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多変数 Padé 近似理論と補間について

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Summary. In [4], we studied multivariate Padé-type and Padé approximants by following similar ways to those of Brezinski[2] in univariate case. Brezinski[2] pointed out the fundamental fact that Padé-type approximants of f(t) can be derived by operating the functional c on an interpolation polynomial of the generating function of f(t). Sablonnière[5] and Arioka[1] extended this fact to the multivariate case by using their own functionals and generating functions. In this paper, we explain this fact from our viewpoint in [4] and study the relations to [1] and [5].

§1. Introduction. In [4], we introduced multivariate Padé-type approximants by the following ways. Let $f(t)=f(t_1,\cdots,t_N)$ be a formal power series in N variables $t_1,\cdots,t_N$ with real coefficients,

$$(1.1)\qquad f(t)=c_0+c_1+c_2+\cdots+c_i+\cdots, \quad t=(t_1,\cdots,t_N),$$

where c_i is a homogeneous polynomial of degree i in $t_1,\cdots,t_N$ with real coefficients. And let P(X) be a "formal Laurent series" in X whose coefficients are polynomials in $t_1,\cdots,t_N$,

$$P(X)=a_nX^n+a_{n+1}X^{n+1}+\cdots, \quad a_i\in R[t_1,\cdots,t_N],\ i=n,n+1,\cdots ,$$

where $R[t_1,\cdots,t_N]$ is the polynomial ring in $t_1,\cdots,t_N$ over the real number field R and n is an integer which may be negative. Let $\mathcal{P}$ be the totality of the above "formal Laurent series". Then $\mathcal{P}$ is an integral domain and contains the polynomial in X whose coefficients are polynomials in $t_1,\cdots,t_N$. The inverse element of a unit P(X) of $\mathcal{P}$ is denoted by 1/P(X). For example,

$$(1.2)\qquad \frac{1}{1-X} = 1+X+X^2+X^3+\cdots\cdots,\qquad \frac{1}{X^n(1-X)} = X^{-n}+X^{-n+1}+X^{-n+2}+\cdots\cdots.$$

For (1.1), an operator c acting on $\mathcal{P}$ is defined by

$$c\Big(\sum_i a_i X^i\Big)=\sum_i a_i c_i \quad \text{(with the convention that } c_i=0 \text{ for } i<0).$$

This operator c has the following property:

For $P(X), Q(X) \in \mathcal{P}$ and $a, b \in R[t_1,\cdots,t_N]$,

$$(1.3)\qquad c\big(aP(X)+bQ(X)\big)= ac\big(P(X)\big)+bc\big(Q(X)\big).$$

We define the operator $c^{(n)}$ by $c^{(n)}\big(P(X)\big)=c\big(X^n P(X)\big)$ for $P(X)\in\mathcal{P}$, where n is an integer. Then $c^{(n)}$ also has the same property as (1.3). Operating c or $c^{(n)}$ on the special element X^i of $\mathcal{P}$, we have

$$c(X^i)=c_i \text{ and } c^{(n)}(X^i)=c_{n+i},$$

where $c_i=0$ for $i<0$ and $c_{n+i}=0$ for $n+i<0$.

We have immediately the following lemma by (1.2).

Lemma 1.1 $\quad c^{(n)}\Big(\frac{1}{1-X}\Big) = f(t),\ t=(t_1,\cdots,t_N),\quad (\ n\leqq 0\ ).$

Here 1/(1-X) is called a generating function of f(t).

The polynomial of $\mathcal{P}$ is called a g-polynomial if it is a homogeneous polynomial with respect to N+1 variables $t_1,\cdots,t_N,X$. A g-polynomial V(X) is expressed as follows,

$$(1.4)\qquad V(X) = b_m X^q+b_{m+1}X^{q-1}+\cdots+b_{m+i}X^{q-i}+\cdots+b_{m+q}\ ,\quad b_m \neq 0,$$

where b_{m+i} is a homogeneous polynomial of degree m+i in $t_1,\cdots,t_N$. Then, we call V(X) a g-polynomial of degree q with shift m. $V(1)=b_m+b_{m+1}+\cdots+b_{m+q}$ ($\in R[t_1,\cdots,t_N]$) is called the reverse polynomial of V(X) and denoted by v(t) for $t=(t_1,\cdots,t_N)$.

Multivariate Padé-type approximants "with shift m" are defined as follows.

