

# On Gevrey Singularities of solutions of equations with non symplectic characteristics

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In this note we shall construct parametrices for a specific class of differential operators with non symplectic characteristics and clarify the structure of Gevrey singularities of solutions of the corresponding equations using constructed parametrices.

## 0. Notation and preliminaries

If  $X$  is an open set of  $\mathbb{R}^N$  and  $\nu \geq 1$ , the Gevrey class of order  $\nu$ ; which we denote by  $G^\nu(X)$ , is the set of all  $u \in C^\infty(X)$  such that for every compact set  $K \subset X$  there is a constant  $C_K$  with

$$|\partial_x^\alpha u(x)| \leq C_K^{|\alpha|+1} (\alpha!)^\nu, \quad x \in K,$$

for all multi-indices  $\alpha \in \mathbb{N}^N$ .

We use the following definition of the Gevrey wave front set given by Hörmander [14].

**Definition 0.1.** If  $X \subset \mathbb{R}^N$  and  $u \in \mathcal{D}'(X)$  we denote by  $WF_\nu(u)$  the complement in  $T^*(X) \setminus 0$  of the set of  $(\hat{x}, \hat{\xi})$  such that there exist a neighborhood  $U \subset X$  of  $\hat{x}$ , a conic neighborhood  $V \subset \mathbb{R}^N \setminus 0$  of  $\hat{\xi}$  and a bounded sequence  $u_k \in \mathcal{D}'(X)$  which is equal to  $u$  in  $U$  and satisfies

$$|\hat{u}_k(\xi)| \leq C^{k+1} (k^\nu / |\xi|)^k, \quad k = 1, 2, \dots$$

for some constant  $C$  when  $\xi \in V$ , where  $\hat{u}_k$  denotes the Fourier transform of  $u_k$ .

$WF_1(u)$  is also denoted by  $WF_A(u)$  since this is one of the

definition of the analytic wave front set known to be equivalent to the others; see e.g. Bony [3].

If  $\pi$  denotes the canonical projection of  $T^*(X)\setminus 0$  on  $X$  then  $u \in G^V(X\setminus\pi(WF_V(u)))$  and for a differential operator  $P$  with analytic coefficients, we have

$$WF_V(Pu) \subset WF_V(u) \subset \text{Char } P \cup WF_V(Pu),$$

where  $\text{Char } P$  denotes the characteristic set of  $P$ . We say that  $P$  is  $G^V$  microhypoelliptic at  $(\hat{x}, \hat{\xi})$  if there is a conic neighborhood  $V \subset T^*(X)\setminus 0$  of  $(\hat{x}, \hat{\xi})$  such that

$$WF_V(Pu) \cap V = WF_V(u) \cap V.$$

### 1. Statement of the results

Let  $\Sigma$  be the submanifold in  $T^*(\mathbb{R}^N)\setminus 0$  of codimension  $2d+d'$  given by

$$\Sigma = \{(x, \xi) \in T^*(\mathbb{R}^N)\setminus 0; x_1 = \dots = x_d = 0, \xi_1 = \dots = \xi_{d+d'} = 0\},$$

where  $0 < d < d+d' < N$ . With this  $\Sigma$  we set

$$\mathbb{R}_x^N = \mathbb{R}_t^d \times \mathbb{R}_y^n = \mathbb{R}_t^d \times \mathbb{R}_{y'}^{d'} \times \mathbb{R}_{y''}^{d''} \quad (d+n=N, d'+d''=n)$$

and denote by  $\xi = (\tau, \eta) = (\tau, \eta', \eta'')$  the dual variables of  $x = (t, y) = (t, y', y'') \in \mathbb{R}_t^d \times \mathbb{R}_{y'}^{d'} \times \mathbb{R}_{y''}^{d''}$ . (In this coordinate  $\Sigma = \{(t, y, \tau, \eta', \eta''); t = \tau = \eta' = 0, \eta'' \neq 0\}$ .)

For a fixed integer  $h \geq 1$  we shall consider a differential operator of order  $m$  with polynomial coefficients of the form:

$$(1.1) \quad P = p(t, D_t, D_y) = \sum_{\substack{|\alpha|+|\beta| \leq m \\ |\gamma| = |\alpha|+|\beta'|+(1+h)|\beta''|-m}} a_{\alpha\beta\gamma} t^\gamma D_y^\beta D_t^\alpha,$$

where  $(\alpha, \beta, \gamma) = (\alpha, \beta', \beta'', \gamma) \in \mathbb{N}^d \times \mathbb{N}^{d'} \times \mathbb{N}^{d''} \times \mathbb{N}^d$  and  $(D_t, D_y) = (-i\partial_t, -i\partial_y)$ . Note that the symbol  $p(t, \tau, \eta)$  has the following quasi-homogeneity:

$$(1.2) \quad p(t/\lambda^\rho, \lambda^\rho \tau, \lambda^\rho \eta', \lambda \eta'') = \lambda^{\rho m} p(t, \tau, \eta', \eta''), \quad \lambda > 0$$

with  $\rho = 1/(1+h)$ .

Let  $p_0$  denote the principal symbol given by

$$(1.3) \quad p_0(t, \tau, \eta) = \sum_{\substack{|\alpha|+|\beta|=m \\ |\gamma|=h|\beta|}} a_{\alpha\beta\gamma} t^\gamma \eta^\beta \tau^\alpha.$$

For a point  $(\hat{x}, \hat{\xi}) = (0, \hat{y}; 0, 0, \hat{\eta}'') \in \Sigma$  ( $|\hat{\eta}''| \neq 0$ ) we suppose:

(H-1) There exists a constant  $c > 0$  such that

$$|p_0(t, \tau, \eta', \hat{\eta}'')| \geq c(|\tau| + |\eta'| + |t|^h)^m, \quad (t, \tau, \eta') \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d'}.$$

We also consider the following condition due to Grusin.

(H-2) For all  $\eta' \in \mathbb{R}^{d'}$ ,  $\text{Ker } p(t, D_t, \eta', \hat{\eta}'') \cap \mathcal{G}(\mathbb{R}_t^d) = \{0\}$ .

Here  $p(t, D_t, \eta', \hat{\eta}'')$  is considered as an operator acting on  $\mathcal{G}(\mathbb{R}_t^d)$  with a parameter  $\eta' \in \mathbb{R}^{d'}$ .

*Remark.* If  $h = 1$ , (H-2) is known to be equivalent to  $C^\infty$  micro-hypoellipticity with loss of  $m/2$  derivatives; see e.g. Boutet de Monvel-Grigis-Helffer [4], see also Grusin [10],[11],[12] and the other authors [8],[15],[28].

**Theorem I.** Let  $P$  be an operator of the form (1.1) satisfying (H-1),

(H-2) for  $(\hat{x}, \hat{\xi}) \in \Sigma$ . If  $\nu \geq 1+h$  then  $P$  is  $G^\nu$  micro-hypoelliptic at  $(\hat{x}, \hat{\xi})$ .

The condition  $\nu \geq 1+h$  is the best in the sense that

**Theorem II.** Let  $P$  be an operator of the form:

$$(1.4) \quad P = p'(t, D_t, D_{y''}) + q(D_{y'}) \\ = \sum_{\substack{|\alpha|+|\beta''| \leq m \\ |\gamma| = |\alpha| + (1+h)|\beta''| - m}} a_{\alpha\beta''\gamma} t^\gamma D_{y''}^{\beta''} D_t^\alpha + \sum_{|\beta'|=m} b_{\beta'} D_{y'}^{\beta'}$$

satisfying (H-1) for  $(0, \dot{\xi}) \in \Sigma$ . Then one can find a neighborhood  $U$  of the origin in  $\mathbb{R}^N$  and a solution  $u \in C^\infty(U)$  of  $Pu = 0$  in  $U$  such that for every  $\nu < 1+h$

$$(1.5) \quad (0, \dot{\xi}) \in WF_\nu(u) \subset WF_A(u) \subset \{(x, \lambda \dot{\xi}); x \in U, \lambda > 0\}.$$

If  $1 \leq \nu < 1+h$  we can get a result on propagation of singularities of solutions for these operators.

Let  $\Lambda$  be the involutive submanifold of  $T^*(\mathbb{R}^N) \setminus 0$  containing  $\Sigma$  given by

$$\Lambda = \{(t, y; \tau, \eta', \eta'') \in T^*(\mathbb{R}^N) \setminus 0; \eta' = 0\}.$$

Then in the canonical way  $\Lambda$  defines a bicharacteristic foliation in  $\Sigma$  as well as in  $\Lambda$ ; that is, each leaf  $\Gamma_0$  is an integral submanifold of dimension  $d'$  of the vector fields generated by  $\{\partial_{y_1}, \dots, \partial_{y_{d'}}\}$ . (Note that  $T_\rho(\Gamma_0) = T_\rho(\Sigma) \cap T_\rho(\Sigma)^\perp$  for all  $\rho \in \Gamma_0$ .)

