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ON A NEW CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the unit disk $U = \{z: |z|<1\}$.

We consider some subclasses of the class A. Let S denote the subclass of A whose functions are univalent in U. A function $f(z)$ belonging to the class A is said to be starlike of order α ($0 \le \alpha < 1$) if it satisfies the inequality

$$\text{Re}\,\frac{z f'(z)}{f(z)} > \alpha \qquad (z \in U)$$

for $0 \le \alpha < 1$. We denote by $S^*(\alpha)$ the subclass of A consisting of all starlike functions of order α in U. On the other hand, a function belonging to the class A is said to be convex of order α ($0 \le \alpha < 1$) if it satisfies the inequality

$$\operatorname{Re}\left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \alpha \qquad (z \in U)$$

for $0 \leq \alpha < 1$. We denote by $K(\alpha)$ the subclass of A consisting of such functions. It is well known that $K(\alpha) \subset S^*(\alpha) \subset S$. These classes were introduced by Robertson [13] in 1936, and studied subsequently by Schild [15], MacGregor [5], Pinchuk [12] and Jack [3].

Let T denote the subclass of A of the form

$$(1.1) \qquad f(z) = z - \sum_{k=2}^{\infty} a_k z^k$$

where a_k are non-negative real numbers for all k. In 1975, Silverman [18] introduced the classes $T^*(\alpha) = T \cap S^*(\alpha)$ and $C(\alpha) = T \cap K(\alpha)$ for some $0 \leq \alpha < 1$, and proved the following lemmas.

Lemma A. *A function* $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ *is in* $T^*(\alpha)$ *if and only if* $\sum_{k=2}^{\infty} (k - \alpha) a_k \leq 1 - \alpha$.

Lemma B. *A function* $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ *is in* $C(\alpha)$ *if and only if* $\sum_{k=2}^{\infty} k (k - \alpha) a_k \leq 1 - \alpha$.

Several other subclasses of T were studied by Sarangi and Uralegaddi [14], Owa [6,7,8,9,10,11], Gupta and Jain [1,2] and Jain and Ahuja [4].

In 1986, Sekine and Owa [17] introduced new subclasses $T^*(\alpha, p_k)$ and $C(\alpha, p_k)$ of $T^*(\alpha)$ and $C(\alpha)$, respectively. They defined the subclass of $T^*(\alpha)$ consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{n} \frac{1-\alpha}{k-\alpha} p_k z^k - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0),$$

where $0 \leq p_k \leq 1$ and $0 \leq \sum_{k=2}^{n} p_k \leq 1$, and denoted it by $T^*(\alpha, P_K)$. They also defined the subclass of $C(\alpha)$ consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{n} \frac{p_k(1-\alpha)}{k(k-\alpha)} z^k - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0),$$

where $0 \leq p_k \leq 1$ and $0 \leq \sum_{k=2}^{n} p_k \leq 1$, and denoted it by $C(\alpha, P_K)$. In 1981, the classes $T^*(\alpha, p_2)$ and $C(\alpha, p_2)$ for $k = 2$ were introduced by Silverman and Silvia [19].

In 1987, Sekine [16] introduced a new generalized subclass of T as follows. Let $\{B_k\}$ denote a sequence of positive real numbers, i.e.

(1.2) $B_k > 0 \quad (k = 2, 3, \cdots)$.

Let $T(\{B_k\})$ denote the subclass of T satisfying the coefficient relation

(1.3) $$\sum_{k=2}^{\infty} B_k a_k \leq 1 .$$

All functions belonging to the class $T(\{B_k\})$ satisfy the coefficient relation

(1.4) $$0 \leq a_k \leq \frac{1}{B_k} \quad (k \geq 2).$$

The classes $T^*(\alpha)$ and $C(\alpha)$ become to special cases of Sekine's new class. Sekine [16] showed many relations among the new class and various subclasses of T.

