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Semilinear equations with exponential nonlinearity

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Our purpose is to review some mathematical study, old and new, on a nonlinear elliptic boundary value problem and to illustrate remarkable structures of its solutions. Thus, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Regarding the positive constant λ as a parameter, we consider the semilinear elliptic eigenvalue problem

$$(0.1) \qquad -\Delta u = \lambda e^u \quad (\text{in } \Omega), \qquad u = 0 \quad (\text{on } \partial\Omega).$$

In the field of mathematical physics, this problem arises in the theory of thermonic emission (Gel'fand [12]), in the problem of the isothermal gas sphere (Chandrasekhar [6]) and in the theory of gas combustion (Mignot-Murat-Puel [21]). Also it has mathematical structures such as differential geometry and complex function theory (Liouville [20], Poincaré [29], Picard [26-28], Bieberbach [5], Lichtenstein [18], Walker [37]).

Existence of solutions — unique existence, multiple existence, and nonexistence — is one of the basic questions for this problem, while property of solutions such as geometrical profile of level sets, stability, and so on, the other. The first task for these questions is to try to continue with respect to λ the trivial solution $(\lambda, u) = (0, 0)$ to obtain "mild" solutions. Next, we pick up "striking" solutions via singular limit and/or singular perturbation approach. Finally, we have to make global analysis about the connectivity in λ of those solutions. In performing these procedures, we believe, three obstructions are there. The first is the space dimension n, the second is the topology of Ω, and the final is the geometry of Ω. However, the most clear answer to those questions seems to have been given when Ω is a ball.

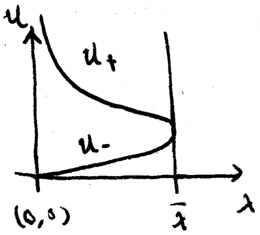
§1. Radial solutions.

First we note $f(t) = \lambda e^t > 0$ and hence any solution $u = u(x) \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ of (0.1) is positive by the maximum principle. Therefore, every solution $u = u(x)$ is radial: $u = u(|x|)$, when the domain Ω is a ball: $B = \{|x| < 1\} \subset \mathbb{R}^n$, so that (0.1) is reduced to

$$(1.1) \quad -\left(\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr}\right) u = \lambda e^u \qquad (0 < r < 1), \quad u'(0) = u(1) = 0.$$

For this fact we refer to Gidas-Ni-Nirenberg [13].

In the case $n = 1$ or 2, the solutions are given explicitly and the following bifurcation diagram is obtained:



Here, we have the asymptotic behavior

$$(1.2) \qquad u_+(\lambda) \sim 4\log\frac{1}{|\lambda|}, \quad u_-(\lambda) \sim 0 \quad \text{as } \lambda \downarrow 0$$

for the case $n=2$, which should be noted for later quotations.

On the other hand when $n>2$ there is a transformation due to Emden [8] for the initial value problem

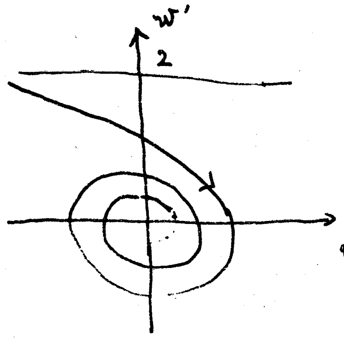
$$(1.3) \qquad (r^{n-1}u')' + \lambda r^{n-1}e^{u} = 0 \quad (0<r), \quad u(0)=A>0, \quad u'(0)=0.$$

Namely, first we set $u(r) = v(s) + A$ for $r = Bs$ with $B = \{2 \times (n-2)/\lambda e^{A}\}^{1/2}$ and next $v(s) = w(t) - 2t$ for $s = e^{t}$. Through that, (1.3) is transformed into

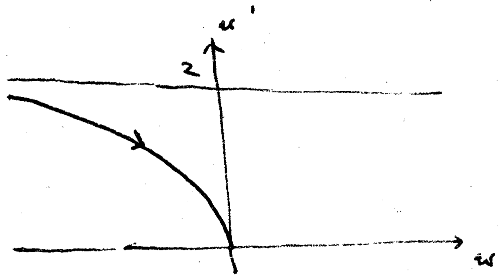
$$(1.4) \qquad \begin{cases} w'' + (n-2)w' + 2(n-2)(e^{w}-1) = 0 \\ \lim\limits_{t\to-\infty}\{w(t)-2t\} = \lim\limits_{t\to-\infty} e^{-t}\{w'(t)-2\} = 0, \end{cases}$$

which is autonomous without any parameters. The orbit $\mathcal{O} = \{(w(t),$

$w'(t)\}$ of (1.4) is illustrated as follows:

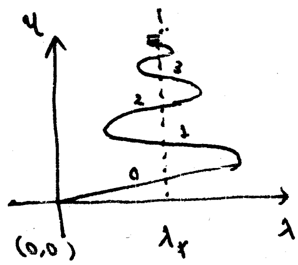


$2<n<10$

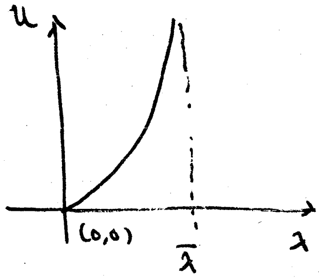


$10 \leqq n$

Each point (w_0, z_0) on the orbit $\mathcal{O}$ corresponds to the time t_0. The condition $u(r)=0$ for $r=1$ is equivalent to $A=-w_0+2t_0$ with $e^{t_0}=B^{-1}$. Hence $\lambda=2(n-2)e^{w_0}$. Namely, each point (w_0, z_0) on the orbit $\mathcal{O}$ corresponds to a solution of (1.3) with $\lambda=2(n-2)e^{w_0}$ and the converse is true. Hence the bifurcation diagram for (0.1) on $\Omega=\{|x|<1\}\subset\mathbb{R}^n$ is given as follows:



$2<n<10$

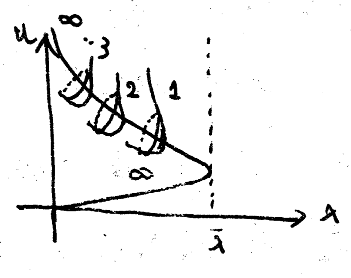


$10 \leqq n$

These results are due to Joseph-Lundgren [14].

The linearized operators of (0.1) around these radial solutions are given as $A_p \equiv -\Delta_D - p$, where $p=\lambda e^u$ is a radial function. Hence its spectrum is divided into $\sigma(A_p)=\bigcup_{m=0}^{\infty}\sigma_m(p)$ via the separation of variables. Namely, here $\sigma_m(p)$ denotes the spectrum of the ordinary differential operator $-\left(\frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}\right)+\frac{\rho_m}{r^2}-p(r)$ with $\frac{d}{dr}\cdot\big|_{r=0} = \cdot\big|_{r=1}=0$, where $\{\rho_m = m(n-2+m) \mid m=0,1,2,\cdots\}$ denotes the spectrum of the Laplace-Beltrami operator on S^{n-1}. We can show that $\sigma_m \cap (-\infty, 0] = \phi$ when $m \geq 1$, while $\ell \stackrel{def}{=} \#\{\sigma_0 \cap (-\infty,0]\}$ increases one by one from zero through the bending of the solution branch, and hence to infinite if $2<n<10$. For these facts, see Suzuki [31].

On the contrary, non-radial solutions may arise when Ω is an annulus $A \equiv \{a<|x|<1\} \subset \mathbb{R}^n$, where $0<a<1$. Up to now, only two-dimensional case has been studied by Nagasaki-Suzuki [25] and Lin [19]. Namely, radial solutions for (0.1) in this case form a similar diagram as for $\Omega = B$, while symmetry-breaking occurs. Those bifurcating corns (i.e., two-dimensional manifolds in $\lambda-u$ plane) have been shown to be only infinitesimally of m-mode, where "mode" denotes the period in the argumental direction. On the other hand from variational approach we know that for each $m=1,2,\cdots$, there exists a family of solutions $\{(u,\lambda)\}$ of (0.1) of mode m with $\int_\Omega e^u dx \to +\infty$. Those families are supposed to be the bifurcating cones from radial solutions. See also Suzuki [32] for details.



Radial singular solutions are studied in Mignot-Puel [22]. Bandle [4] studied perturbation of domains around radial cases.

§2. Mild solutions

Henceforth Ω denotes a general domain $\mathbb{R}^n$. Still we have the following facts for solutions to (0.1) via the maximum principle.

<u>Fact 1</u>. There exists a $\bar{\lambda} \in (0, +\infty)$ such that (0.1) has no solution when $\lambda > \bar{\lambda}$, while (0.1) has at least one solution when $0 < \lambda < \bar{\lambda}$.

<u>Outline of proof</u>: Non-existence part follows from Kaplan's method. Namely, taking the first eigenfunction $\varphi_1 > 0$ of $-\Delta_D$, we apply Jensen's inequality for $J = \int_\Omega u \varphi_1 dx > 0$.