**Theorem III.** *Let  $\Gamma_0$  be the bicharacteristic leaf passing through  $(\dot{x}, \dot{\xi}) \in \Sigma$  defined as above and  $W$  be an open set containing  $(\dot{x}, \dot{\xi})$  such that  $\Gamma_0 \cap W$  is connected. Suppose that  $P$  is an operator of the form (1.1) satisfying (H-1) for  $(\dot{x}, \dot{\xi})$  and that  $1 \leq \nu < 1+h$ . If  $u \in \mathcal{D}'(\mathbb{R}^N)$  and  $WF_\nu(Pu) \cap \Gamma_0 \cap W = \emptyset$  then either  $\Gamma_0 \cap W \cap WF_\nu(u) = \emptyset$  or  $\Gamma_0 \cap W \subset WF_\nu(u)$ .*

*Remark.* If  $h = 1$  and  $\nu = 1$  this is a special case of Theorem 2 in Grigis-Schapira-Sjöstrand [9]. See also Sjöstrand [29],[30] and Hasegawa [13] in this connexion.

*Example.* Let

$$(1.6) \quad P = \sum_{j=1}^{N-1} \partial_{x_j}^2 + \sum_{j=1}^d x_j^{2h} \partial_{x_N}^2 + h \sum_{j=1}^d c_j x_j^{h-1} \partial_{x_N},$$

where  $c = (c_1, \dots, c_d) \in \mathbb{C}^d$ . If

$$(1.7) \quad \sup_{\substack{\langle \sigma, \text{Im} c \rangle = 0 \\ |\sigma|_1 = 1}} |\langle \sigma, \text{Im} c \rangle| < 1 \quad (\sigma \in \mathbb{R}^d, |\sigma|_1 = |\sigma_1| + \dots + |\sigma_d|),$$

then  $p$  is  $G^{1+h}$  microhypoelliptic at every point in  $T^*(\mathbb{R}^N) \setminus 0$ .

In fact, noticing that  $hx_j^{h-1} \partial_{x_N} = [\partial_{x_j}, x_j^h \partial_{x_N}]$  we get by Theorem 1' of Rothschild-Stein [25]

$$(1.8) \quad \sum_{j=1}^{N-1} \|\partial_{x_j} u\|^2 + \sum_{j=1}^d \|x_j^h \partial_{x_N} u\|^2 \leq C |(Pu, u)|$$

if (1.7) is fulfilled. This implies (H-2) while (H-1) is evident.

## 2. A study of the Grushin operator

We shall construct a right parametrix  $K$  for a self-adjoint operator  $Q = (P^*P)^k$  with  $2km \geq d+1$ . (Note that the quasi-homogeneity (1.1), (1.2) and the conditions (H-1), (H-2) are preserved for  $Q$  with the order  $m$  replaced by  $M = 2km$ .) Then clearly  $K^*(P^*P)P^*$  is a left parametrix of  $p$  and the microhypoellipticity of  $p$  follows immediately from that of  $Q$ .

In the construction of the parametrix we follow closely Métivier [21] and Ōkaji [22]. In this section we shall derive the estimates for the inverse of  $\hat{Q} = \mathcal{F}_y Q \mathcal{F}_y^{-1}$ .

**2.1. Grushin operator.** Let  $Q = q(t, D_t, D_y)$  be an operator of the form (1.1) satisfying (H-1), (H-2) for  $(\dot{x}, \dot{\xi}) \in \Sigma$ . We may assume  $(\dot{x}, \dot{\xi}) = (0, e_N) = (0; 0, \dots, 0, 1)$  without loss of generality; henceforth we let  $\dot{\xi} = (0, \dot{\eta}) = (0, 0, \dot{\eta}'' ) = (0, \dots, 0, 1) \in \mathbb{R}^N$ .

By the fourier transform in  $y$ , we consider the equation:

$$(2.1) \quad q(t, D_t, \eta) v(t, \eta) = u(t, \eta)$$

in a conic neighborhood  $U_\varepsilon \times V_\varepsilon$  of  $(0; \dot{\eta}) \in \mathbb{R}^d \times (\mathbb{R}^n \setminus 0)$  given by

$$(2.2) \quad U_\varepsilon = \{t \in \mathbb{R}^d; |t| < 1\},$$

$$V_\varepsilon = \{\eta = (\eta', \eta'') \in \mathbb{R}^n \setminus \{0\}; |\eta'| < \varepsilon \eta_n, |\eta'' - \hat{\eta}'' \eta_n| < \varepsilon \eta_n\}.$$

$q(t, D_t, \eta)$  is essentially the same operator that was studied by Grušin [12]; so we call it *Grušin operator*.

Now we shall start with the following lemma due to Grušin (: Lemma 3.4 in [12]).

**Lemma 2.1.** *Let  $Q = q(t, D_t, D_y)$  be an operator of order  $M$  of the form (1.1) satisfying (H-1) and (H-2) for  $\xi = (0, 0, \hat{\eta}'')$ . Then there exist a conic neighborhood  $V''$  of  $\hat{\eta}''$  and a constant  $C$  such that for all  $\eta = (\eta', \eta'') \in \mathbb{R}^{d'} \times V''$*

$$(2.3) \quad \sum_{|\beta| \leq M} \int |(|\eta''|^\rho + |\eta'| + |t|^h |\eta''|)^{M-|\beta|} |D_t^\beta v(t)|^2 dt \\ \leq C \int |q(t, D_t, \eta) v(t)|^2 dt$$

for  $v \in \mathcal{G}(\mathbb{R}_t^d)$ , where  $\rho = 1/(1+h)$ .

Let us introduce new variables

$$\bar{t} = t \eta_n^\rho, \quad \bar{\eta}' = \eta' / \eta_n^\rho, \quad \bar{\eta}'' = \eta'' / \eta_n, \quad (\eta_n > 0)$$

and set

$$\bar{v}(\bar{t}, \bar{\eta}, \eta_n) = v(\bar{t} / \eta_n, \bar{\eta}' \eta_n, \bar{\eta}'' / \eta_n).$$

Then in view of (1.2) we have

$$(2.4) \quad q(\bar{t}, D_{\bar{t}}, \bar{\eta}) \bar{v}(\bar{t}, \bar{\eta}, \eta_n) = \eta_n^{-\rho M} q(t, D_t, \eta) v(t, \eta),$$

and the conic neighborhood  $U_\varepsilon \times V_\varepsilon$  blows up into  $\mathbb{R}_{\bar{t}}^d \times \bar{V}_\varepsilon$ , where  $\bar{V}_\varepsilon = \mathbb{R}_{\bar{t}}^d \times \{\bar{\eta}'' \in \mathbb{R}^{d''}; |\bar{\eta}'' - \hat{\eta}''| < \varepsilon\}$ .

By multiplying  $\eta_n^{-\rho(M-d)}$ , (2.3) becomes

$$(2.5) \quad \sum_{|\beta| \leq M} \int |(1 + |\bar{\eta}'| + |\bar{t}|^h |\bar{\eta}''|)^{M-|\beta|} |D_{\bar{t}}^\beta \bar{v}(\bar{t}, \eta_n)|^2 d\bar{t} \\ \leq C \int |q(\bar{t}, D_{\bar{t}}, \bar{\eta}) \bar{v}(\bar{t}, \eta_n)|^2 d\bar{t}.$$

Moreover, we have

**Proposition 2.2.** *Let*

$$\bar{V}_\varepsilon^{\mathbb{C}} = \{ \bar{\eta} = (\bar{\eta}', \bar{\eta}'') \in \mathbb{C}^{d'} \times \mathbb{C}^{d''}; |\operatorname{Im} \bar{\eta}'| < \varepsilon(1 + |\operatorname{Re} \bar{\eta}'|), |\bar{\eta}'' - \hat{\eta}''| < \varepsilon \}.$$

If  $\varepsilon$  is chosen sufficiently small then for all  $\bar{\eta} \in \bar{V}_\varepsilon^{\mathbb{C}}$  we have (2.5) with another constant  $C$  and there exists a left inverse  $\bar{K}(\bar{\eta})$  of  $q(\bar{t}, D_{\bar{t}}, \bar{\eta})$  depending holomorphically on  $\bar{\eta} \in \bar{V}_\varepsilon^{\mathbb{C}}$ .

