \ell \le 13},$$

which is a real symmetric matrix. Consider the decomposition

(2.16) $$V(\xi) = \begin{pmatrix} V_{11}(\xi) & V_{12}(\xi) \\ V_{21}(\xi) & V_{22}(\xi) \end{pmatrix},$$

where $V_{11}(\xi)$, $V_{12}(\xi)$, $V_{21}(\xi)$ and $V_{22}(\xi)$ are 5×5, 5×8, 8×5 and 8×8 matrices, respectively. We have $V_{11}(\xi)^T = V_{11}(\xi)$, $V_{12}(\xi)^T = V_{21}(\xi)$ and $V_{22}(\xi)^T = V_{22}(\xi)$, where the superscript T denotes transpose. By straightfoward calculation, using (2.9) and (2.13), we have the following expressions:

$$(2.17)\qquad V_{11}(\xi) = \left(\begin{array}{c|ccc|c} 0 & \xi_1 & \xi_2 & \xi_3 & 0 \\ \hline \xi_1 & & & & a_1\xi_1 \\ \xi_2 & & 0 & & a_1\xi_2 \\ \xi_3 & & & & a_1\xi_3 \\ \hline 0 & a_1\xi_1 & a_1\xi_2 & a_1\xi_3 & 0 \end{array}\right)$$

$$(2.18)\qquad V_{12}(\xi) = \left(\begin{array}{c|ccc|c} 0 & a_{21}\xi_1 & a_{22}\xi_2 & a_{23}\xi_3 & 0 \\ 0 & a_{31}\xi_1 & a_{32}\xi_2 & a_{33}\xi_3 & 0 \\ \hline 0 & \xi_2 & \xi_1 & 0 & 0 \\ 0 & 0 & \xi_3 & \xi_2 & 0 \\ 0 & \xi_3 & 0 & \xi_1 & 0 \\ \hline 0 & & & & a_4\xi_1 \\ 0 & & 0 & & a_4\xi_2 \\ 0 & & & & a_4\xi_3 \end{array}\right)$$

where $\xi = (\xi_1,\xi_2,\xi_3)$, $a_1 = \sqrt{2/3}$, $a_{kj} = \sqrt{2}c_{kj}$, $k=2,3$, $j=1,2,3$, and $a_4 = \sqrt{3/5}$.

3. Construction of a compensating function

We introduce the notion of a compensating function for the Boltzmann equation (2.3). Let $\mathbb{B}(L^2(v))$ be the Banach space of bounded linear operators on $L^2(v)$, with the operator norm.

Definition 3.1. Let $S(\omega)$ be a bounded linear operator on $L^2(v)$ with a parameter $\omega \in S^2$, i.e., $S(\omega) \in \mathbb{B}(L^2(v))$ for each $\omega \in S^2$. $S(\omega)$ is called a compensating function for the Boltzmann equation (2.3), if the following conditions are satisfied:

(i) $S(\cdot) \in C^\infty(S^2; B(L^2(v)))$ and $S(-\omega) = -S(\omega)$ for each $\omega \in S^2$.

(ii) $iS(\omega)$ is a selfadjoint operator on $L^2(v)$ for each $\omega \in S^2$.

(iii) There exists a positive constant δ such that for any $\omega \in S^2$ and $f \in L^2_1(v)$, the following inequality holds.

$$\mathrm{Re}\,(S(\omega)(v\cdot\omega)f,f) + (Lf,f) \geq \delta|f|_2^2 .$$

In order to show the existence of a compensating function for the Boltzmann equation, we prepare the following