Existence part is verified through the method of super- and sub-solutions. Namely, if there exist smooth functions $\underline{u}$ and $\bar{u}$ such that

$$-\Delta \underline{u} \leq f(\underline{u}), \quad -\Delta \bar{u} \geq f(\bar{u}), \quad \underline{u} \leq \bar{u} \ (\text{in } \Omega), \quad \underline{u} \leq 0, \ \bar{u} \geq 0 \ (\text{on } \partial\Omega),$$

then there exists a solution u for

$$(2.1) \qquad -\Delta u = f(u) \quad (\text{in } \Omega), \qquad u = 0 \quad (\text{on } \partial\Omega),$$

satisfying $\underline{u} \leq u \leq \bar{u}$. //

In particular, now $f(t)=\lambda e^t$ is monotone increasing and the iterations

$$\overline{u}_{n+1}=(-\Delta_D)^{-1}f(\overline{u}_n),\ \overline{u}_0=\overline{u}\ ;\quad \underline{u}_{n+1}=(-\Delta_D)^{-1}f(\underline{u}_n),\ \underline{u}_0=\underline{u}$$

form the monotone iterations: $\underline{u}\leqq \underline{u}_1\leqq\cdots\leqq\underline{u}_n\leqq\cdots\leqq\overline{u}_n\leqq\cdots\leqq\overline{u}_1\leqq\overline{u}$ to reach u. This property yields the following

<u>Fact 2</u>. For each fixed λ, the set $\mathcal{S}_\lambda=\{u\in C^0(\overline{\Omega})\cap C^2(\Omega)\mid u \text{ solves } (0.1)\}$ has a minimal element $u=\underline{u}_\lambda$ if $\mathcal{S}_\lambda\neq\phi$.

If u is a minimal solution $(\underline{u}_\lambda)$, then $\mu_1(p)\geqq 0$. Conversely, $\mu_1(p)>0$ implies the minimality of u. Here and henceforth, $\{\mu_j(p)\}_{j=1}^{\infty}$ $(-\infty<\mu_1(p)<\mu_2(p)\leqq\cdots\to+\infty)$ denotes the spectrum of the linearized operator $A_p\equiv-\Delta-p$ for $p=\lambda e^u$.

<u>Fact 3</u>. There exists no triple $\{u_1,u_2,u_3\}\subset\mathcal{S}_\lambda$ satisfying $u_1\lneqq u_2\lneqq u_3$.

<u>Outline of proof</u>: Taking such a triple $\{u_1,u_2,u_3\}\subset\mathcal{S}_\lambda$, we see that $\varphi\equiv u_2-u_1$ and $\chi\equiv u_3-u_2$ are first eigenfunctions for $L\equiv-\Delta_D-c(x)$ and $H\equiv-\Delta_D-d(x)$, respectively, with zero eigenvalues, where $c(x)=f'((1-\theta)u_2+\theta u_1)$ and $d(x)=f'((1-\gamma)u_3+\gamma u_2)$ for some $\theta=\theta(x)$ and $\gamma=\gamma(x)\in(0,1)$. However, $f(t)=\lambda e^t$ is convex, so that $d\gneqq c$, which gives a contradiction. //

These facts have been shown by Keller-Cohen [16], Fujita [10], Laetsch [17] and Keener-Keller [15].

The following fact holds by the implicit function theory.

Fact 4. Minimal solutions $\{(\lambda, \underline{u}_\lambda) \mid 0<\lambda<\bar{\lambda}\}$ form a branch $\underline{\mathcal{S}}$, i.e., one-dimensional manifold in $\lambda-u$ plane, starting from $(\lambda, u) = (0,0)$.

Furthermore, we have

Fact 5. In the case of $n \leqq 9$, $\underline{\mathcal{S}}$ continues up to $\lambda=\bar{\lambda}$ and then bends back.