**2.2 Commutator estimates.** We consider the operators

$$T_j = \partial_{\bar{t}_j} \quad \text{and} \quad T_{-j} = i\bar{t}_j \quad (j = 1, 2, \dots, d).$$

For a sequence  $I = (j_1, \dots, j_k) \in \{\pm 1, \dots, \pm d\}^k$  we denote by  $T_I$  the operator

$$T_I = T_{j_1} T_{j_2} \cdots T_{j_k}$$

and  $\langle I \rangle = |I_+| + (1/h)|I_-| = \#\{j_l > 0\} + (1/h)\#\{j_l < 0\}$ .

We define the space

$$B^k(\bar{\eta}) = \{ u \in L^2(\mathbb{R}^d); \forall I, \langle I \rangle \leq k, T_I u \in L^2(\mathbb{R}^d) \}$$

for  $k \in \mathbb{N}/h$  equipped with the norm:

$$|u|_{k, \bar{\eta}} = \max_{\langle I \rangle + j \leq k} (1 + |\bar{\eta}'|)^j \|T_I u\|_{L^2(\mathbb{R}^d)}$$

depending on  $\bar{\eta} \in \bar{V}_\varepsilon^{\mathbb{C}}$ . Note that  $|u|_{0, \bar{\eta}}$  is the usual  $L^2$  norm independent of  $\bar{\eta}$ ; hence denoted by  $|u|_0$ . We also define  $B^{-k}(\bar{\eta})$  the dual space of  $B^k(\bar{\eta})$ .

If  $L$  is an operator acting from  $\mathcal{G}(\mathbb{R}^d)$  into  $\mathcal{G}'(\mathbb{R}^d)$  we set

$$(\operatorname{ad} T_j)(L) = [T_j, L] = T_j L - L T_j \quad (j = \pm 1, \dots, \pm d)$$

and because the  $\operatorname{ad} T_j$ 's commute, we denote for a multi-index  $\alpha =$

$$(\alpha_+, \alpha_-) = (\alpha_1, \dots, \alpha_d; \alpha_{-1}, \dots, \alpha_{-d}) \in \mathbb{N}^d \times \mathbb{N}^d$$

$$(\operatorname{ad} T)^\alpha = \prod_j (\operatorname{ad} T_j)^{\alpha_j}.$$

If the operator  $L$  from  $\mathcal{G}(\mathbb{R}^d)$  into  $\mathcal{G}'(\mathbb{R}^d)$  can be extended as a bounded operator in  $L^2(\mathbb{R}^d)$  we denote by  $\|L\|_0$  the norm of this extension, otherwise we agree with  $\|L\|_0 = +\infty$ .

At last, we introduce the norm:

$$\|L\|_{k, \bar{\eta}} = \text{Max}_{\langle I \rangle + \langle J \rangle + j \leq k} (1 + |\bar{\eta}'|)^j \|T_I L T_J\|_0$$

for  $k \in \mathbb{N}/h$ , then  $\|L\|_{k, \bar{\eta}} < +\infty$  only means that  $L$  is bounded from  $B^{-p}(\bar{\eta})$  to  $B^{-p+k}(\bar{\eta})$  for all  $p = 0, 1/h, 2/h, \dots, k$ .

Now let  $\bar{Q} = \bar{Q}(\bar{\eta}) = q(\bar{t}, D_{\bar{t}}, \bar{\eta})$ . Then we can write

$$(2.6) \quad \bar{Q}(\bar{\eta}) = \sum_{\langle I \rangle + |\beta| \leq M} b_{I, \beta} \bar{\eta}^\beta T_I$$

and (2.5) by:

$$(2.7) \quad |u|_{M, \bar{\eta}} \leq C_0 |\bar{Q}u|_0 \quad \text{for } \bar{\eta} \in \bar{V}_\varepsilon^{\mathbb{C}}$$

We obtain as in Ōkaji [22]

**Lemma 2.3.** *If  $\bar{Q}$  is a self-adjoint operator satisfying (2.7) then there exists a constant  $C_1$  such that*

$$(2.8) \quad \|L\|_{M, \bar{\eta}} \leq C_1 (\|\bar{Q}L\|_0 + \|L\bar{Q}\|_0) \quad \text{for } \bar{\eta} \in \bar{V}_\varepsilon^{\mathbb{C}}.$$

Let  $p$  be an integer. For real  $R > 1$ ,  $\mathcal{L}_R^p(\bar{V}_\varepsilon^{\mathbb{C}})$  denotes the space of operators  $L$  for which there is a constant  $C$  such that for all  $\alpha = (\alpha_+, \alpha_-) \in \mathbb{N}^d \times \mathbb{N}^d$  and  $\bar{\eta} \in \bar{V}_\varepsilon^{\mathbb{C}}$

$$\|(\text{ad } T)^\alpha(L)\|_{\langle \alpha \rangle + p, \bar{\eta}} \leq C |\alpha|! R^{|\alpha|},$$

where  $\langle \alpha \rangle = (1/h)|\alpha_+| + |\alpha_-|$ . Then  $\mathcal{L}_R^p(\bar{V}_\varepsilon^{\mathbb{C}})$  becomes a Banach space in an obvious way.

**Lemma 2.4.** *Let  $\bar{Q}$  be as in Lemma 2.3. Then there are constants  $R_0$  and  $C_2$  depending only on  $C_1$  and  $\text{Max } |b_{I, \beta}|$  such that if  $R > R_0$  and both  $\bar{Q}L$  and  $L\bar{Q}$  are in  $\mathcal{L}_R^0(\bar{V}_\varepsilon^{\mathbb{C}})$  then  $L$  is in  $\mathcal{L}_R^M(\bar{V}_\varepsilon^{\mathbb{C}})$ ; moreover*

$$(2.9) \quad \|L\|_{\mathcal{L}_R^M(\bar{V}_\varepsilon)} \leq C_2 (\|\bar{Q}L\|_{\mathcal{L}_R^0(\bar{V}_\varepsilon)} + \|L\bar{Q}\|_{\mathcal{L}_R^0(\bar{V}_\varepsilon)}).$$

Proof is parallel to that of Métivier [21] Proposition 2.3 or Ōkaji [22] Lemma 7.2 and can be found in [27]. The following proposition is just a consequence of this lemma.

**Proposition 2.5.** *Let  $\bar{Q}$  be a self-adjoint operator satisfying (2.7) and let  $\bar{K}$  be the inverse of  $\bar{Q}$  such that  $\bar{K}\bar{Q} = \bar{Q}\bar{K} = Id$ . Then, if  $R$  is large enough,  $\bar{K}$  is in  $\mathcal{L}_R^M(\bar{V}_\varepsilon)$ .*

**2.3. Kernel of the inverse.** For an operator  $K$  from  $\mathcal{G}(\mathbb{R}^d)$  to  $\mathcal{G}'(\mathbb{R}^d)$ , we denote by  $K(\bar{t}, \bar{s})$  its distribution kernel.

**Lemma 2.6.** *If  $K$  is in  $\mathcal{L}_R^M(\bar{V}_\varepsilon)$  with  $M \geq d+1$  then  $K(\bar{t}, \bar{s})$  is in  $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ , moreover there exist constants  $\bar{C}$  and  $\bar{R}$  such that for all  $\alpha = (\alpha_+, \alpha_-) \in \mathbb{N}^d \times \mathbb{N}^d$*

$$(2.10) \quad \|(\bar{t}-\bar{s})^{\alpha_-} (\partial_{\bar{t}} + \partial_{\bar{s}})^{\alpha_+} K(\bar{t}, \bar{s})\|_{L^2} \leq \bar{C} \|K\|_{\mathcal{L}_R^M(\bar{V}_\varepsilon)} \bar{R}^{|\alpha|} (\alpha_+!)^{1-\rho} (\alpha_-!)^\rho,$$

where  $\rho = 1/(1+h)$ .

*Proof.* Note that if  $K$  and  $K^*$  are bounded from  $L^2(\mathbb{R}^d)$  into  $B^{d+1}(\bar{\eta})$  then  $K$  is a Hilbert-Schmidt operator with the continuous kernel such that

$$\|K(\bar{t}, \bar{s})\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \leq C \|K\|_{d+1, \bar{\eta}}.$$

To prove (2.10) we consider

$$(2.11) \quad \begin{aligned} & \left( (\bar{t}-\bar{s})^{\alpha_-} (\partial_{\bar{t}} + \partial_{\bar{s}})^{\alpha_+} \right)^{1+h} K(\bar{t}, \bar{s}) \\ &= \sum (\pm)^{\beta'_- \beta''_- \beta'_+ \beta''_+} \bar{t}^{\beta'_-} \bar{s}^{\beta''_-} \partial_{\bar{t}}^{\beta'_+} \partial_{\bar{s}}^{\beta''_+} (\bar{t}-\bar{s})^{\alpha_-} (\partial_{\bar{t}} + \partial_{\bar{s}})^{h\alpha_+} K(\bar{t}, \bar{s}), \end{aligned}$$

where the sum consists of  $2^{h|\alpha_-| + |\alpha_+|}$  terms of the coefficients 1 or -1 with the multi-indices  $\beta'_-, \beta''_-, \beta'_+, \beta''_+$  such that  $\beta'_- + \beta''_- = h\alpha_-, \beta'_+ + \beta''_+ = \alpha_+$ .