Definition 1.1. Let $V(X)$ be a g-polynomial of degree q with shift m and

$$(1.5)\qquad w(t) = c\left(\frac{v(t)-X^{p-q+1}V(X)}{1-X}\right), \quad t=(t_1,\cdots,t_N),$$

where $v(t)$ is the reverse polynomial of $V(X)$. Then the rational function $w(t)/v(t)$ is called the (p/q) Padé-type approximant with shift m and denoted by $(p/q)_f^m(t)$. We call the g-polynomial $V(X)$ a generating polynomial of the Padé-type approximant $(p/q)_f^m(t)$.

Theorem 1.1 (Th.2.1 in [4]) In Definition 1.1, $v(t)$ and $w(t)$ are polynomials of degree $m+q$ and $m+p$ respectively. Moreover,

$$(1.6)\qquad f(t)v(t)-w(t)= c^{(p-q+1)}\left(\frac{V(X)}{1-X}\right)= O(m+p+1), \quad t=(t_1,\cdots,t_N).$$

Theorem 1.2 (Th.2.4 in [4]) Let $\overline{w}(t)$ be a function in $t_1,\cdots,t_N$,

$$\overline{w}(t)= c^{(p-q+1)}\left(\frac{v(t)-V(X)}{1-X}\right), \quad t=(t_1,\cdots,t_N).$$

(a) If $p < q$, then $(p/q)_f^m(t)= \dfrac{\overline{w}(t)}{v(t)}$.

(b) If $p \geqq q$, then $(p/q)_f^m(t)= c_0+\cdots+c_{p-q}+ \dfrac{\overline{w}(t)}{v(t)}$.

§2. Relation between Padé-type approximation and polynomial interpolation. Let Ω be a function field which contains all polynomials in $t_1,\cdots,t_N$ (i.e. the rational function field $R(t_1,\cdots,t_N)$ or its extension field). $\Omega[X]$ and $\Omega(X)$ are the polynomial ring and the rational function field in X over the field Ω respectively. In considering the interpolation problem, since it is algebraically meaningless to substitute an element of Ω into the variable X of the formal infinite series $g(X)=1+X+X^2+\cdots$, we need regard the generating function $g(X)$ as an element $1/(1-X)$ of $\Omega(X)$.

Let us now define the Hermite interpolation polynomial of the generating function $g(X)=1/(1-X)$.

Definition 2.1 Let $\alpha_1,\cdots,\alpha_s$ be s given distinct 'points' of Ω. If the polynomial $P_n(X)$ $(\in\Omega[X])$ of degree n in X satisfies the following condition,

(2.1) $P_n^{(j)}(\alpha_i)=g^{(j)}(\alpha_i)$, $0\leqq j\leqq k_i-1$, $i=1,\cdots,s$, $\sum_{i=1}^{s} k_i=n+1$, $k_i\geqq 1$,

where $P_n^{(j)}(X)$ and $g^{(j)}(X)$ denote the j-th formal algebraic derivatives $\frac{d^j}{dX^j}P_n(X)$ and $\frac{d^j}{dX^j}g(X)$ respectively, then the polynomial $P_n(X)$ is called the Hermite interpolation polynomial of $g(X)$ at the nodes $\alpha_1,\cdots,\alpha_s$.

We can prove the uniqueness of such interpolating polynomial in the same way as in the ordinary interpolation problem for a real valued function. If there exists an element α (a function in $t_1,\cdots,t_N$) of Ω such that $P(\alpha)=0\in\Omega$, then the function α is called the zero of $P(X)$. We denote $\frac{1}{u(t)}c^{(n)}\left(P(X)\right)$ by $c^{(n)}\left(\frac{P(X)}{u(t)}\right)$ for the sake of convenience.

The following theorem gives the relation between polynomial interpolation and Padé-type approximation.

Theorem 2.1 Let $V(X)=b_mX^q+b_{m+1}X^{q-1}+\cdots+b_{m+q}$ be a g-polynomial of degree q with shift m. Suppose that s distinct functions (in $t_1,\cdots,t_N$) $\alpha_1,\cdots,\alpha_s$ of Ω are the zeros of multiplicity k_i of $V(X)$, that is, $V(X)=b_m\prod_{i=1}^{s}(X-\alpha_i)^{k_i}$, $k_1+\cdots+k_s=q$, $k_i\geqq 1$.