Outline of proof : The first part follows from an apriori estimate. Namely, supposing that (u, λ) solves

$$(2.2) \qquad -\Delta u = \lambda f(u) \quad (\text{in } \Omega), \quad u=0 \quad (\text{on } \partial\Omega)$$

with the positive linearized operator $-\Delta_D - \lambda f'(u)$, we have

$$\int_\Omega |\nabla w|^2 dx \geq \lambda \int_\Omega f'(u) w^2 dx \qquad \text{for } w \in H_0^1(\Omega).$$

Take $w = \varphi(u)$ with $\varphi \in C^1(\mathbb{R} \to \mathbb{R})$ and $\varphi(0)=0$. Then, (2.3) $$\int_\Omega \varphi'(u)^2 |\nabla u|^2 dx \geq \lambda \int_\Omega f'(u)\varphi(u)^2 dx.$$

On the other hand, multiplying $\chi(u)$ in (2.2) for $\chi \in C^1(\mathbb{R}\to\mathbb{R})$ with $\chi(0)=0$, we get

(2.4) $$\int_\Omega \chi'(u)\, |\nabla u|^2 dx = \lambda \int_\Omega f(u)\chi(u) dx .$$

Hence

(2.5) $$\int_\Omega f(u)\chi(u)\, dx \geq \int_\Omega f'(u)\, \varphi(u)^2 dx$$

provided that $\chi' \geq (\varphi')^2$.

Now for the case $f(t)=e^t$, we take $\varphi(t)=e^{mt}-1$ and $\chi(t)=\int_0^t \varphi'(s)^2 ds = \frac{m}{2}(e^{mt}-1)$. Then,

$$\frac{m}{2}\int_\Omega \{e^{(2m+1)u} - e^u\} dx \geq \int_\Omega \{e^{(2m+1)u} - 2e^{(m+1)u} + e^u\} dx.$$

Therefore, $\|e^u\|_{L^{2m+1}} \in O(1)$ if $m<2$, i.e., $\|f(u)\|_{L^p} \in O(1)$ for $p<5$. Hence $\|u\|_{W^{2,p}} \in O(1)$ by $\lambda \leq \bar\lambda$ so that $\|u\|_{L^\infty} \in O(1)$ of $n<2p$, i.e., $n<10$.

Now at $\lambda=\bar\lambda$ we have $\mu_1(p)=0$. Then again the implicit function theory gives a detailed description of the solution set of (0.1) around

$(\bar\lambda, \underline{u}_{\bar\lambda})$. In fact, the bending back can be proven. //

Those approaches from nonlinear functional analysis were taken by Crandall-Rabinowitz [7]. Regarding the radial solutions, we see that $n \le 9$ is optimal in Fact 5. Mignot-Murat-Puel [22] and Gallouet-Mignot-Puel [11] have also studied the behavior of solutions around $\lambda = \bar\lambda$. Bandle [1-2] has established an isoperimetric inequality for $\bar\lambda = \bar\lambda(\Omega)$.

§3. Striking solutions.

In the case $n=2$, the nonlinearity $f(t) = e^t$ is of sub-critical, and hence the Mountain pass lemma applies. Consequently, Crandall-Rabinowitz [7] has shown the following

Fact 6. For each $\lambda \in (0, \bar\lambda)$, there exists some $u \in S_\lambda \setminus \{\underline{u}_\lambda\}$.

More remarkably, (0.1) has integrals when $n=2$. In fact, $-\Delta u = \lambda e^u$ is written as

$$\frac{\partial^2}{\partial x \partial y} \log f = c f^2$$

via a complex transformation, which is the Liouville equation ([20]).

In particular, $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ solves (0.1) if and only if there exists an analytic function $F = F(z)$ on $\Omega \subset \mathbb{C}$ such that $\rho(F) \equiv |F'| / \{1 + |F|^2\}$ is single-valued, positive, continuous on $\overline{\Omega}$, and

$$(3.1) \qquad \rho(F)\Big|_{\partial\Omega} = \left(\frac{\lambda}{8}\right)^{1/2},$$

satisfying

$$(3.2) \qquad \left(\frac{\lambda}{8}\right)^{1/2} e^{u/2} = \rho(F) \quad (\text{in } \Omega).$$

See Bandle [3], for example.

This fact was taken up by Weston [39] to construct striking solutions for (0.1) through the singular perturbation method. The theory is divided into two parts. First, asymptotic solutions $\{u = u_n\}$ are constructed as $\lambda \downarrow 0$ for $n = 1, 2, \cdots$. That is, $u_n = u_n(x)$ satisfies

$$-\Delta u_n = \lambda e^{u_n} \quad (\text{in } \Omega), \qquad \|u_n\|_{L^\infty(\partial\Omega)} \in O(\lambda^n).$$

Next, a modified Newton method is shown to converge for (0.1) if the starting point is taken to be the n-th asymptotic solution u_n. We shall review briefly his theory.