I'd like to introduce a new subclass of $T(\{B_k\})$ by using the inequality (1.4). For a finite sequence $\{p_k\}_{k=2}^{n}$ of real numbers satisfying the condition

(1.5) $\quad 0 \le p_k \le 1 \quad (k = 2,3,\cdots,n), \qquad 0 \le \sum_{k=2}^{n} p_k \le 1,$

we define by $T(\{B_k\},\{p_k\}_2^n)$ the subclass of $T(\{B_k\})$ consisting of functions $f(z)$ of the form :

$$f(z) = z - \sum_{k=2}^{n} \frac{p_k}{B_k} z^k - \sum_{k=n+1}^{\infty} a_k z^k.$$

2. Fundamental results

THEOREM 1. *Let a function f be in the class $T(\{B_k\})$. Then $f \in T(\{B_k\},\{p_k\}_2^n)$ if and only if*

(2.1) $$\sum_{k=n+1}^{\infty} B_k a_k \le 1 - \sum_{k=2}^{n} p_k .$$

The result (2.1) is sharp.

Proof. Since $f \in T(\{B_k\})$, the function f has the form (1.1) and the relation (1.3) holds for a_k and B_k. We put

$a_k = \dfrac{p_k}{B_k} \quad (k = 2,\cdots,n)$, then

$f \in T(\{B_k\},\{p_k\}_2^n)$ if and only if $\displaystyle\sum_{k=2}^{n} B_k \times \frac{p_k}{B_k} + \sum_{k=n+1}^{\infty} B_k a_k \le 1$

This shows the result (2.1). The function $f(z)$ of the form

$$f(z) = z - \sum_{k=2}^{n} \frac{p_k}{B_k} z^k - \frac{1 - \sum_{k=2}^{n} p_k}{B_{N+1}} z^N$$

for $N \geq n + 1$ shows that the result (2.1) is sharp.

The following corollary is a kind of coefficient estimates for f.

COROLLARY 1. *Let a function f be in the class $T(\{B_k\},\{p_k\}_2^n)$. Then,*

$$0 \leq a_k \leq \frac{1 - \sum_{j=2}^{n} p_j}{B_k} \qquad (k \geq n+1). \tag{2.2}$$

The result (2.2) is sharp.

The following theorem shows an inclusion relation.

THEOREM 2. *Let sequences $\{B_k\}_2^\infty$ and $\{p_k\}_2^n$ satisfy (1.2) and (1.5), respectively. Then we have*

$$T(\{B_k\},\{p_k\}_2^n) \subset T(\{B_k d_k\},\{p_k d_k\}_2^m)$$

for positive integers m and n and a sequence $\{d_k\}_2^n$ such that $2 \leq m \leq n$ and $0 < d_k \leq 1$.

We can obtain the proof of Theorem 2 by using the following two lemmas.

LEMMA 1. *Under the same hypotheses as in Theorem 2, we have*

$$T(\{B_k\},\{p_k\}_2^n) \subset T(\{B_k\},\{p_k\}_2^m)$$

for positive integers m and n such that $2 \leq m \leq n$.

LEMMA 2. *Under the same hypotheses as in Theorem 2, we have*

$$T(\{B_k\},\{p_k\}_2^n) \subset T(\{B_kd_k\},\{p_kd_k\}_2^n)$$

for a sequence $\{d_k\}_2^n$ *such that* $0 < d_k \le 1$.

Proof. Let f denote an element of $T(\{B_k\},\{p_k\}_2^n)$. Then we obtain the form

$$f(z) = z - \sum_{k=2}^{n} \frac{p_kd_k}{B_kd_k} z^k - \sum_{k=n+1}^{\infty} a_k z^k$$

and the relation (2.1). The hypotheses $0 < d_k \le 1$, $B_k > 0$ and $a_k \ge 0$ show

(2.3) $\quad 0 \le p_kd_k \le 1 \quad (k = 2,3,\cdots,n), \quad 0 \le \sum_{k=2}^{n} p_kd_k \le 1$

and

$$0 < \sum_{k=n+1}^{\infty} B_kd_ka_k \le \sum_{k=n+1}^{\infty} B_k a_k \le 1 - \sum_{k=2}^{n} p_k \le 1 - \sum_{k=2}^{n} p_kd_k,$$

which prove $f \in T(\{B_kd_k\},\{p_kd_k\}_2^n)$.

Theorem 2' is an analogous result as Theorem 2.