Now

$$(2.12) \quad \frac{\beta'_- \beta''_-}{\bar{t} \bar{s}} \frac{\beta'_+ \beta''_+}{\partial_{\bar{t}} \partial_{\bar{s}}} (\bar{t} - \bar{s})^{\alpha_-} (\partial_{\bar{t}} + \partial_{\bar{s}})^{h\alpha_+} K(\bar{t}, \bar{s})$$

is the distribution kernel of

$$\frac{\beta'_- \beta''_-}{T_- T_+} (\text{ad } T_-)^{\alpha_-} (\text{ad } T_+)^{\alpha_+} (K) T_- T_+;$$

which is bounded from  $L^2(\mathbb{R}^d)$  into  $B^M(\bar{\eta})$  together with its adjoint. Since  $M \geq d+1$  we know (2.12) is a continuous function with  $L^2$  norm bounded by

$$C \|K\|_{\mathcal{L}_R^M(\bar{V}_\varepsilon^{\mathbb{C}})} R^{|\alpha_-| + h|\alpha_+|} (|\alpha_-| + h|\alpha_+|)!.$$

Adding up these estimates we have

$$(2.13) \quad \begin{aligned} & \|(\bar{t} - \bar{s})^{\alpha_-} (\partial_{\bar{t}} + \partial_{\bar{s}})^{\alpha_+} K(\bar{t}, \bar{s})\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \\ & \leq C \|K\|_{\mathcal{L}_R^M(\bar{V}_\varepsilon^{\mathbb{C}})} \bar{R}^{|\alpha_-| + h|\alpha_+|} (|\alpha_-| + h|\alpha_+|)! \end{aligned}$$

provided that  $\bar{R} \geq (2R)^h$ . Also we have

$$(2.14) \quad \|K(\bar{t}, \bar{s})\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \leq C \|K\|_{\mathcal{L}_R^M(\bar{V}_\varepsilon^{\mathbb{C}})}.$$

Then a simple interpolation argument yields (2.10) in view of the Stirling formula.  $\square$

**2.4. Symbol of the inverse.** We write the operator  $K$  of kernel  $K(\bar{t}, \bar{s})$  with a symbol  $k = \sigma(K)$  in the way that

$$(2.15) \quad K(\bar{t}, \bar{s}) = (2\pi)^{-d} \int e^{i\langle \bar{t} - \bar{s}, \bar{\tau} \rangle} k(\bar{t}, \bar{\tau}) d\bar{\tau}.$$

That is,  $k$  is the distribution on  $\mathbb{R}^{2d}$  given by

$$(2.16) \quad k(z^+, z^-) = \int e^{i\langle u, z^- \rangle} K(z^+, z^+ + u) du.$$

Here and below we use the notation  $z = (z^+, z^-) =$

$$(z_1, \dots, z_d; z_{-1}, \dots, z_{-d}) \in \mathbb{R}^{2d}.$$

Since (2.15), (2.16) have a sense as the partial Fourier transform

the mapping  $\sigma$  is clearly an isomorphism between  $L^2(\mathbb{R}^d \times \mathbb{R}^d)$  and  $L^2(\mathbb{R}_z^{2d})$ . Also by the definition of  $\sigma$  we have

$$\sigma((\text{ad } T_j)(K)) = \partial_{z_j} \sigma(K).$$

Hence Lemma 2.6 is restated as follows:

**Lemma 2.7.** Let  $k = k(\bar{\eta}) = \sigma(K(\bar{\eta}))$ ; the symbol of  $K(\bar{\eta}) \in \mathcal{L}_R^M(\bar{V}_\varepsilon^{\mathbb{C}})$  with  $M \geq d+1$ . Then there exist constants  $\bar{C}, \bar{R}$  such that for all  $\alpha = (\alpha_+, \alpha_-) \in \mathbb{N}^d \times \mathbb{N}^d$  and  $\bar{\eta} \in \bar{V}_\varepsilon^{\mathbb{C}}$

$$(2.17) \quad \|\partial_z^\alpha k(\bar{\eta})\|_{L^2(\mathbb{R}_z^{2d})} \leq \bar{C} \|K\|_{\mathcal{L}_R^M(\bar{V}_\varepsilon^{\mathbb{C}})} \bar{R}^{|\alpha|} (\alpha_+!)^{1-\rho} (\alpha_-!)^\rho$$

where  $\rho = 1/(1+h)$ .

Now suppose that  $K(\bar{\eta}) \in \mathcal{L}_R^M(\bar{V}_\varepsilon^{\mathbb{C}})$  ( $M \geq d+1$ ) depends holomorphically on  $\bar{\eta}$ . Then we have

**Proposition 2.8.** Let  $K(\bar{\eta})$  be as above and let  $k(z, \bar{\eta}) = \sigma(K(\bar{\eta}))(z)$ . Then there exists a constant  $C$  such that for  $(z, \bar{\eta}) \in \mathbb{R}^{2d} \times \bar{V}_{\varepsilon'}^{\mathbb{C}}$  with  $0 < \varepsilon' < \varepsilon$  and for all  $(\alpha, \beta) = (\alpha_+, \alpha_-, \beta', \beta'') \in \mathbb{N}^d \times \mathbb{N}^d \times \mathbb{N}^{d'} \times \mathbb{N}^{d''}$

$$(2.18) \quad |\partial_z^\alpha \partial_{\bar{\eta}}^\beta k(z, \bar{\eta})| \leq C |\alpha| + |\beta| + 1 \left(\frac{1}{\varepsilon - \varepsilon'}\right)^{|\beta|} (\alpha_+!)^{1-\rho} (\alpha_-!)^\rho \beta! (1 + |\bar{\eta}'|)^{-|\beta'|},$$

where  $\rho = 1/(1+h)$ .

*Proof.* Recall that

$$\bar{V}_\varepsilon^{\mathbb{C}} = \{\bar{\eta} = (\bar{\eta}', \bar{\eta}'') \in \mathbb{C}^{d'} \times \mathbb{C}^{d''}; |\text{Im } \bar{\eta}'| < \varepsilon(1 + |\text{Re } \bar{\eta}'|), |\bar{\eta}'' - \hat{\eta}''| < \varepsilon\}.$$

Then we use the Cauchy inequality to obtain

$$\|\partial_{\bar{\eta}}^\beta K\|_{\mathcal{L}_R^M(\bar{V}_{\varepsilon'}^{\mathbb{C}})} \leq \|K\|_{\mathcal{L}_R^M(\bar{V}_\varepsilon^{\mathbb{C}})} \left(\frac{n}{\varepsilon - \varepsilon'}\right)^{|\beta|} \beta! (1 + |\bar{\eta}'|)^{-|\beta'|}.$$

Applying Lemma 2.7 to  $\partial_{\bar{\eta}}^\beta K(\bar{\eta})$  we get (2.18) by means of the Sobolev

lemma.  $\square$

## 3. Parametrix; proof of Theorem I

In Section 2 we have showed that there is the inverse  $\bar{K}(\bar{\eta})$  of  $\bar{Q}(\bar{\eta}) = q(\bar{t}, D_{\bar{t}}, \bar{\eta}) = (P^*P)^k(t, D_t, \eta)$  ( $2km \geq d+1$ ) for  $\bar{\eta} \in \bar{V}_\varepsilon^C$  such that

$$q(\bar{t}, D_{\bar{t}}, \bar{\eta}) \bar{K}(\bar{t}, \bar{s}, \bar{\eta}) d\bar{s} = \delta(\bar{t} - \bar{s}) d\bar{s}$$

with the kernel

$$\bar{K}(\bar{t}, \bar{s}, \bar{\eta}) = (2\pi)^{-d} \int e^{i\langle \bar{t} - \bar{s}, \bar{\tau} \rangle} \bar{k}(\bar{t}, \bar{\tau}, \bar{\eta}) d\bar{\tau},$$

where  $\bar{k}$  satisfies (2.18) in  $\mathbb{R}_{\bar{t}}^d \times \mathbb{R}_{\bar{\tau}}^d \times (\bar{V}_\varepsilon^C \cap \mathbb{R}_{\bar{\eta}}^{d'})$  for  $0 < \varepsilon' < \varepsilon$ .