First, through a conformal mapping $g : \Omega \longrightarrow B = \{|\zeta| < 1\}$, (3.1) is transformed into finding a holomorphic function $G = G(\zeta)$ on B such that

$$(3.3)\qquad \rho(G) = \left(\frac{\lambda}{8}\right)^{1/2} |g'| \qquad (\text{on } \partial B).$$

Putting $G = \lambda^{-1/2} G_0$, we have $\frac{8}{\lambda}\rho(G)^2 = \frac{8|G_0'|^2}{|G_0|^4}\{1 + O(\lambda)\}$ and the first approximate equation for (3.3),

$$(3.4)\qquad |g'| = \sqrt{8}\, |G_0'| \big/ |G_0|^2 \qquad (\text{on } \partial B)$$

is obtained. Substitution of the formal expression $G \sim \lambda^{-1/2} \sum_{q=0}^{\infty} \lambda^q G_q$ into (3.3) gives also higher approximate equations.

In the case $\Omega = B$, the mapping g is takes as identity. Then we can take $G^+ = G^+(z) \sim -\sqrt{8}\, z$ for (3.3) corresponding to the explicit solution $u^+(z) \sim 4 \log \frac{1}{|z|}$ as $\lambda \downarrow 0$. On the other hand (3.4) follows from

$$(3.5)\qquad \sqrt{8}\, \frac{d}{dz} [G_0(z)]^{-1} = A(z)\, g'(z) \qquad (\text{in } B),$$

$A = A(z)$ satisfying $|A(z)| = 1$ on ∂B. Taking acount of the above radial case, we put $A(z) = \left(\frac{1 - \bar{\delta} z}{z - \delta}\right)^2$ for some $\delta \in B$. Then, for $G_0 = G_0(z)$ to be single-valued in (3.5), that is, to eliminate the $\log \zeta$ term in integration, we arrive at the equation

$$(3.6)\qquad \bar{\delta} = \frac{1}{2}(1-|\delta|^2)\, g''(\delta)/g'(\delta)$$

for $\delta \in B$. Though a Möbius transformation on B, δ is reduce to 0.

Higher order equations are to determine holomorphic functions in B for given real parts of their derivatives on ∂B. Hence in the integration procedure, a freedom by the integral constant appears, while simultaneously the necessity to eliminate the $\log \zeta$ term arises. The integral constant of n-th approximate solution is utilized to make $(n+1)$-th approximate solution to be single-valued.

Adopting another expression of the Liouville integral, Moseley [23] has succeeded in reducing the assumptions for the asymptotic solutions to be constructed. That is,

$$(3.7)\qquad \alpha \equiv |g_N'''(0)/g_N'(0)| \neq 2,$$

where $g_N = g \circ \varphi_\delta$ with $\varphi_\delta(\zeta) = \dfrac{\zeta+\delta}{1-\bar{\delta}\zeta}$.

Genuine solutions for (0.1) are constructed through a modified Newton method. Thus, we note that (0.1) is equivalent to the integral equation

(3.8) $$U = K(U)$$

on B, where $(KU)(x) = \int_B k(x,y)\,(\lambda|g'|^2 e^U)(y)\,dy$, $k = k(x,y)$ being the Green function on B. Writing (3.8) as

(3.9) $$U = S(U)$$

with $S(U) = (1-K'_{U_0})^{-1}(K(U) - K'_{U_0}(U))$, we can give a modified Newton scheme

(3.10) $$U_{n+1} = S(U_n) \qquad (n = 0, 1, 2, \cdots).$$

The following abstract theorem is utilized to establish the convergence of (3.10): Let X be a Banach space and $S \in C^1(X \to X)$. Suppose that there exists a $\varphi \in C^1(\mathbb{R} \to \mathbb{R})$ and some $t^* > 0$ satisfying $\varphi(t^*) = t^*$, $\|S(U_0) - U_0\| \le \varphi(0)$ and $\|S'(U)\| \le \varphi'(t)$ for $\|U - U_0\| \le t$. Then $\{U_n\}$ of (3.10) converges to U^* with the relation $\|U^* - U_0\| \le t^*$.

For the proof, see Vainberg [36]. Here, we adopt $X = C^0(\bar{B})$, $\varphi(t) = \varphi(0) + P(e^t - t - 1)$, $P \ge \|(1-K'_{U_0})^{-1} K'_{U_0}\|$ and $\varphi(0) = (1+P) \times \|U_0 - K(U_0)\|$ for (3.10). Taking U_0 as the n-th asymptotic solution, we have $\|U_0 - K(U_0)\| \in O(\lambda^n)$. On the other hand, the estimate

$$(3.10) \qquad \|(1-K'_{U_0})^{-1}\| \in O(1/\lambda)$$

holds under some generic condition for Ω. Thus we obtain

Fact 7. Under a generic condition for Ω other than (3.7), we have a solution $u^* = U^* \circ g$ for (0.1) if $\lambda > 0$ is small, satisfying $\|U^* - U_0\| \in O(\lambda^{\frac{1}{2}(n-1)})$, where U_0 denotes the n-th asymptotic solution with $n \geq 3$.