THEOREM 2'. *Let sequences* $\{B_k\}_2^{\infty}$, $\{p_k\}_2^n$ *and* $\{d_k\}_2^n$ *satisfy* (1.2), (2.3) *and* $d_k \ge 1$. *Then we have*

$$T(\{B_kd_k\},\{p_kd_k\}_2^n) \subset T(\{B_k\},\{p_k\}_2^m)$$

for positive integers m *and* n *such that* $2 \le m \le n$.

3. Convexity of the class $T(\{B_k\},\{p_k\}_2^n)$

THEOREM 3. *Let a sequence* $\{n_j\}_1^m$ *consist of integers larger* *than 1 and n denote the minimum of the numbers* $n_1, \cdots, n_m$. *Let each function*

$$(3.1) \qquad f_j(z) = z - \sum_{k=2}^{n_j} \frac{p_k^{(j)}}{B_k} z^k - \sum_{k=n_j+1}^{\infty} a_k^{(j)} z^k \qquad (a_k^{(j)} \geq 0)$$

be in each class $T(\{B_k\}, \{p_k^{(j)}\}_2^{n_j})$ *for each* $j = 1, \cdots, m$. *Then the function* $F(z)$ *defined by*

$$(3.2) \qquad F(z) = \sum_{j=1}^{m} \lambda_j f_j(z),$$

where $\lambda_j \geq 0$, $\sum_{j=1}^{\infty} \lambda_j = 1$, *is in the class* $T(\{B_k\}, \{\sum_{j=1}^{m} \lambda_j p_k^{(j)}\}_2^{n})$.

Proof. By (3.1) and Theorem 1, we have inequalities

$$(3.3) \qquad \sum_{k=n_j+1}^{\infty} B_k\, a_k^{(j)} \leq 1 - \sum_{k=2}^{n_j} p_k^{(j)} \qquad (j = 1, \cdots, m)$$

An easy calculation shows from (3.1) and (3.2) that

$$F(z) = z - \sum_{k=2}^{n} \frac{\sum_{j=1}^{m} \lambda_j p_k^{(j)}}{B_k} z^k - \sum_{k=n+1}^{\infty} \left(\sum_{j=1}^{m} \lambda_j a_k^{(j)} \right) z^k ,$$

where $a_k^{(j)} = \dfrac{p_k^{(j)}}{B_k}$ for $k = n+1, n+2, \cdots, n_j$.

By (3.3) and the definition of $T(\{B_k\}, \{p_k^{(j)}\}_2^{n_j})$, we observe that

$$0 \leq \sum_{j=1}^{m} \lambda_j a_k^{(j)}$$

$$0 \leq \sum_{j=1}^{m} \lambda_j p_k^{(j)} \leq \sum_{j=1}^{m} \lambda_j = 1,$$

$$0 \le \sum_{k=2}^{n}\left(\sum_{j=1}^{m} \lambda_j p_k^{(j)}\right) \le \sum_{j=1}^{m}\left(\lambda_j \sum_{k=2}^{n_j} p_k^{(j)}\right) \le \sum_{j=1}^{m} \lambda_j = 1,$$

and

$$\sum_{k=n+1}^{\infty}\left(B_k \sum_{j=1}^{m} \lambda_j a_k^{(j)}\right) = \sum_{j=1}^{m} \lambda_j \left(\sum_{k=n+1}^{\infty} B_k a_k^{(j)}\right)$$

$$= \sum_{j=1}^{m} \lambda_j \left(\sum_{k=n+1}^{n_j} p_k^{(j)} + \sum_{k=n_j+1}^{\infty} B_k a_k^{(j)}\right)$$

$$\le \sum_{j=1}^{m} \lambda_j \left(\sum_{k=n+1}^{n_j} p_k^{(j)} + 1 - \sum_{k=2}^{n_j} p_k^{(j)}\right)$$

$$= 1 - \sum_{k=2}^{n}\left(\sum_{j=1}^{m} \lambda_j p_k^{(j)}\right),$$

which prove $F \in T(\{B_k\},\{\sum_{j=1}^{m} \lambda_j p_k^{(j)}\}_2^n)$ with the aid of Theorem 1.

Immediately, the following corollaries are obtained by Theorem 3.