(a) The case of $p>q-1$. Let $P(X)$ be the Hermite interpolation polynomial of degree p of the generating function $1/(1-X)$ at the nodes $\alpha_1,\cdots,\alpha_s$ and 0 (with multiplicity p-q+1). Then

$$c\bigl(P(X)\bigr)=(p/q)_f^m(t),\quad t=(t_1,\cdots,t_N).$$

(b) The case of $p\leqq q-1$. Let P(X) be the Hermite interpolation polynomial of degree q-1 of 1/(1-X) at the nodes $\alpha_1,\cdots,\alpha_s$. Then

$$c^{(p-q+1)}\bigl(P(X)\bigr)=(p/q)_f^m(t),\quad t=(t_1,\cdots,t_N).$$

In both cases, the denominator of the approximant is the reverse polynomial of V(X) i.e. $V(1)=v(t)=b_m\prod_{i=1}^{s}(1-\alpha_i)^{k_i}$, $t=(t_1,\cdots,t_N)$.

Proof. (a) Put $\overline{P}(X)=\dfrac{v(t)-X^{p-q+1}V(X)}{v(t)(1-X)}$. Then, from the expression

$$\overline{P}(X)=\frac{b_m+\cdots+b_{m+q}-b_mX^{p+1}-\cdots-b_{m+q}X^{p-q+1}}{v(t)(1-X)}$$

$$=\frac{1}{v(t)}\Bigl\{b_m(1+X+\cdots+X^p)+\ \cdots\ +b_{m+q}(1+x+\cdots+X^{p-q})\Bigr\},$$

it follows that $\overline{P}(X)$ is a polynomial of degree p with respect to X. We are going to show that the polynomial $\overline{P}(X)$ satisfies the condition (2.1) for n=p. $\overline{P}(X)$ can be written as follows,

$$\overline{P}(X)=\frac{1}{1-X}-\frac{X^{p-q+1}V(X)}{v(t)(1-X)}=\frac{1}{1-X}-\frac{b_m}{v(t)}\frac{\prod_{i=1}^{s+1}(X-\alpha_i)^{k_i}}{1-X},$$

where $\alpha_{s+1}=0$ and $k_{s+1}=p-q+1$. Differentiating j times with respect to X and substituting α_i into X,

$$\bigl(\overline{P}(X)\bigr)^{(j)}_{X=\alpha_i}=\Bigl(\frac{1}{1-X}\Bigr)^{(j)}_{X=\alpha_i}-\frac{b_m}{v(t)}\Bigl(\frac{\prod_{i=1}^{s+1}(X-\alpha_i)^{k_i}}{1-X}\Bigr)^{(j)}_{X=\alpha_i},$$

where $0\leqq j\leqq k_i-1$, $i=1,\cdots,s+1$ and $\sum_{i=1}^{s+1}k_i=q+(p-q+1)=p+1$. As the last terms equal zero, the condition (2.1) holds. By the uniqueness of the interpolation polynomial, the polynomial $\overline{P}(X)$ coincides with P(X). On the other hand,

$$c\bigl(\overline{P}(X)\bigr)=\frac{1}{v(t)}c\Bigl(\frac{v(t)-X^{p-q+1}V(X)}{1-X}\Bigr)=(p/q)_f^m(t),$$

which implies the result of the case (a).

(b) Put

$$\bar{P}(X)=\frac{v(t)-V(X)}{v(t)(1-X)}=\frac{1}{1-X}-\frac{b_m}{v(t)}\frac{\prod_{i=1}^{s}(X-\alpha_i)^{k_i}}{1-X}.$$

Then, from the similar consideration to that of (a), it follows that $\bar{P}(X)$ is a polynomial of degree q-1 with respect to X and coincides with the Hermite interpolation polynomial of 1/(1-X) at the nodes $\alpha_1,\cdots,\alpha_s$. On the other hand, by Theorem 1.2 (a),

$$c^{(p-q+1)}\left(\bar{P}(X)\right)=\frac{1}{v(t)}c^{(p-q+1)}\left(\frac{v(t)-V(X)}{1-X}\right)=(p/q)_f^m(t),$$

which proves the result of the case (b).