Now we return to the original variables:

$$t = \bar{t}/\eta_n^\rho, \quad \tau = \bar{\tau}/\eta_n^\rho, \quad \eta' = \bar{\eta}'/\eta_n^\rho, \quad \eta'' = \bar{\eta}''/\eta_n^\rho, \quad (\eta_n > 0)$$

and set

$$\hat{K}(t, s, \eta) = (2\pi)^{-d} \int e^{i\langle t-s, \tau \rangle} \hat{k}(t, \tau, \eta) d\tau,$$

where

$$\begin{aligned} (3.1) \quad \hat{k}(t, \tau, \eta) &= \eta_n^{-2\rho km} \bar{k}(\bar{t}, \bar{\tau}, \bar{\eta}) \\ &= \eta_n^{-2\rho km} \bar{k}(t\eta_n^\rho, \tau/\eta_n^\rho, \eta'/\eta_n^\rho, \eta''/\eta_n^\rho). \end{aligned}$$

Then in view of (2.4)

$$q(t, D_t, \eta) \hat{K}(t, s, \eta) ds = \delta(t-s) ds$$

for  $\eta \in V_\varepsilon = \{(\eta', \eta'') \in \mathbb{R}^n \setminus 0; |\eta'| < \varepsilon \eta_n, |\eta'' - \bar{\eta}'' \eta_n| < \varepsilon \eta_n\}$ .

Let us introduce a cut off function given by Métivier:

**Lemma 3.1.** For given two cones  $V_1 \subset V_2 \subset \mathbb{R}^N \setminus 0$  and  $0 < \rho < 1$  there exist  $g \in C^\infty(\mathbb{R}^N)$  and  $C$  such that

$$(3.2) \quad g(\xi) = 0 \quad \text{for } \xi \notin V_2 \quad \text{or } |\xi| \leq 1$$

$$g(\xi) = 1 \quad \text{for } \xi \in V_1 \text{ and } |\xi| \geq 2.$$

and

$$(3.3) \quad |\partial_{\xi}^{\alpha} g(\xi)| \leq C^{|\alpha|+1} \left(\frac{|\alpha|}{|\xi|}\right)^{\rho|\alpha|}$$

for all  $\alpha, \xi$  such that  $|\alpha| \leq |\xi|$ . (Lemma 3.1 in [21].)

With  $\rho = 1/(1+h)$  and  $\xi \in V_1 \subset V_2 = \{\xi = (\tau, \eta) \in \mathbb{R}^d \times \mathbb{R}^n; |\tau| < \varepsilon', \eta \in V_{\varepsilon'}\}$ , we take  $g(\xi) = g(\tau, \xi)$  as above and set  $k_g(t, \tau, \eta) = \hat{\chi}(t, \tau, \eta)g(\tau, \eta)$ . Then

**Proposition 3.2.** *There exists a constant  $C_0$  such that*

$$(3.4) \quad |\partial_{\eta}^{\beta} \partial_{\tau}^{\alpha_{-}} \partial_t^{\alpha_{+}} k_g(t, \tau, \eta)| \leq C_0^{|\alpha|+|\beta|+1} (1+|t|)^{|\beta|} (|\alpha_{+}|^{1-\rho} |\xi|^{\rho})^{|\alpha_{+}|} \\ \times \left(\frac{|\alpha_{-}|}{|\xi|}\right)^{\rho|\alpha_{-}|} \left(\frac{|\beta'|}{|\xi|^{\rho+|\eta'|}} + \chi_g(\xi) \left(\frac{|\beta'|}{|\xi|}\right)^{\rho}\right)^{|\beta'|} \left(\frac{|\beta''|}{|\xi|}\right)^{\rho|\beta''|}$$

for  $|\alpha_{-}|+|\beta| \leq |\xi|$ , where  $\xi = (\tau, \eta) = (\tau, \eta', \eta'') \in \mathbb{R}^N$ ,  $(\alpha, \beta) = (\alpha_{+}, \alpha_{-}, \beta', \beta'') \in \mathbb{N}^d \times \mathbb{N}^d \times \mathbb{N}^{d'} \times \mathbb{N}^{d''}$ ,  $\rho = 1/(1+h)$  and  $\chi_g$  is the characteristic function of the support of  $\nabla_{\eta} g$ .

Now let  $K_g = k_g(t, D_t, D_y) = \text{Op}(k_g)$ ; that is,  $K_g$  is the operator defined by the kernel:

$$(3.5) \quad K_g(t, y, s, w) = (2\pi)^{-N} \int e^{i\langle t-s, \tau \rangle + i\langle y-w, \eta \rangle} k_g(t, \tau, \eta) d\tau d\eta.$$

Then we have

$$(3.6) \quad QK_g = K_g^*Q = g(D_t, D_y) = \text{Op}(g)$$

and the following

**Proposition 3.3.**

$$(3.7) \quad \text{WF}_A(K_g) \subset \{(t, y, t, w; \tau, \eta, -\tau, -\eta) \in T^*(\mathbb{R}^{2N}) \setminus 0; y'' = w'', (\tau, \eta) \in \bar{V}_2\}.$$

$$(3.8) \quad WF_{1+h}(K_g) \subset \{(t, y, t, y; \tau, \eta, -\tau, -\eta) \in T^*(\mathbb{R}^{2N}) \setminus 0; (\tau, \eta) \in \bar{V}_2\}.$$

*Proof.* By Lemma 3.3 and Remark 3.4 in Métivier [21] we obtain (3.7). Hence to prove (3.8) it suffices to show that  $K$  is in  $C^{1+h}$  for  $y' \neq 0$ . Using the vector field  $(1/|y'|^2) \langle y', D_{\eta'} \rangle$  for integrating by parts we can prove this as in Case 2 in the proof of Lemma 3.3 in [21].  $\square$

For any set  $V$  we write  $\text{diag}(V) = \{(\rho, \rho) \in V \times V\}$ . We have therefore proved the following theorem; from which Theorem I follows immediately.

**Theorem 3.4.** *Let  $P$  be an operator of the form (1.1) satisfying (H-1), (H-2) for  $(\dot{x}, \dot{\xi}) \in \Sigma$  and let  $Q = (P^*P)^k$  with  $2km \geq d+1$ . Then there are a conic neighborhood  $V \subset \mathbb{R}^N \setminus 0$  of  $\dot{\xi}$  and an operator  $K: \mathcal{D}'(\mathbb{R}^N) \longrightarrow \mathcal{D}'(\mathbb{R}^N)$  such that for every  $u \in \mathcal{D}'(\mathbb{R}^N)$*

$$(3.9) \quad WF_A(QKu - u) \cap (\mathbb{R}^N \times V) = \phi,$$

$$(3.10) \quad WF_A(K^*Qu - u) \cap (\mathbb{R}^N \times V) = \phi$$

and that

$$(3.11) \quad WF'_{1+h}(K) \subset \text{diag}(T^*(\mathbb{R}^N) \setminus 0),$$

where  $WF'_{1+h}(K) = \{(x, \xi; \tilde{x}, \tilde{\xi}); (x, \tilde{x}; \xi, -\tilde{\xi}) \in WF_{1+h}(K)\}$ .

#### 4. Proof of Theorem II

Let  $\dot{\xi} = (0, 0, \hat{\eta}'')$  with  $\hat{\eta}'' \neq 0$ . We consider the operator  $p'(t, D_t, \hat{\eta}'')$ ; which is precisely the same one that was studied by Grušin [10].

From the result of Grušin [10] we can take  $c \in \mathbb{C}$  and  $0 \neq v \in \mathcal{G}(\mathbb{R}^N)$  such that

$$(4.1) \quad p'(t, D_t, \hat{\eta}'')v(t) = -c^m q(\bar{\eta}'),$$

where  $\bar{\eta}' \in \mathbb{R}^{d'}$  is fixed with  $|\bar{\eta}'| = 1$ . Then

$$u_\lambda(t, y) = \exp(i\lambda^\rho c \langle y', \bar{\eta}' \rangle + i\lambda \langle y'', \bar{\eta}'' \rangle) v(\lambda^\rho t), \quad \rho = 1/(1+h)$$

is a solution of  $pu = 0$  for every  $\lambda \geq 0$ . Hence

$$u(t, y) = \int_0^{+\infty} u_\lambda(t, y) e^{-\lambda} d\lambda$$

is a  $C^\infty$  solution in  $U = \{(t, y', y'') \in \mathbb{R}^N; |\operatorname{Im} c| |y'| < 1\}$ .