Lemma 3.1. *There exist matrices* R^j, $j=1,2,3$, *which satisfy the following properties: Each* R^j *is a* 13×13 *real skew-symmetric matrix with constant entries. Moreover, there exist positive constants* c_1 *and* C_1 *such that for any* $\omega \in S^2$ *and* $w = (w_1,\cdots,w_{13})^T \in \mathbb{C}^{13}$,

$$\text{(3.1)} \qquad \mathrm{Re} < R(\omega)V(\omega)w,w > \;\geq c_1|w_I|^2 - C_1|w_{II}|^2 ,$$

where $R(\omega) = \sum R^j\omega_j$ *for* $\omega = (\omega_1,\omega_2,\omega_3)$, $V(\omega)$ *is the matrix defined by* (2.15) *with* ξ *replaced by* ω, $w_I = (w_1,\cdots,w_5)^T$, $w_{II} = (w_6,\cdots,w_{13})^T$, *and* $<\,,\,>$ *denotes the standard inner product of* $\mathbb{C}^{13}$.

Proof. We define R^j, $j=1,2,3$, by

$$\text{(3.2)} \qquad \sum_{j=1}^{3} R^j\xi_j = R(\xi) = \begin{pmatrix} \alpha\tilde{R}_{11}(\xi) & V_{12}(\xi) \\ -V_{21}(\xi) & 0 \end{pmatrix} ,$$

where α is a positive constant which will be determined later, $V_{12}(\xi)$ and $V_{21}(\xi)$ are the matrices in (2.16), and

$$(3.3)\qquad \tilde{R}_{11}(\xi) = \left(\begin{array}{c|ccc|c} 0 & \xi_1 & \xi_2 & \xi_3 & 0 \\ \hline -\xi_1 & & & & 0 \\ -\xi_2 & & 0 & & 0 \\ -\xi_3 & & & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

By the definition, each R^j is a 13×13 real skew-symmetric matrix with constant entries. We shall show (3.1). Put $U(\xi) = R(\xi)V(\xi)$ and let $U(\xi) = (U_{pq}(\xi))_{1\le p,q\le 2}$ be the decomposition of the same type as in (2.16). From (2.16) and (3.2) we have

$$U_{11}(\xi) = \alpha\tilde{R}_{11}(\xi)V_{11}(\xi) + V_{12}(\xi)V_{21}(\xi).$$

By a simple calculation, using (2.7) and (3.3), we know that for $\omega \in S^2$ and $w_I = (w_1,\cdots,w_5)^T \in \mathbb{C}^5$,

$$(3.4)\qquad \mathrm{Re} < \tilde{R}_{11}(\omega)V_{11}(\omega)w_I, w_I > \ \ge c_2|w_1|^2 - C_2\sum_{k=2}^{5}|w_k|^2 ,$$

where c_2 and C_2 are positive constants. On the other hand, it follows from (2.18) that $\mathrm{rank}\, V_{21}(\omega) = 4$ for any $\omega \in S^2$ and hence

$$(3.5)\qquad < V_{12}(\omega)V_{21}(\omega)w_I, w_I > \ = |V_{21}(\omega)w_I|^2 \ge c_3\sum_{k=2}^{5}|w_k|^2 ,$$

where c_3 is a positive constant. We multiply (3.4) by $\alpha > 0$ and then add the resulting inequality to (3.5). Choosing α such that $\alpha C_2 = c_3/2$, we obtain

$$(3.6)\qquad \mathrm{Re} < U_{11}(\omega)w_I, w_I > \ \ge c|w_I|^2$$

with $c = \min\{\alpha c_2, c_3/2\}$. The desired estimate (3.1) is an easy consequence of (3.6). Therefore the proof of Lemma 3.1 is complete.

We denote the components of the matrix $R(\omega)$ in Lemma 3.1 by $r_{k\ell}(\omega)$, $k,\ell = 1,\cdots,13$, and define the operator $S(\omega)$ with a parameter $\omega \in S^2$ by

$$(3.7) \qquad S(\omega)f = \sum_{k,\ell=1}^{13} \beta r_{k\ell}(\omega)(f,e_\ell)e_k , \qquad f \in L^2(v) ,$$

where β is a positive constant. We shall show that the above $S(\omega)$ is a compensating function for the Boltzmann equation.