The proof of (3.10) is the most crucial part of Weston [39] and Wente [38] has refined some of his arguments. Anyhow, we can show that for the essential part of the operator K'_{U_0}, which is denoted by K_0, the relation $K_0 \varphi = \Lambda \varphi$ is equivalent to

$$(3.11) \qquad \Delta \varphi + \frac{1}{\Lambda} V_0(x) \varphi = 0 \quad (\text{in } B), \qquad \varphi = 0 \quad (\text{on } \partial B),$$

where $V_0(x) = \dfrac{8\mu}{(r^2+\mu)^2}$ with $\mu = \frac{\lambda}{8} |g_N'(0)|^2$ and $r = |x|$. This eigenvalue problem (3.11) is transformed into the associated Legendre equation

$$(3.12)\quad \begin{cases} [(1-\zeta^2)\Phi_\zeta]_\zeta + [2/\Lambda - m^2/(1-\zeta^2)]\Phi = 0 & (\zeta_\mu < \zeta < 1) \\ \Phi(1) : \text{bounded}, \quad \Phi(\zeta_\mu) = 0, \end{cases}$$

where $\zeta_\mu = \dfrac{\mu-1}{\mu+1}$, through the separation of variables $\varphi = \Phi(r)e^{im\theta}$ and the Bandle transformation $\zeta = (\mu - r^2)/(\mu + r^2)$ ([3]). Now (3.10) is reduced to an asymptotic analysis for (3.12).

As we have seen, the solution u^* blows up as $\lambda \downarrow 0$ at the single point $\kappa = g(\delta) = g_\lambda(0) \in \Omega$. This point $\kappa \in \Omega$ is characterized through the Green function $K = K(x,y)$ of $-\Delta_D$ in Ω. Namely, κ is nothing but a critical point of $\ell(x) = L(x,x)$, where $L(x,y) = K(x,y) + \frac{1}{2\pi}\log|x-y|$. Such a point $\kappa \in \Omega$ we call a core. When $\Omega \subset \mathbb{R}^2$ is simply connected, then cores are finite. Furthermore, core is unique when Ω is convex. For these facts, see for example Friedman [9].

Spruck [30] has studied conversely the asymptotic behavior of the solutions for

$$(3.13)\qquad -\Delta u = \lambda \sinh u, \quad u > 0 \ (\text{in } \Omega), \qquad u = 0 \ (\text{on } \partial\Omega).$$

when Ω is a rectangle. Inspired by this, Suzuki-Nagasaki [35] has shown the following

<u>Fact 8</u>. In the case that $\Omega \subset \mathbb{R}^2$ is star-shaped, the solutions $\{u\}$ of (0.1) accumulate as $\lambda \downarrow 0$ to $v=0$ or $v = 8\pi E_\kappa(x)$ in $W^{1,p}(\Omega)$ with $1<p<2$, where $\kappa \in \Omega$ is a core and $-\Delta E_\kappa = \delta(\kappa)$ with $E_\kappa|_{\partial\Omega} = 0$. In particular, unless $\{u\}$'s approach to the trivial solution 0, they make one-point blow up at some core $\kappa \in \Omega$ and $\Sigma \to 8\pi$ as $\lambda \downarrow 0$, as far as $\{\Sigma\}$ is bounded.

<u>Outline of proof</u>: The proof is done in three steps. First is to show that when $\Omega \subset \mathbb{R}^2$ is star-shaped, then $\Sigma \overset{\text{def}}{=} \lambda \int_\Omega e^u dx$ is bounded. This is an immediate consequence of the Rellich-Pohozaev identity. Next, when $\{\Sigma\}$ is bounded, then the blow-up points are finite. This is proven in a complex function theoretic way via the Liouville integral. Finally, when Ω is simply connected, then only one blow-up point κ is permitted with κ being a core and $v = 8\pi E_\kappa$, unless $v \equiv 0$. This can be shown by modifying some arguments of Spruck [30]. //

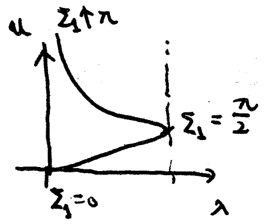
We expect that when (3.7) holds, the function $v = 8\pi E_\kappa$ admits only one branch of solutions for (0.1) approaching it as $\lambda \downarrow 0$. On the other hand, we expect that each m-mode non-radial solution of (0.1) on $A = \{a < |x| < 1\} \subset \mathbb{R}^2$ makes m-points blow-up as $\lambda \downarrow 0$. In fact in this case $\Sigma \to +\infty$ along the branch of radial solutions, which make entire blow-up as $\lambda \downarrow 0$. Incidentally, Facts 7 and 8 holds for other semilinear equations with exponentially dominated nonlinearities.