By Lemma 3.7 in Ōkaji [22],  $v$  satisfies the estimate

$$|\partial_t^\alpha v(t)| \leq C^{|\alpha|+1} (\alpha!)^{1-\rho}.$$

Hence we have

$$(4.2) \quad WF_A(u) \subset \{(t, y; 0, 0, \lambda \hat{\eta}'') \in T^*(\mathbb{R}^N) \setminus 0; \lambda > 0\}$$

in the same way as (3.7).

On the other hand, since  $v$  is analytic,  $\partial_t^\alpha v(0) \neq 0$  for some  $\alpha \in \mathbb{N}^d$ . Therefore,

$$(4.3) \quad |\langle \hat{\eta}'', D_{y''} \rangle^k \partial_t^\alpha u(0, 0)| = \int_0^{+\infty} |\hat{\eta}''|^{2k} \lambda^{\rho|\alpha|+k} |\partial_t^\alpha v(0)| e^{-\lambda} d\lambda \\ = \text{const. } \Gamma((k+1)/\rho + |\alpha|).$$

This combined with (4.2) implies  $(0; 0, 0, \hat{\eta}'') \in WF_V(u)$  for every  $v < 1+h$ , and proof is now complete.  $\square$

## 5. Second microlocalization in Gevrey class

Following Sjöstrand [29] we introduce the Fourier-Bros-Iagolnitzer transform (F.B.I. tr.):

$$(5.1) \quad T^{(1)}f(z, \lambda) = \int e^{-\lambda(z-x)^2/2} f(x) dx, \quad (f \in \mathcal{G}'(\mathbb{R}^N))$$

associated to  $\kappa: T^*(\mathbb{R}^N) \setminus 0 \ni (x, \xi) \longmapsto x - i\xi \in \mathbb{C}_z^N$ .

$T^{(1)}f$  is defined on  $\mathbb{C}_z^N \times \mathbb{R}_\lambda^+$ , holomorphic with respect to  $z$  and bounded by  $Ce^{\lambda|\operatorname{Im}z|^2/2} (\lambda + |y|)^k$  for some  $C, k$  real.

In terms of the F.B.I. tr. we can characterize the Gevrey wave

front set as follows: For  $f \in \mathcal{G}'(\mathbb{R}^N)$ ,  $(\dot{x}, \dot{\xi}) \notin WF_\nu(f)$  if and only if there are constants  $C, c > 0$  such that

$$(5.2) \quad |T^{(1)}f(z, \lambda)| \leq C e^{\frac{\lambda}{2} |\operatorname{Im} z|^2 - c \lambda^{1/\nu}} \quad \text{for } |z - (\dot{x} - i \dot{\xi})| < c.$$

Let  $\Lambda$  be the involutive submanifold of  $T^*(\mathbb{R}^N)$ :

$$\Lambda = \{(x, \xi) \in T^*(\mathbb{R}^N); \xi_1 = \dots = \xi_{d'} = 0\} \quad (1 \leq d' < N),$$

and  $\Gamma_0$  be the bicharacteristic leaf pathing through  $(\dot{x}, \dot{\xi}) \in \Lambda$ . Then  $\Lambda$  and  $\Gamma_0$  can be identified with  $\kappa(\Lambda) = \{z \in \mathbb{C}^N; \operatorname{Im} z' = 0\}$  and  $\kappa(\Gamma_0) = \{z \in \mathbb{C}^N; \operatorname{Im} z' = 0, z'' = \dot{x}'' - i \dot{\xi}''\}$  respectively, where  $z = (z', z'') \in \mathbb{C}^{d'} \times \mathbb{C}^{N-d'}$ .

We set  $\varphi_\Lambda(z) = |\operatorname{Im} z''|^2/2$ ; which is the pluri-subharmonic function canonically associated to  $\Lambda$ . If  $\Omega$  is a neighborhood of  $\dot{z} \in \kappa(\Lambda)$ , we denote by  $H_\Lambda^{\nu, \text{loc}}(\Omega)$  the space of holomorphic functions  $u(z, \lambda)$  in  $\Omega$  with a parameter  $\lambda > 0$  such that for all  $K \subset\subset \Omega$  and  $\varepsilon > 0$  there exists  $C_{K, \varepsilon}$  with the estimate:

$$(5.3) \quad |u(z, \lambda)| \leq C_{K, \varepsilon} e^{\lambda \varphi_\Lambda + \varepsilon \lambda^{1/\nu}} \quad \text{for } z \in K, \lambda \geq 1.$$

For  $\dot{z} \in \Lambda$  we also use the notation:  $u \in H_{\Lambda, \dot{z}}^\nu$  if there is a neighborhood  $\omega_{\dot{z}}$  of  $\dot{z}$  such that  $u \in H_\Lambda^\nu(\omega_{\dot{z}})$ .

If  $u \in H_\Lambda^{\nu, \text{loc}}(\Omega)$  we denote by  $S_\Lambda^\nu(u)$  the subset in  $\Omega$  defined by:

$$(5.4) \quad \dot{z} \notin S_\Lambda^\nu(u) \quad \text{if and only if there exist a neighborhood } \omega_{\dot{z}} \text{ of } \dot{z} \text{ and constants } C, c > 0 \text{ such that}$$

$$|u(z, \lambda)| \leq C e^{\lambda \varphi_\Lambda - c \lambda^{1/\nu}} \quad \text{for } z \in \omega_{\dot{z}}, \lambda \geq 1.$$

By applying the maximum principle to  $z' \mapsto \lambda^{-1/\nu} (\log |u(z, \lambda)| - \lambda |\operatorname{Im} z''|^2/2)$  it can be seen easily the following two lemmas.

**Lemma 5.1.** *Let  $\Gamma_0$  be a bicharacteristic leaf in  $\Lambda$  and  $\omega$  be a connected open set in  $\Gamma_0$  containing  $(\dot{x}, \dot{\xi})$ . If  $u \in H_{\Lambda, \dot{z}}^\nu$  for all*

$z \in \kappa(\omega)$  and  $\kappa(\hat{x}, \hat{\xi}) = \hat{x} - i\hat{\xi} \notin S_{\Lambda}^{\nu}(u)$  then  $\kappa(\omega) \cap S_{\Lambda}^{\nu}(u) = \emptyset$ .

**Lemma 5.2.** Let  $(\hat{x}, \hat{\xi}) \in \Lambda$ ,  $f \in \mathcal{G}'(\mathbb{R}^N)$ . If  $(\hat{x}, \hat{\xi}) \notin WF_{\nu}(f)$  and  $T^{(1)}f \in H_{\Lambda, \hat{x}-i\hat{\xi}}^{\nu}$  then  $\hat{x}-i\hat{\xi} \notin S_{\Lambda}^{\nu}(T^{(1)}f)$ .

Let us introduce the F.B.I. tr. of second kind along  $\Lambda$  following Lebeau [19]:

$$(5.5) \quad T_{\Lambda}^{(2)}f(w, \mu, \lambda) = \int e^{-\lambda(w''-x'')^2/2 - \lambda\mu(w'-x')^2/2} f(x) dx \quad (f \in \mathcal{G}'(\mathbb{R}^N)).$$

Then  $T_{\Lambda}^{(2)}f(w, \mu, \lambda)$  is a holomorphic function with respect to  $w \in \mathbb{C}^N$  with the bound:

$$|T_{\Lambda}^{(2)}f(w, \mu, \lambda)| \leq C e^{\frac{\lambda}{2} |\operatorname{Im} w''|^2 + \frac{\lambda}{2} \mu |\operatorname{Im} w'|^2} (\lambda + |w|)^k.$$

It was shown in [20] and [2] that the relation between  $T^{(1)}f$  and  $T_{\Lambda}^{(2)}f$  is

$$(5.6) \quad T_{\Lambda}^{(2)}f(w, \mu, \lambda) = \left( \frac{\lambda}{2\pi(1-\mu)} \right)^{\frac{d'}{2}} \int_{\mathbb{R}^{d'}} e^{-\lambda\rho(w'-x')^2/2} T^{(1)}f(x', w'', \lambda) dx',$$

where  $\rho = \mu/(1-\mu)$  with an inversion formula:

$$(5.7) \quad T^{(1)}f(z, \lambda) = \frac{1}{2} \left( \frac{1}{2\pi\lambda} \right)^{\frac{d'}{2}} \int_{\mathbb{R}^{d'}} e^{-\lambda R |\xi'|/2} \left( 1 - i \frac{\langle \xi', \nabla' \rangle}{\lambda |\xi'|} \right) T_{\Lambda}^{(2)}f\left(z' - i \frac{R\xi'}{|\xi'|}, z'', \mu, \lambda\right) \frac{R d\xi'}{R + |\xi'|},$$

where  $\mu = |\xi'|/(R + |\xi'|)$ .