Example 2.1 The functions $\alpha_1=\sqrt{t^2+s^2}$ and $\alpha_2=-\sqrt{t^2+s^2}$ are the zeros of a g-polynomial $V(X)=X^2-t^2-s^2$, $t,s\in R$. Let $P_1(X)$ be the interpolation polynomial of first degree of 1/(1-X) at the nodes α_1, α_2 and $P_2(X)$ the Hermite interpolation polynomial of third degree of 1/(1-X) at the nodes $\alpha_1,\alpha_2,0,0$. Then

$$P_1(X)=\frac{X+1}{1-t^2-s^2},\quad P_2(X)=\frac{X^3+X^2+(1-t^2-s^2)X+1-t^2-s^2}{1-t^2-s^2}$$

and

$$c\left(P_1(X)\right)=\frac{c_1+1}{1-t^2-s^2}=(1/2)_f(t,s),\quad t,s\in R,$$

$$c\left(P_2(X)\right)=\frac{c_3+c_2+(1-t^2-s^2)(c_1+c_0)}{1-t^2-s^2}=(3/2)_f(t,s),\quad t,s\in R,$$

where

$$f(t,s)=\sum_{i=0}^{\infty}\left(\sum_{j+k=i}c_{jk}t^js^k\right)=\sum_{i=0}^{\infty}c_i,\quad c_i=\sum_{j+k=i}c_{jk}t^js^k,\quad c_{jk},t,s\in R.$$

Now let us consider the particular case in Theorem 2.1 such that m=0, $b_0=1$, $\alpha_i=\alpha(i)\cdot t$, $i=1,\cdots,s$, where $\alpha(i)=(\alpha_1^{(i)},\cdots,\alpha_N^{(i)})$, $t=(t_1,\cdots,t_N)$, $\alpha_j^{(i)},t_i\in R$ and $*\cdot*$ denotes the scalar product in R^N. Then, since the polynomial,

$$V(X)=\prod_{i=1}^{s}(X-\alpha_i)^{k_i}=\prod_{i=1}^{s}(X-\alpha(i)\cdot t)^{k_i}, \quad k_1+\cdots+k_s=q,$$

is a homogeneous polynomial of degree q in $t_1,\cdots,t_N,X$, that is, a g-polynomial of degree q with shift 0, we have the following corollary.

Corollary 2.1 (a) The case of $p>q-1$. Let P(X) be the Hermite interpolation polynomial of degree p of the generating function $1/(1-X)$ at the p+1 nodes, $\alpha(1)\cdot t, \cdots ,\alpha(q)\cdot t,0,\cdots,0$. Then

$$c\big(P(X)\big)=(p/q)_f^m(t), \quad t=(t_1,\cdots,t_N).$$

(b) The case of $p\leqq q-1$. Let P(X) be the Hermite interpolation polynomial of degree q-1 of $1/(1-X)$ at the q nodes $\alpha(1)\cdot t, \cdots ,\alpha(q)\cdot t$ Then

$$c^{(p-q+1)}\big(P(X)\big)=(p/q)_f^m(t), \quad t=(t_1,\cdots,t_N).$$

In both cases, the denominator of the approximant is a polynomial of degree q, $\prod_{i=1}^{q}\big(1-\alpha(i)\cdot t\big)$, where $\{\alpha(i)\}$ are not always distinct.

In this corollary, the cases of $p=q-N$ and $p=q-1$ correspond to [5] and [1] respectively (See §3 in detail).

Remark 2.1 In one variable case in [2], the polynomial $v(x)=\prod_i(x-\alpha_i)$ always becomes a generating polynomial of a Padé-type approximant for any given finite points $\{\alpha_i\}$. But in our case, this fact does not hold. In order to be applied Theorem 2.1 to the given functions $\{\alpha_i\}$, it is necessary that the polynomial $V(X)=b_m\prod_i(X-\alpha_i)$ is a g-polynomial. Let us give a simple counter example. The polynomial in X, $V(X)=(X-t^2)(X-s^2)$, $t,s\in R$, is not a g-polynomial. Let P(X) be the Hermite interpolation polynomial of first degree of $1/(1-X)$ at the nodes t^2,s^2. Then

$$c\big(P(X)\big)=c\Big(\frac{X+1-t^2-s^2}{(1-t^2)(1-s^2)}\Big)=\frac{c_1+(1-t^2-s^2)c_0}{(1-t^2)(1-s^2)},\quad t,s\in R.$$

On the other hand, as the denominator $(1-t^2)(1-s^2)$ is the reverse polynomial of the g-polynomial $X^4-(t^2+s^2)X^2+t^2s^2$ and the numerator has second degree, we have, by the definition,

$$(2/4)_f(t,s)=\frac{1}{(1-t^2)(1-s^2)}\,c\Big(\frac{1-t^2-s^2+t^2s^2-X^{-1}(X^4-(t^2+s^2)X^2+t^2s^2)}{1-X}\Big)$$

$$=\frac{1}{(1-t^2)(1-s^2)}\,c\big(1+X+X^2-(t^2+s^2)\big)=\frac{c_2+c_1+(1-t^2-s^2)c_0}{(1-t^2)(1-s^2)}.$$

They are not coincident.