Proposition 3.2. *The operator* $S(\omega)$ *defined by* (3.7) *is a compensating function for the Boltzmann equation, provided that* $\beta > 0$ *is sufficiently small. Moreover, for each* $\omega \in S^2$, $S(\omega)$ *maps* $L^2(v)$ *into the subspace* W *defined by* (2.12).

Proof. Since $\{e_1,\cdots,e_{13}\}$ is an orthonormal basis of W, the last statement of the proposition is obvious from the definition (3.7). We shall check conditions (i), (ii) and (iii) of Definition 3.1. Condition (i) is an easy consequence of $R(\omega) = \sum R^j\omega_j$. Let $f, g \in L^2(v)$. We have from (3.7),

$$(3.8) \qquad (S(\omega)f,g) = \sum_{k,\ell=1}^{13} \beta r_{k\ell}(\omega)(f,e_\ell)\overline{(g,e_k)} .$$

Let w and u be the vectors in $\mathbb{C}^{13}$ whose k-th components are (f,e_k) and (g,e_k), respectively. The equality (3.8) then gives $(S(\omega)f,g) = \beta < R(\omega)w,u >$, where $< , >$ is the standard inner product of $\mathbb{C}^{13}$. This relation shows that $iS(\omega)$ is a selfadjoint operator on $L^2(v)$, since $R(\omega)$ is real skew-symmetric. Thus condition (ii) is verified. Finally, we check condition (iii). Let $f \in L^2_1(v)$. From (3.8) we have

$$(3.9) \qquad (S(\omega)(v\cdot\omega)f,f) = \sum_{k,\ell=1}^{13} \beta r_{k\ell}(\omega)((v\cdot\omega)f,e_\ell)\overline{(f,e_k)} .$$

We denote the orthogonal projection from $L^2(v)$ onto W by P, namely,

$$Pf = \sum_{k=1}^{13} (f,e_k)e_k .$$

We substitute the decomposition $f = Pf + (I-P)f$ into the right hand side of (3.9) to obtain

$$(3.10) \qquad (S(\omega)(v\cdot\omega)f,f) = \beta < R(\omega)V(\omega)w,w > + \sum_{k,\ell=1}^{13} \beta r_{k\ell}(\omega)((I-P)f,(v\cdot\omega)e_\ell)\overline{(f,e_k)} ,$$

where $V(\omega)$ is the matrix defined by (2.15), and w is the vector in $\mathbb{C}^{13}$ whose k-th component is (f,e_k). The second term on the right hand side of (3.10) is bounded by $\beta C|(I-P_0)f|_2|f|_2$, where C is a constant independent of β and P_0 is the orthogonal projection from $L^2(v)$ onto $N(L)$. On the other hand, by virtue of Lemma 3.1, the real part of the first term on the right side of (3.10) is bounded from below by $\beta c_1|P_0f|_2^2 - \beta C_1|(I-P_0)f|_2^2$, where c_1 and C_1 are the positive constants in (3.1) and hence do not depend on β. Therefore we obtain

$$(3.11) \qquad \mathrm{Re}\,(S(\omega)(v\cdot\omega)f,f) \geq \beta(c_1-\varepsilon)|P_0f|_2^2 - \beta C_\varepsilon|(I-P_0)f|_2^2$$

for any $\varepsilon > 0$, where C_ε is a constant depending on ε but not on β. We add (2.10) to (3.11) and choose ε and β such that $\varepsilon = c_1/2$ and $\beta C_\varepsilon = \delta_1/2$. Then we get the inequality

$$(3.12) \qquad \mathrm{Re}\,(S(\omega)(v\cdot\omega)f,f) + (Lf,f) \geq \delta_2|f|_2^2$$

with $\delta_2 = \min\{\beta c_1/2, \delta_1/2\}$. Thus condition (iii) has been checked. This completes the proof of Proposition 3.2.