COROLLARY 2. *Let functions f and g be in the class $T(\{B_k\},\{p_k\}_2^n)$ and $T(\{B_k\},\{p_k'\}_2^{n'})$, respectively. Then we have*

$$\lambda f + \lambda' g \in T(\{B_k\},\{\lambda p_k + \lambda' p_k'\}_2^n)$$

where $0 \le \lambda \le 1$, $0 \le \lambda' \le 1$, $\lambda + \lambda' = 1$ and $n \le n'$.

The next corollary shows convexity of the class $T(\{B_k\},\{p_k\}_2^n)$.

COROLLARY 3. *If f and g are functions in the class $T(\{B_k\},\{p_k\}_2^n)$ and λ is a real number such that $0 \le \lambda \le 1$, then the function $\lambda f + (1 - \lambda) g$ is also in the class $T(\{B_k\},\{p_k\}_2^n)$.*

We like to obtain a generalization of Corollary 2.

THEOREM 4. *Let f and g be functions in the class*

$T(\{B_k'\},\{p_k'\}_2^{n'})$ *and* $T(\{B_k''\},\{p_k''\}_2^{n''})$, *respectively. Then the function* $\lambda' f + \lambda'' g$, *where* $0 \leq \lambda' \leq 1$, $0 \leq \lambda'' \leq 1$ *and* $\lambda' + \lambda'' = 1$, *is in the class*

$$T\left(\left\{ \frac{B_k' B_k''}{B_k} \right\}, \left\{ \frac{B_k'' p_k' \lambda' + B_k' p''\lambda''}{B_k} \right\}_2^n \right),$$

where $B_k = \max\{B_k', B_k''\}$ *and* $n = \min\{n', n''\}$.

Proof. We may consider the case of $n' = n'' = n$, by virtue of Lemma 1. We can put, with the definitions of f and g and aid of Theorem 1,

$$f(z) = z - \sum_{k=2}^{n} \frac{p_k'}{B_k'} z^k - \sum_{k=n+1}^{\infty} a_k' z^k$$

and

$$g(z) = z - \sum_{k=2}^{n} \frac{p_k''}{B_k''} z^k - \sum_{k=n+1}^{\infty} a_k'' z^k,$$

where

(3.4) $$\sum_{k=n+1}^{\infty} B_k' a_k' \leq 1 - \sum_{k=2}^{n} p_k', \quad \sum_{k=n+1}^{\infty} B_k'' a_k'' \leq 1 - \sum_{k=2}^{n} p_k''.$$

Then we have

$$\lambda' f(z) + \lambda'' g(z)$$

$$= z - \sum_{k=2}^{n} \left(\frac{p_k'}{B_k'} \lambda' + \frac{p_k''}{B_k''} \lambda'' \right) z^k - \sum_{k=n+1}^{\infty} \left(\lambda' a_k' + \lambda'' a_k'' \right) z^k$$

$$= z - \sum_{k=2}^{n} \frac{q_k}{c_k} z^k - \sum_{k=n+1}^{\infty} b_k z^k,$$

where $c_k = \dfrac{B_k' B_k''}{B_k}$, $b_k = \lambda' a_k' + \lambda'' a_k''$ and $q_k = \dfrac{B_k'' p_k' \lambda' + B_k' p_k'' \lambda''}{B_k}$.

Since, by (3.4) and a simple calculation,

$$0 \le q_k \le p_k'\lambda' + p_k''\lambda'' \le \lambda' + \lambda'' = 1,$$

$$0 \le \sum_{k=2}^{n} q_k \le \lambda' \sum_{k=2}^{n} p_k' + \lambda'' \sum_{k=2}^{n} p_k'' \le \lambda' + \lambda'' = 1$$

and

$$\sum_{k=n+1}^{\infty} c_k\, b_k \le \sum_{k=n+1}^{\infty} \left(B_k'\lambda' a_k' + B_k''\lambda'' a_k'' \right)$$

$$\le \lambda' \left(1 - \sum_{k=2}^{n} p_k' \right) + \lambda'' \left(1 - \sum_{k=2}^{n} p_k'' \right)$$

$$= 1 - \sum_{k=2}^{n} \left(\lambda' p_k' + \lambda'' p_k'' \right) \le 1 - \sum_{k=2}^{n} q_k,$$

we obtain that

$$\lambda' f + \lambda'' g \in T(\{c_k\}, \{q_k\}_2^n)$$

with virture of Theorem 1.

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