Remark 2.2 Let $P(X)$ be the polynomial such that $c\big(P(X)\big)=(p/q)^{\mathbf{m}}_f(t)$, $t=(t_1,\cdots,t_N)$ for any formal power series $f(t)$, provided that the denominator is fixed. We note that the polynomial $P(X)$ is uniquely determined for $p\geqq q-1$, but for $p<q-1$, such polynomial is not unique. In fact, putting $P_1(X)=\frac{V(t)-V(X)}{V(t)(1-X)}$, then $c^{(p-q+1)}\big(P_1(X)\big)=(p/q)^{\mathbf{m}}_f(t)$ by Theorem 1.2(a). On the other hand, putting $P_2(X)=\frac{X^{q-p-1}V(t)-V(X)}{V(t)(1-X)}$, then $c^{(p-q+1)}\big(P_2(X)\big)=c\Big(\frac{V(t)-X^{p-q+1}V(X)}{V(t)(1-X)}\Big)=(p/q)^{\mathbf{m}}_f(t)$ by the definition. Here, $P_1(X)$ and $P_2(X)$ are different polynomials of degree $q-1$ in X. We derived Theorem 2.1(b) by using the polynomial $P_1(X)$. By taking $P_2(X)$, we can also get the different result from Theorem 2.1(b):

"In the case of $p\leqq q-1$, let $P(X)$ be the Hermite interpolation polynomial of degree $q-1$ of $X^{q-p-1}/(1-X)$ at the nodes $\alpha_1,\cdots,\alpha_q$. Then $c^{(p-q+1)}\big(P(X)\big)=(p/q)^{\mathbf{m}}_f(t)$."

By operating c or $c^{(p-q+1)}$ on the determinantal expression of the Hermite interpolation polynomial $P(X)$ in Theorem 2.1, we can obtain the determinantal expression of Padé-type approximants by the zeros of the generating polynomial.

Theorem 2.2 Let $\alpha_1,\cdots,\alpha_q$ be the distinct zeros of a g-polynomial V(X) of degree q with shift m, i.e $V(X)=b_m\prod_{i=1}^{q}(X-\alpha_i)$, $b_m\neq 0$. Then

$$(p/q)_f^m(t)=\frac{w(t)}{v(t)}=\begin{vmatrix}\sum_{i=0}^{p-q}c_i & c_{p-q+1} & c_{p-q+2}\cdots & c_p\\ -\frac{1}{1-\alpha_1} & 1 & \alpha_1 & \cdots\alpha_1^{q-1}\\ \vdots & \vdots & \vdots & \vdots\\ -\frac{1}{1-\alpha_q} & 1 & \alpha_q & \cdots\alpha_q^{q-1}\end{vmatrix}\Bigg/\begin{vmatrix}1 & \alpha_1\cdots & \alpha_1^{q-1}\\ \vdots & \vdots & \vdots\\ 1 & \alpha_q\cdots & \alpha_q^{q-1}\end{vmatrix}.$$

where $v(t)=V(1)=b_m\prod_{i=1}^{s}(1-\alpha_i)^{k_i}$, $t=(t_1,\cdots,t_N)$ and $\sum_{i=0}^{p-q}c_i=0$ for $p-q<0$.

Proof. Let g(X) be the generating function 1/(1-X).

(a) For p<q, the interpolation polynomial in Theorem 2.1(b) is expressed by the determinant as follows,

$$P(X)=\begin{vmatrix}0 & 1 & X & \cdots & X^{q-1}\\ -g(\alpha_1) & 1 & \alpha_1 & \cdots & \alpha_1^{q-1}\\ \vdots & \vdots & \vdots & & \vdots\\ -g(\alpha_q) & 1 & \alpha_q & \cdots & \alpha_q^{q-1}\end{vmatrix}\Bigg/\begin{vmatrix}1 & \alpha_1 & \cdots & \alpha_1^{q-1}\\ \vdots & \vdots & & \vdots\\ 1 & \alpha_q & \cdots & \alpha_q^{q-1}\end{vmatrix}.$$

Operating $c^{(p-q+1)}$ on P(X), the result immediately follows.