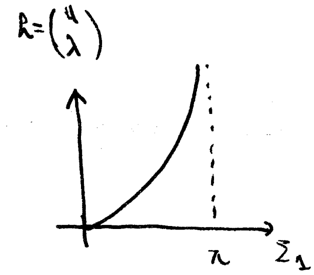
§4. Connections.

It is quite interesting whether and how these solutions connect with mild solutions in λ–u plane. One way to approach to this question is the topological degree argument after having established some a priori estimates and explicit structure of singular limit of solutions as $\lambda \downarrow 0$. The other approach is to appeal to the rearrangement theory to establish a similar bifurcation diagram to the radial case when Ω is close to a ball. The latter method has been taken up by Suzuki–Nagasaki ([24], [33], [34]).

They note that $\rho(F) = \dfrac{|F'|}{1+|F|^2}$ is nothing but the sperical derivative of F in the Liouville integral (3.2). Namely, regarding the analytic function F as a conformal mapping F^* from Ω into the Riemann sphere $\mathbb{C}^*$ with diameter 1, we have $\dfrac{d\sigma}{ds} = \rho(F)$ under it, where ds and $d\sigma$ denote the natural metrics on Ω and $\mathbb{C}^*$, respectively. Thus, $\ell \equiv \int_{\partial\Omega} \rho(F)\, ds = \left(\frac{\lambda}{8}\right)^{1/2} |\partial\Omega|$ and $\Sigma_1 = \int_\Omega \rho(F)^2 dx = \frac{\lambda}{8}\int_\Omega e^u dx$ denote the length and area of $F^*(\partial\Omega)$ and $F^*(\Omega)$, respectively, as immersions.

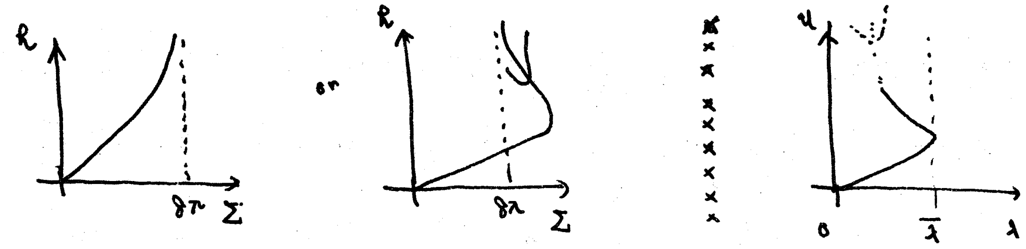
In the case that Ω is the ball $B = \{|x|<1\} \subset \mathbb{R}^2$, we can take $F(z) = cz$. Then, Σ_1 grows from 0 to π along the solution branch starting from $(\lambda, u) = (0, 0)$:





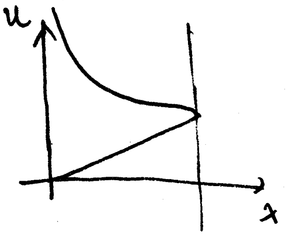
Then they have taken the idea to parametrize the solutions $\{h \equiv \binom{u}{\lambda}\}$ of (0.1) in terms of $\Sigma' \equiv \lambda \int_\Omega e^u dx$. The following is regarded as the final result by this approach.

<u>Fact 9</u>. In the Σ-h plane with $0 < \Sigma < 8\pi$, there exists only one branch $\mathcal{C}$ starting from $(\Sigma, h) = (0,0)$ without bending which satisfies (0.1) with $\Sigma = \lambda \int_\Omega e^u dx$. The branch $\mathcal{S}$ in the λ-u plane corresponding to $\mathcal{C}$ starts from $(\lambda, u) = (0,0)$ and bends at most once.