Now we define second wave front sets adapted to the Gevrey class. (See also Esser [7].)

**Definition 5.3.** If  $1 \leq \nu < +\infty$  and  $f \in \mathcal{G}'(\mathbb{R}^N)$ , the second wave front set along  $\Lambda$  of  $f$ ; denoted by  $WF_{\Lambda, \nu}^{(2)}(f)$ , is the subset in  $T_{\Lambda}(T^*(\mathbb{R}^N) \setminus 0)$  defined by the following condition:

$$(5.8) \quad (\hat{x}, 0, \hat{\xi}''; \hat{\sigma}') \notin WF_{\Lambda, \nu}^{(2)}(f)$$

if and only if there exist  $C, c > 0, 0 < \mu_0 < 1$  and a decreasing function  $o(\lambda)$  with  $\lim_{\lambda \rightarrow +\infty} o(\lambda) = 0$  such that

$$(5.9) \quad |T_{\Lambda}^{(2)} f(w, \mu, \lambda)| \leq C e^{\frac{\lambda}{2} |\operatorname{Im} w''|^2 + \frac{\lambda}{2} \mu |\operatorname{Im} w'|^2 - c \lambda \mu}$$

for

$$(5.10) \quad 0 < \mu < \mu_0, \lambda \mu > o(\lambda) \lambda^{1/\nu}, |w' - (\dot{x}' - i\dot{\sigma}')| + |w'' - (\dot{x}'' - i\dot{\xi}'')| < c.$$

Using (5.6) and (5.7) we can show the following:

**Lemma 5.4.** *Let  $(\dot{x}, \dot{\xi}) \in \Lambda$  and  $f \in \mathcal{G}'(\mathbb{R}^N)$ . Then  $T^{(1)} f \in H_{\Lambda, \dot{x} - i\dot{\xi}}^{\nu}$  if and only if  $\pi_{\Lambda}^{-1}(\dot{x}, \dot{\xi}) \cap WF_{\Lambda, \nu}^{(2)}(f) = \emptyset$ , where  $\pi_{\Lambda}: T_{\Lambda}(T^*(\mathbb{R}^N) \setminus 0) \rightarrow \Lambda$  is the canonical projection.*

At last, we introduce the space of partially holomorphic Gevrey functions  $G_{\mathcal{A}_x}^{\nu}$ , as follows:  $f(x) \in G_{\mathcal{A}_x}^{\nu}(\Omega)$  if and only if for every compact set  $K \subset\subset \Omega$  there is a constant  $C$  such that

$$(5.11) \quad |\partial_x^{\alpha'} \partial_{x''}^{\alpha''} f(x)| \leq C^{|\alpha|+1} \alpha'! (\alpha''!)^{\nu} \quad \text{for } x \in K.$$

We have

**Lemma 5.5.** *If  $f \in \mathcal{G}'(\mathbb{R}^N) \cap G_{\mathcal{A}_x}^{\nu}(\Omega)$  and  $1 \leq \nu' < \nu$  then  $T^{(1)} f \in H_{\Lambda, z}^{\nu'}$  for every  $z \in \kappa(\pi^{-1}(\Omega) \cap \Lambda)$ .*

## 6. Proof of Theorem III

As in Section 2 we suppose that  $\dot{x} = 0, \dot{\xi} = (0, 0, \dot{\eta}'') = (0, \dots, 0, 1) \in \mathbb{R}^N \setminus 0$  and set  $Q = (p^* p)^k$  with  $2km \geq d+1$ . Here we also introduce the pseudo-differential operator:

$$(6.1) \quad \operatorname{Op}(r) = \operatorname{Op}_{\eta} \left( \eta_n^{2km/(1+h)} e^{-|\eta'|^{2l(1+h)}/\eta_n^{2l}}, \right),$$

where  $l$  is a positive integer to be determined. Then  $\operatorname{Op}(r)$  has the same quasi-homogeneity in its symbol as  $Q$  has.

Consider the operator  $Q + \operatorname{Op}(r)$ . Then it satisfies (H-2) since  $Q$  is non negative self-adjoint operator at  $\dot{\xi}$ . We also note that

though not being polynomial,  $r$  is holomorphic with the uniform bound  $O(|\xi|^{2km/(1+h)})$  in a small quasi-homogeneous neighborhood of  $\xi$  of the form:

$$V_{\varepsilon}^{\mathbb{C}} = \{(\eta', \eta'') \in \mathbb{C}^{d'} \times \mathbb{C}^{d''}; |\operatorname{Im} \eta'| < \varepsilon(|\eta_n|^{1/(1+h)} + |\operatorname{Re} \eta'|), |\eta''/\eta_n - \dot{\eta}''| < \varepsilon\}.$$

Now all the results in Section 2 are remain valid for  $Q + \operatorname{Op}(r)$  and we get a symbol  $k_g(t, \tau, \eta)$  satisfying (3.4) such that

$$(6.2) \quad \operatorname{Op}(k_g)^*(Q + \operatorname{Op}(r)) = \operatorname{Op}(g).$$

Here  $g$  is an arbitrary cut off function satisfying (3.3) for  $\rho = 1/(1+h)$  with its support in

$$(6.3) \quad V_{\varepsilon_0} = \{(\tau, \eta) \in T^*(\mathbb{R}^N) \setminus \{0\}; |\tau| < \varepsilon_0 \eta_n, |\eta'| < \varepsilon_0 \eta_n, |\eta''/\eta_n - \dot{\eta}''| < \varepsilon_0\}.$$

If  $(\dot{x}, \dot{\xi}) = (0; 0, 0, \dot{\eta}'') \in \Sigma$  then the bicharacteristic leaf is  $\Gamma_0 = \{(0, y', 0; 0, 0, \dot{\eta}''); y' \in \mathbb{R}^{d'}\}$ . For any compact set  $F \subset \pi(\Gamma_0 \cap W)$  there exist a neighborhood  $U \subset \subset O_R = \{x \in \mathbb{R}^N; |x| < R\}$  of  $F$  and a conic neighborhood  $V$  of  $\xi$  such that

$$(6.4) \quad WF_V(Pu) \cap \bar{U} \times (\bar{V} \setminus \{0\}) = \emptyset,$$

where  $\bar{U}, \bar{V}$  denote the closures of  $U, V$  respectively.

After replacing  $u$  by  $\varphi u$  with a suitable  $\varphi \in C_0^\infty(O_R)$  we can suppose  $u \in \mathcal{E}'(O_R)$  with no influence on (6.4).

We fix a conic neighborhood  $V_2$  of  $\xi$  with  $V_2 \subset \subset V \cap V_{\varepsilon_0}$ . If we choose another conic neighborhood  $V_1$  of  $\xi$  sufficiently small then the cut off function  $g$  in Lemma 3.1 can be taken in the form:  $g(\xi) = g'(\eta', \eta_n)g''(\tau, \eta'')$  so that  $\operatorname{supp} \nabla_{\eta'} g \subset \{(t, \eta', \eta''); |\eta'| > \delta|\xi|\}$  for some  $\delta > 0$ .

As in Proposition 3.3 one can see the following:

**Proposition 6.1.** *If  $k_g$  satisfies (3.4) with  $\chi_g(\xi) = 0$  for  $|\eta'| < \delta|\xi|$  ( $\delta > 0$ ), then*

$$(6.5) \quad K_g(t, y, s, w) \in G_{y', w}^{1+h}((\mathbb{R}^N \times \mathbb{R}^N) \setminus \operatorname{diag}(\mathbb{R}^N)).$$

where  $K_g$  denotes the distribution kernel of  $\text{Op}(k_g)$ .

Now we let  $g$  be taken as above and write for  $u \in \mathcal{S}'(O_R)$

$$(6.6) \quad \begin{aligned} \text{Op}(g)u &= \text{Op}(k_g)^*Qu + \text{Op}(k_g)^*\text{Op}(r)u \\ &= \text{Op}(k_g)^*Qu + \text{Op}(r)\text{Op}(k_g)^*u. \end{aligned}$$

We shall apply the theory of second microlocalization along the involutive submanifold:

$$\Lambda = \{(t, y; \tau, \eta', \eta'') \in T^*(\mathbb{R}^N) \setminus 0; \eta' = 0\}.$$

Hereafter, we also denote the coordinate in  $T^*(\mathbb{R}^N)$  by

$$x' = y', \quad x'' = (t, y'') \quad \text{and} \quad \xi' = \eta', \quad \xi'' = (\tau, \eta'')$$

and use the notation in Section 5 without mentioning it.