4. Decay estimate for the linearized equation

We consider the initial value problem for the linearized Boltzmann equation:

(4.1) $f_t + v\cdot\nabla_x f + Lf = g$,

(4.2) $f(0,x,v) = f_0(x,v)$,

where g is a given function of $(t,x,v) \in [0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3$. Our aim is to show a decay estimate of solutions of (4.1),(4.2) by an energy method similar to the one employed in [4] (see also [8]) for the discrete Boltzmann equation. Our method is based on the existence of a compensating function for the Boltzmann equation.

Let us introduce function spaces. $H^\ell(x)$ denotes the usual Sobolev space on $\mathbb{R}^3_x$ of order ℓ. We denote by $\mathbb{H}^\ell$ the space of $L^2(v)$-functions with values in $H^\ell(x)$, with the norm $\|\cdot\|_\ell$. $\mathbb{H}^\ell_1$ is the space of $L^2_1(v)$-functions with values in $H^\ell(x)$. $L^{p,2}$ denotes the space of $L^2(v)$-functions with values in $L^p(x)$. The norm of $L^{p,2}$ is denoted by $[\![\cdot]\!]_{p,2}$.

Our result is then stated as follows.

Theorem 4.1. *Let* $\ell \geq 0$ *and* $p, q \in [1,2]$. *Suppose that* $f_0 \in \mathbb{H}^\ell \cap L^{p,2}$. *Moreover we assume that* $g \in L^\infty([0,\infty);\mathbb{H}^\ell \cap L^{q,2})$ *and* $(P_0 g)(t,x,v) = 0$ *for* $(t,x,v) \in [0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3$, *where* P_0 *is the orthogonal projection from* $L^2(v)$ *onto* $N(L)$. *Let* f *be a solution of the problem* (4.1),(4.2) *satisfying* $f \in L^\infty([0,\infty);\mathbb{H}^\ell_1)$ *and* $f_t \in L^\infty([0,\infty);\mathbb{H}^{\ell-1})$. *Then we have*

$$\|f(t)\|_\ell^2 \leq C(1+t)^{-2\gamma}(\|f_0\|_\ell + [\![f_0]\!]_{p,2})^2 + \tag{4.3}$$
$$+ C\int_0^t (1+t-\tau)^{-2\gamma'}(\|g(\tau)\|_\ell + [\![g(\tau)]\!]_{q,2})^2 d\tau$$

for $t \in [0,\infty)$, *where* $\gamma = (3/2)(1/p - 1/2)$, $\gamma' = (3/2)(1/q - 1/2)$ *and* C *is a constant.*

The decay estimate (4.3) with $g = 0$ has been obtained in [6],[7] by a method based on the spectral theory for the linearized Boltzmann operator.

In order to prove Theorem 4.1, we consider (4.1),(4.2) in the Fourier transform:

$$(4.4) \qquad \hat{f}_t + i|\xi|(v\cdot\omega)\hat{f} + L\hat{f} = \hat{g}\ , \qquad \omega = \xi/|\xi| \in S^2\ ,$$

$$(4.5) \qquad \hat{f}(0,\xi,v) = \hat{f}_0(\xi,v)\ ,$$

where $\hat{f} = \hat{f}(t,\xi,v)$ denotes the Fourier transform of $f = f(t,x,v)$. Let $S(\omega)$ be the compensating function for the Boltzmann equation constructed in Proposition 3.2 and let μ be a positive constant. Put

$$(4.6) \qquad E[\hat{f}](t,\xi) = |\hat{f}(t,\xi)|_2^2 - \frac{\mu|\xi|}{1+|\xi|^2}(iS(\omega)\hat{f}(t,\xi),\hat{f}(t,\xi))\ , \qquad \omega = \xi/|\xi|.$$

We shall show that for a suitably chosen $\mu > 0$, $E[\hat{f}]$ is a Ljapunov function of (4.4), which is regarded as an ordinary differential equation in $L^2(v)$ with a parameter $\xi \in \mathbb{R}^3$. More precisely we have