(b) For $p\geqq q$, P(X) in Theorem 2.1(a) is written by

$$P(X)=\begin{vmatrix}0 & 1 & X & X^2\cdots & X^{p-q} & \cdots X^p\\ -1 & 1 & 0 & 0\cdots & 0 & \cdots 0\\ -1! & 0 & 1! & 0\cdots & 0 & \cdots 0\\ -2! & 0 & 0 & 2!\cdots & 0 & \cdots 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ -(p-q)! & 0 & 0 & 0\cdots & (p-q)! & \cdots 0\\ -g(\alpha_1) & 1 & \alpha_1 & \cdots & \alpha_1^{p-q} & \cdots\alpha_1^p\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ -g(\alpha_q) & 1 & \alpha_q & \cdots & \alpha_q^{p-q} & \cdots\alpha_q^p\end{vmatrix}\Bigg/\begin{vmatrix}1 & 0 & 0\cdots & 0 & \cdots 0\\ 0 & 1! & 0\cdots & 0 & \cdots 0\\ 0 & 0 & 2!\cdots & 0 & \cdots 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0\cdots & (p-q)! & \cdots 0\\ 1 & \alpha_1 & \cdots & \alpha_1^{p-q} & \cdots\alpha_1^p\\ \vdots & \vdots & & \vdots & \vdots\\ 1 & \alpha_q & \cdots & \alpha_q^{p-q} & \cdots\alpha_q^p\end{vmatrix}.$$

Taking account of $1-g(\alpha_i)=-\alpha_i g(\alpha_i)$, we get

$$P(X)=\left|\begin{matrix} \sum_{i=0}^{p-q} X_i & X^{p-q+1} & X^{p-q+2} & \cdots & X^p \\ -g(\alpha_1) & 1 & \alpha_1 & \cdots & \alpha_1^{q-1} \\ \vdots & \vdots & \vdots & & \vdots \\ -g(\alpha_q) & 1 & \alpha_q & \cdots & \alpha_q^{q-1} \end{matrix}\right| \Bigg/ \left|\begin{matrix} 1 & \alpha_1 & \cdots & \alpha_1^{q-1} \\ \vdots & \vdots & & \vdots \\ 1 & \alpha_q & \cdots & \alpha_q^{q-1} \end{matrix}\right| .$$

Operating c on P(X), we obtain the result.

§3. Relation to [1] and [5].

As an interpolating polynomial in many variables, Sablonnière[5] and Arioka[1] take up the Hakopian interpolation polynomial and the Kergin one respectively. We are going to study the relation between these polynomials and the polynomial P(X) in §2.

(A) The relation to [1]. Let f(t) be a formal power series,

$$(3.1) \qquad f(t)=\sum_{n\geq 0}\sum_{|i|=n}\bar{c}_i t^i, \quad t=(t_1,\cdots,t_N)\in R^N, \ i=(i_1,\cdots,i_N)\in N^N,$$

where $|i|=i_1+\cdots+i_N$. He defines the functional $\bar{c}$ by $\bar{c}(x^i)=\bar{c}_i/\binom{n}{i}$, $x=(x_1,\cdots,x_N)$, $n=|i|$ and shows that $f(t)=\bar{c}\left(\frac{1}{1-x\cdot t}\right)$, where $x\cdot t=x_1t_1+\cdots+x_Nt_N$, that is, the generating function in [1] is $1/(1-x\cdot t)$. Then it holds that

$$c(X^n)=c_n=\sum_{|i|=n}\bar{c}_i t^i=\bar{c}\left((x\cdot t)^n\right), \quad x=(x_1,\cdots,x_N), \ t=(t_1,\cdots,t_N),$$

which implies that to obtain the expression in [1], it is sufficient to change c into $\bar{c}$ after putting $X=x\cdot t$. For example, $c\left(\frac{1}{1-X}\right)=\bar{c}\left(\frac{1}{1-x\cdot t}\right)$.