First we study  $\text{Op}(r)\text{Op}(k_g)^*u$ , where

$$r(\xi) = \eta_n^{2km/(1+h)} e^{-|\eta'|^{2l(1+h)}/\eta_n^{2l}}$$

was given in (6.1). Now we choose  $l$  so that  $(1+h) - (1/2l) > \nu$ .

Then

$$(6.7) \quad |\eta'|^{2l(1+h)}/\eta_n^{2l} \geq |\eta'| \quad \text{for} \quad |\eta'| \geq \eta_n^{-\varepsilon} \eta_n^{1/\nu}, \quad \eta_n > 0,$$

where  $\varepsilon = (1/\nu) - (2l/(2l(1+h)-1)) > 0$ . We can see easily the following:

**Lemma 6.2.** *If  $r = O(e^{-c|\eta'|})$ ,  $c > 0$  for  $|\eta'| \geq \eta_n^{-\varepsilon} \eta_n^{1/\nu}$ ,  $\eta_n > 0$  then for every  $u \in \mathcal{S}'(\mathbb{R}^N)$*

$$(6.8) \quad \text{WF}_{\Lambda, \nu}^{(2)}(\text{Op}(r)u) \cap \pi_{\Lambda}^{-1}(\Gamma_0) = \emptyset.$$

Since  $\text{Op}(k_g)(\mathcal{S}) \subset \mathcal{S}$ : equivalently  $\text{Op}(k_g)^*(\mathcal{S}') \subset \mathcal{S}'$ , (6.8) holds for  $\text{Op}(r)\text{Op}(k_g)^*u$ . Therefore we have

$$(6.9) \quad \Gamma^{(1)}(\text{Op}(r)\text{Op}(k_g)^*u) \in H_{\Lambda, z}^{\nu} \quad \text{for all } z \in \kappa(\Gamma_0)$$

in view of Lemma 5.4.

Next we study  $\text{Op}(k_g)^*Qu$ . Let  $\tilde{g}$  be another cut off function given by Lemma 3.1 with two cones  $\tilde{V}_1, \tilde{V}_2$  such that

$$V_2 \subset\subset \tilde{V}_1 \subset\subset \tilde{V}_2 = V.$$

Noticing that  $WF_V(Qu) \subset WF_V(Pu)$ , we then get by (6.4)

$$(6.10) \quad WF_V(\text{Op}(\tilde{g})Qu) \subset WF_V(Pu) \cap (\mathbb{R}^N \times \bar{V}) \subset \pi^{-1}(O_R \setminus U),$$

$$(6.11) \quad WF_V(\text{Op}(1-\tilde{g})Qu) \subset WF_V(Pu) \cap (\mathbb{R}^N \times (\mathbb{R}^N \setminus \tilde{V}_1)) \subset O_R \times (\mathbb{R}^N \setminus \bar{V}_2).$$

Hence we can write

$$(6.12) \quad Qu = \chi_{F_\varepsilon} \text{Op}(\tilde{g})Qu + \chi_{O_R} (1-\chi_{F_\varepsilon}) \text{Op}(\tilde{g})Qu + \chi_{O_R} \text{Op}(1-\tilde{g})Qu \\ (\equiv v_1 + v_2 + v_3),$$

where  $\chi_B$  denotes the characteristic function of each set  $B$  and

$$F_\varepsilon = \{(x', x'') \in \mathbb{R}^N; (x', 0) \in F, |x''| \leq \varepsilon\}$$

with  $\varepsilon > 0$  so small that  $F_\varepsilon \subset U$ .

In the following we assume further that

$$(6.13) \quad F \text{ is convex with an analytic boundary in } \pi(\Gamma_0),$$

By (6.10) we see that

$$WF_V(v_1) \subset \{(x, \xi); (x', \xi') \in T_{\partial F}^*(\pi(\Gamma_0)), |x''| < \varepsilon, \xi'' = 0\} \cup \pi^{-1}(\{x; |x''| \geq \varepsilon\}).$$

Hence by (3.7)

$$(6.14) \quad \text{Op}(k_g)^*v_1 \in G^V(\text{Int}(F_\varepsilon)),$$

where  $\text{Int}(F_\varepsilon)$  denotes the interior of  $F_\varepsilon$ .

Since  $\text{supp}(v_2) \subset \bar{O}_R \setminus F_\varepsilon$ , it follows by Proposition 6.1

$$(6.15) \quad \text{Op}(k_g)^*v_2 \in G^{1+h_{d_x}}(\text{Int}(F_\varepsilon)).$$

Thus by Lemma 5.5,

$$(6.16) \quad T^{(1)}(\text{Op}(k_g)^*v_2) \in H_{\Lambda, z}^{\nu} \quad \text{for all } z \in \kappa(\pi^{-1}(\text{Int}(F_{\varepsilon}) \cap \Lambda)).$$

In view of (6.11),

$$WF_{\nu}(v_3) \subset O_R \times (\mathbb{R}^N \setminus \bar{V}_1) \cup T_{\partial O_R}^*(\mathbb{R}^N).$$

Again by (3.7) this yields

$$(6.17) \quad \text{Op}(k_g)^*v_3 \in G^{\nu}(\text{Int}(F_{\varepsilon})).$$

Consequently, by (6.9) and (6.14)–(6.17), we have

$$(6.18) \quad \text{Op}(g)u = u_1 + u_2,$$

where

$$u_1 = \text{Op}(k_g)^*(v_1 + v_3) \in G^{\nu}(\text{Int}(F_{\varepsilon}))$$

and

$$u_2 = \text{Op}(k_g)^*v_2 + \text{Op}(r)\text{Op}(k_g)^*u$$

with

$$T^{(1)}(u_2) \in H_{\Lambda, z}^{\nu} \quad \text{for all } z \in \kappa(\pi^{-1}(\text{Int}(F_{\varepsilon}) \cap \Gamma_0)).$$

Now we apply Lemma 5.1, 5.2 and obtain

$$(6.19) \quad \text{If } (\dot{x}, \dot{\xi}) \in \pi^{-1}(\text{Int}(F_{\varepsilon})) \cap \Gamma_0 \quad \text{and} \quad (\dot{x}, \dot{\xi}) \notin WF_{\nu}(u_2) \\ \text{then } \pi^{-1}(\text{Int}(F_{\varepsilon})) \cap \Gamma_0 \cap WF_{\nu}(u_2) = \emptyset.$$

Because  $g \equiv 1$  in the neighborhood  $V_1$  of  $\dot{\xi}$ ,

$$WF_{\nu}(u_2) \cap \pi^{-1}(\text{Int}(F_{\varepsilon})) \cap \Gamma_0 = WF_{\nu}(u) \cap \pi^{-1}(\text{Int}(F_{\varepsilon})) \cap \Gamma_0.$$

Therefore (6.19) implies Theorem III for  $\tilde{W} = \pi^{-1}(\text{Int}(F_{\varepsilon}))$ .

Since any compact set in  $\Gamma_0 \cap W$  can be covered by a finite number of such  $\tilde{W}$ 's we have actually proved Theorem III.  $\square$

## 7. Remarks

The problem to determine the Gevrey class in which certain  $C^{\infty}$  hypoelliptic operators still remain hypoelliptic, has its origin in a

celebrated example given by Baouendi-Goulaouic [1]:

$$P_1 = \partial_t^2 + \partial_x^2 + t^2 \partial_y^2;$$

which has a solution  $u$  of  $P_1 u = 0$  in a neighborhood of the origin only belonging to  $G^2$ .

Deridj-Zuily [5] and Durand [6] have studied Gevrey hypo-ellipticity for second order operators and proved, for example,  $G^{1+h+0}$  and  $G^{1+h}$  hypoellipticity of the operator (1.6) in Section 1 respectively.

However, as was shown by Parenti-Rodino [24], hypoellipticity does not always imply microlocal one. In this respect, Iwasaki [17] proved among others  $G^2$  microhypoellipticity for double characteristic operators. Our Theorem I is an extension of this in some sense, though the operators are much restricted.

Recently, Kajitani-Wakabayashi also studied Gevrey microhypoellipticity in [18] but for more general classes of operators and obtained the results including our Theorem I as a special case.

However our proof by constructing parametrices reveals how the quasi-homogeneity of operators relate to the lowest order of Gevrey class in which the operators remain hypoelliptic and gives a more precise information on the singularities of solutions (: Proposition 6.1 and Theorem III).

At last, we remark the following: Since  $\text{Op}(k_g)$  act on the space of ultra-distributions  $(G^{1+h})'$  preserving local  $G^{1+h}$  regularities Theorem I and III are valid for  $u \in (G^{1+h})'$  without any change.

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