<u>Lemma 4.2</u>. *For a suitably chosen* $\mu > 0$, *the function* $E[\hat{f}]$ *defined by* (4.6) *satisfies the following inequalities.*

$$(4.7) \qquad \frac{1}{2}|\hat{f}|_2^2 \le E[\hat{f}] \le 2|\hat{f}|_2^2\ ,$$

$$(4.8) \qquad \frac{\partial}{\partial t}E[\hat{f}] + \delta\rho(\xi)E[\hat{f}] \le C|\hat{g}|_2^2\ ,$$

where δ *and* C *are positive constants, and* $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$.

Theorem 4.1 can be proved by using Lemma 4.2. In fact, applying Gronwall's inequality to (4.8) and using (4.7), we obtain for $(t,\xi) \in [0,\infty)\times\mathbb{R}^3$,

$$(4.9) \qquad |\hat{f}(t,\xi)|_2^2 \le Ce^{-\delta\rho(\xi)t}|\hat{f}_0(\xi)|_2^2 + C\int_0^t e^{-\delta\rho(\xi)(t-\tau)}|\hat{g}(\tau,\xi)|_2^2\,d\tau\ ,$$

where δ is the constant in (4.8) and C is some constant. The desired estimate (4.3) is then obtained from (4.9) by the standard technique and so we omit the arguments. See, for example, [6] or [8].

<u>Proof of Lemma 4.2.</u> We first note that (4.7) holds true for sufficiently small $\mu > 0$, say, $\mu \in (0,\mu_1]$. To show (4.8), we use the argument analogous to that employed in [4],[8]. We take the inner product (of $L^2(v)$) of (4.4) and $\hat{f}$. Its real part is

$$(4.10)\qquad (\tfrac{1}{2}|\hat{f}|_2^2)_t + (L\hat{f},\hat{f}) = \mathrm{Re}\,(\hat{g},\hat{f}) .$$

We apply $-i|\xi|S(\omega)$, $\omega = \xi/|\xi|$, to (4.4) and then take the inner product with $\hat{f}$. Since $iS(\omega)$ is a selfadjoint operator, the real part of the resulting equality is

$$(4.11)\qquad \{-\tfrac{1}{2}|\xi|(iS(\omega)\hat{f},\hat{f})\}_t + |\xi|^2\mathrm{Re}\,(S(\omega)(v\cdot\omega)\hat{f},\hat{f})$$
$$= |\xi|\,\mathrm{Re}\,\{(iS(\omega)L\hat{f},\hat{f}) - (iS(\omega)\hat{g},\hat{f})\} .$$

We calculate $(4.10)\times(1+|\xi|^2)+(4.11)\times\mu$ with a positive constant μ to obtain

$$(4.12)\qquad \{\tfrac{1}{2}(1+|\xi|^2)E[\hat{f}]\}_t + \{1+(1-\mu)|\xi|^2\}(L\hat{f},\hat{f}) +$$
$$+ \mu|\xi|^2\{\mathrm{Re}\,(S(\omega)(v\cdot\omega)\hat{f},\hat{f}) + (L\hat{f},\hat{f})\}$$
$$= (1+|\xi|^2)\,\mathrm{Re}\,(\hat{g},\hat{f}) + \mu|\xi|\,\mathrm{Re}\,\{(iS(\omega)L\hat{f},\hat{f}) - (iS(\omega)\hat{g},\hat{f})\} ,$$

where $E[\hat{f}]$ is the function defined by (4.6). We assume that $\mu \in (0,1]$. Then the second term on the left hand side of (4.12) is bounded from below by $(1-\mu)(1+|\xi|^2)(L\hat{f},\hat{f}) \geq (1-\mu)\delta_1(1+|\xi|^2)|(I-P_0)\hat{f}|_2^2$, where we used (2.10). On the other hand, by virtue of (3.12), the third term on the left side is bounded from below by $\mu\delta_2|\xi|^2|\hat{f}|_2^2$. Therefore we have the following lower bound for the left side of (4.12).