Now, applying Corollary 2.1 for $p=q-1$ and q distinct points $\alpha(1),\cdots,\alpha(q)$ of R^N, and putting $X=x\cdot t$, we have

$$(q-1/q)_f(t)=c\left(P(X)\right)=\bar{c}\left(P(x\cdot t)\right), \quad x=(x_1,\cdots,x_N), \ t=(t_1,\cdots,t_N),$$

where

$$P(X)=\frac{v(t)-V(X)}{v(t)(1-X)}, \ V(X)=\prod_{i=1}^{q}(X-\alpha_i), \ v(t)=\prod_{i=1}^{q}(1-\alpha_i), \ \alpha_i=\alpha(i)\cdot t.$$

Here, $P(x\cdot t)$ is a polynomial of degree q-1 in x because P(X) is one of degree q-1 in X. Moreover the condition of the interpolation,

$$P(\alpha_i)=\frac{1}{1-\alpha_i},\qquad i=1,\cdots,q,$$

means the condition with respect to x,

$$P(\alpha(i)\cdot t)=\frac{1}{1-\alpha(i)\cdot t},\quad i=1,\cdots,q,$$

that is, the polynomial $P(x\cdot t)$ is the interpolating polynomial of degree q-1 in x of $1/(1-x\cdot t)$ at the nodes $\alpha(1),\cdots,\alpha(q)$ of R^N and it is nothing else but the Kergin interpolation polynomial $K(x)$ of $1/(1-x\cdot t)$. In fact, from the expression of $K(x)$ (in the proof of Theorem 4.3 in [1]), we have

$$K(x)=\frac{1}{1-x\cdot t}\Big(1-\frac{\prod_{i=1}^{q}(x\cdot t-\alpha(i)\cdot t)}{\prod_{i=1}^{q}(1-\alpha(i)\cdot t)}\Big)=\frac{v(t)-V(X)}{v(t)(1-X)}\Big|_{X=x\cdot t}=P(x\cdot t).$$

(B) The relation to [5]. For a formal power series (3.1), Sablonnière[5] defines the functional $\bar{c}$ by $\bar{c}_i=\binom{n+N-1}{N-1}\binom{n}{i}\bar{c}(x^i)$, $|i|=n$, $x=(x_1,\cdots,x_N)$ and shows that $f(t)=\bar{c}\Big(\frac{1}{(1-x\cdot t)^N}\Big)$, that is, the function $1/(1-x\cdot t)^N$ is the generating function in [5]. Thus there is the following relation between our operation c and $\bar{c}$.

$$c(X^n)=c_n=\sum_{|i|=n}\bar{c}_i t^i=\bar{c}\Big(\binom{n+N-1}{N-1}(x\cdot t)^n\Big).$$

Taking account of the fact that $\frac{d^{N-1}}{dX^{N-1}}\Big(\frac{X^{N-1}}{(N-1)!}\cdot X^n\Big)=\binom{n+N-1}{N-1}X^n$, we have

$$(3.2)\qquad c(X^n)=\bar{c}\Big(\Big(\frac{X^{N-1}}{(N-1)!}X^n\Big)^{(N-1)}\Big|_{X=x\cdot t}\Big),$$

where $(\cdots)^{(N-1)}$ denotes the (N-1)th derivative with respect to X. By (3.2) we can obtain the expression in [5]. For examples,

$$(3.3)\qquad c\Big(\frac{1}{1-X}\Big)=\sum_{n=0}^{\infty}c(X^n)=\sum_{n=0}^{\infty}\bar{c}\Big(\Big(\frac{X^{N-1}}{(N-1)!}X^n\Big)^{(N-1)}\Big|_{X=x\cdot t}\Big)$$

$$=\bar{c}\Big(\Big(\frac{X^{N-1}}{(N-1)!}\frac{1}{1-X}\Big)^{(N-1)}\Big|_{X=x\cdot t}\Big)=\bar{c}\Big(\frac{1}{(1-X)^N}\Big|_{X=x\cdot t}\Big)=\bar{c}\Big(\frac{1}{(1-x\cdot t)^N}\Big),$$

Moreover, (3.2) holds also for n such that $1-N \leq n < 0$ since the both sides equal zero. Thus we have

$$(3.4) \qquad c\left(\frac{P(X)}{X^{N-1}}\right) = \bar{c}\left(\left(\frac{X^{N-1}}{(N-1)!}\frac{P(X)}{X^{N-1}}\right)^{(N-1)}_{|X=x\cdot t}\right) = \bar{c}\left(\left(\frac{P(X)}{(N-1)!}\right)^{(N-1)}_{|X=x\cdot t}\right),$$

where P(X) is a polynomial in X.

Now let us apply Corollary 2.1 for q=r+1, p=r-N+1 and $\alpha(i+1)=x(i)$ $(i=0,1,\cdots,r)$. Then,

$$(r-N+1/r+1)_f(t) = c^{(-N+1)}(P(X)) = c\left(\frac{P(X)}{X^{N-1}}\right), \quad t=(t_1,\cdots,t_N),$$

$$=\bar{c}\left(\left(\frac{P(X)}{(N-1)!}\right)^{(N-1)}_{|X=x\cdot t}\right) \quad \text{(by (3.4))},$$

where

$$P(X) = \frac{v(t)-V(X)}{v(t)(1-X)}, \quad V(X)=\prod_{i=0}^{r}(X-x(i)\cdot t), \quad v(t)=\prod_{i=0}^{r}(1-x(i)\cdot t).$$

Put $p(x,t)=\left(\frac{P(X)}{(N-1)!}\right)^{(N-1)}_{|X=x\cdot t}$. Then it is a polynomial of degree r-N+1 in x because P(X) is one of degree r in X. We are going to show that this polynomial p(x,t) is nothing else but the Hakopian interpolation polynomial in [5]. From the expression

$$P(X) = \frac{1}{1-X} - \frac{1}{v(t)}\cdot\frac{\prod_{i=0}^{r}(X-x(i)\cdot t)}{1-X},$$

we have

$$(3.5) \qquad p(x,t) = \frac{1}{(N-1)!}\left(\frac{1}{1-X}\right)^{(N-1)}_{|X=x\cdot t} - \frac{1}{(N-1)!\,v(t)}\left(\frac{\prod_{i=0}^{r}(X-x(i)\cdot t)}{1-X}\right)^{(N-1)}_{|X=x\cdot t}$$

$$= g(x,t) - \frac{1}{(N-1)!\,v(t)}U^{(N-1)}(x\cdot t),$$

where

$$g(x,t) = \frac{1}{(1-x\cdot t)^N} \quad \text{and} \quad U(X) = \frac{\prod_{i=0}^{r}(X-x(i)\cdot t)}{1-X}.$$

We prepare some notations. Let $i=(i_0,i_1,\cdots,i_{N-1})$ be a subset of $\{0,1,\cdots,r\}$ and $X_i=\{x(i_0),x(i_1),\cdots,x(i_{N-1})\}$ a subset of points $\{x(0),x(1),\cdots,x(r)\}$ in R^N. For a function h(x), $h\{X_i\}$ is defined by

$$(3.6) \qquad h\{X_i\} = (N-1)!\int_{Q^{N-1}} h\left(\lambda_0 x(i_0)+\cdots+\lambda_{N-1}x(i_{N-1})\right)d\lambda,$$

where $Q^{N-1}=\{(\lambda_1,\cdots,\lambda_{N-1})\in R^{N-1};\ \lambda_1+\cdots+\lambda_{N-1}\leqq 1,\ \lambda_i\geqq 0\}$, $\lambda_0=1-\sum_{i=1}^{N-1}\lambda_i$ and $d\lambda=d\lambda_1\cdots d\lambda_{N-1}$. Now let us prove that $p(x,t)$ satisfies the condition of the Hakopian interpolation, i.e.

$$(3.7)\qquad p(\{X_i\},t)=g(\{X_i\},t)\quad \text{for every multi-index } i=(i_0,i_1,\cdots,i_{N-1}).$$

From the expression (3.5) and the definition (3.6), we obtain that

$$p(\{X_i\},t)=g(\{X_i\},t)-\frac{1}{v(t)}\int_{Q^{N-1}}U^{(N-1)}(\{\lambda_0 x(i_0)+\cdots+\lambda_{N-1}x(i_{N-1})\}\cdot t)d\lambda$$

$$=g(\{X_i\},t)-\frac{1}{v(t)}\int_{Q^{N-1}}U^{(N-1)}(\lambda_0\{x(i_0)\cdot t\}+\cdots+\lambda_{N-1}\{x(i_{N-1})\cdot t\})d\lambda,$$

by the Hermite-Gennochi formula,

$$=g(\{X_i\},t)-\frac{(-1)^{N-1}}{v(t)}U[x(i_0)\cdot t,\ \cdots\ ,x(i_{N-1})\cdot t],$$

where $U[x(i_0)\cdot t,\ \cdots\ ,x(i_{N-1})\cdot t]$ denotes the divided difference of U at $x(i_0)\cdot t,\cdots,x(i_{N-1})\cdot t$. In the last expression, the second term in the right hand side is vanished by the fact $U(x(i)\cdot t)=0$ $(i=0,1,\cdots,r)$, which implies the result.

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