(4.13) $\{\frac{1}{2}(1+|\xi|^2)E[\hat{f}]\}_t + (1-\mu)\delta_1(1+|\xi|^2)|(I-P_0)\hat{f}|_2^2 + \mu\delta_2|\xi|^2|\hat{f}|_2^2$.

Next we estimate the right side of (4.12). Since $P_0\hat{g} = 0$ by the assumption, the first term on the right side of (4.12) is majorized by $(1+|\xi|^2)|(I-P_0)\hat{f}|_2|\hat{g}|_2$. Also, from (3.8), we see that the second term on the right side is estimated by $\mu C|\xi||\hat{f}|_2(|(I-P_0)\hat{f}|_2 + |\hat{g}|_2)$, where C is a constant independent of μ. Therefore, the right side of (4.12) is bounded by

(4.14) $(\varepsilon+\mu C_\varepsilon)(1+|\xi|^2)|(I-P_0)\hat{f}|_2^2 + \mu\varepsilon|\xi|^2|\hat{f}|_2^2 + C_\varepsilon(1+|\xi|^2)|\hat{g}|_2^2$

for any $\varepsilon > 0$, where C_ε is a constant depending on ε but not on $\mu \in (0,1]$. We choose ε and μ_2 such that $\varepsilon = \min\{\delta_1/6, \delta_2/2\}$ and $\mu_2 = \min\{1/6, \delta_1/6C_\varepsilon\}$. Then we have for $\mu \in (0,\mu_2]$,

(4.15) $\frac{\partial}{\partial t}E[\hat{f}] + \delta_1|(I-P_0)\hat{f}|_2^2 + \mu\delta_2\rho(\xi)|\hat{f}|_2^2 \le C|\hat{g}|_2^2$,

where C is a constant. Now we put $\mu = \min\{\mu_1, \mu_2\}$. For this choice of μ, the inequality (4.15) combined with (4.7) gives (4.8) with $\delta = \mu\delta_2/2$. This completes the proof of Lemma 4.2.

5. Global solutions of the nonlinear equation

With the aid of the decay estimate (4.3) we can show the existence of global solutions to the problem (2.3),(2.4) in the same way as in [6],[7]. Of course the result obtained is the same as that in [6],[7].

Theorem 5.1. *Let* $\ell > 3/2$, $\beta > 5/2$ *and* $p \in [1,2]$. *We assume that* $f_0 \in \dot{\mathbb{B}}^\ell_\beta \cap L^{p,2}$. *If* $\|f_0\|_{\ell,\beta} + [\![f_0]\!]_{p,2}$ *is suitably small, then the problem* (2.3),(2.4) *has a unique global solution* f *in* $C^0([0,\infty);\dot{\mathbb{B}}^\ell_\beta) \cap C^1([0,\infty);\dot{\mathbb{B}}^{\ell-1}_{\beta-1})$. *Moreover, the solution satisfies*

(5.1) $\|f(t)\|_{\ell,\beta} \leq C(1+t)^{-\gamma}(\|f_0\|_{\ell,\beta} + [\![f_0]\!]_{p,2})$

for $t \in [0,\infty)$, *where* $\gamma = (3/2)(1/p - 1/2)$ *and* C *is a constant.*

Here we have employed the following notations: $\dot{\mathbb{B}}^{\ell}_{\beta}$ is the space of $\dot{L}^{\infty}_{\beta}(v)$-functions with values in $H^{\ell}(x)$, where $\dot{L}^{\infty}_{\beta}(v)$ denotes the space of functions $f = f(v)$ such that $(1+|v|)^{\beta}f \in L^{\infty}(v)$ and $(1+|v|)^{\beta}f(v) \to 0$ uniformly as $|v| \to \infty$. The norm of $\dot{\mathbb{B}}^{\ell}_{\beta}$ is denoted by $\|\cdot\|_{\ell,\beta}$.

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