We recall that when Σ is bounded, which is the case for star-shaped domains Ω, then it must converge to 8π or 0 as $\lambda \downarrow 0$ as far as Ω is simply-connected. About Weston's solutions, we can determine when Σ approaches to 8π from below. Then, the following is verified from Bieberbach's area theorem:

Corollary 1. When $K|g_\nu'|<2$ holds everywhere on $\partial\Omega$, then Weston's branch exists uniquely and connects to the branch of minimal solutions. The branch $\mathcal{S}$ generated in this way bends just once in λ-u plane. Here, K denotes the curvature of $\partial\Omega$.



Note that $K|g_\nu'|\equiv 1$ if Ω is a ball.

In view of the Rellich-Pohozaev identity

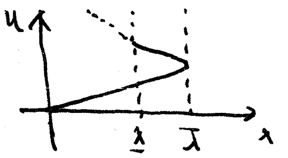
$$2\lambda\int_\Omega e^u\,dx = \int_{\partial\Omega}\left\{\frac{1}{2}(x\cdot n)\left(\frac{\partial u}{\partial n}\right)^2 + \lambda(x\cdot n)\right\}ds$$

and the Schwarz inequality

$$\int_{\partial\Omega}(x\cdot n)\left(\frac{\partial u}{\partial n}\right)^2 ds \geq \left\{\int_{\partial\Omega}\left(\frac{\partial u}{\partial n}\right)ds\right\}^2 \Big/ \left\{\int_{\partial\Omega}\frac{ds}{|x\cdot n|}\right\} \equiv \Sigma^2/B,$$

we have that $\Sigma \geq 8\pi$ implies $\lambda \leq \underline{\lambda} \equiv 8\pi(B-2\pi)/|\Omega|\cdot B$. Hence

Corollary 2. If Ω is star-shaped with respect to the origin and $B\equiv\int_{\partial\Omega}\frac{ds}{n\cdot x} \leq 4\pi$, then for $\lambda\in(\underline{\lambda},\overline{\lambda})$ the solutions u of (0.1) are just two.



Concluding this article, we describe

<u>Outline of proof for Fact 9</u>: The first part about the bifurcation diagram in Σ-h plane follows from a priori estimates and the invertibility of the linearized operator $d_h\Phi$ about a zero (h, Σ) with $0<\Sigma<8\pi$ of the mapping $\Phi : \hat{X}\times\mathbb{R}\to\hat{Y}$ with

$$\Phi(h,\Sigma)=\begin{pmatrix}\Delta u+\lambda e^{u}\\ \int_\Omega e^{u}dx-\frac{\Sigma}{\lambda}\end{pmatrix} \quad \text{for} \quad h=\begin{pmatrix}u\\ \lambda\end{pmatrix},$$

where $\hat{X}=X\times\mathbb{R}$ and $\hat{Y}=Y\times\mathbb{R}$ with $X=\{v\in C^{2+\alpha}(\bar{\Omega}) \mid v=0 \text{ on } \partial\Omega\}$ and $Y=C^{\alpha}(\bar{\Omega})$ $(0<\alpha<1)$. Since $\lambda\in(0,\bar{\lambda}]$ is bounded, we have to give an estimate for $\|u\|_{L^\infty}$. We can prove

$$\|u\|_{L^\infty}\leq -2\log\left(1-\frac{\Sigma}{8\pi}\right)$$

by the mean value theorem for $\Delta\log p+p\geq 0$ ([3]), the isoperimetric inequality on the plane ([33]), or Bol's inequality ([24])

$$\ell(\omega)^2\geq\frac{1}{2}(8\pi-m(\omega))m(\omega)$$

where $\ell(\omega)=\int_{\partial\omega}p^{1/2}ds$ and $m(\omega)=\int_\omega p\,dx$ for $p=\lambda e^{u}$.

On the other hand the invertibility of the linearized operator $d_h\Phi$ can eventually be reduced to showing the second eigenvalue $\hat{\nu}_2(p)$ is larger than 1 in

$$\varphi \in \hat{V}, \quad \int_\Omega \nabla\varphi \cdot \nabla\chi \, dx = \nu \int_\Omega \varphi \chi p \, dx \quad (\chi \in \hat{V}),$$

provided that $\Sigma = \int_\Omega p\,dx < 8\pi$, where $p = \lambda e^u$ and $\hat{V} = \{ v \in H^1(\Omega) \mid v =$ constant on $\partial\Omega \}$. This fact is proven by modifying Bandle's generalized Schwarz symmetrization ([3]).

Finally, $0 < \Sigma < 8\pi$ implies $\mu_2(p) > 0$ because $0 < \Sigma < 4\pi$ implies $\mu_1(p) > 0$ by Bandle's isoperimetric inequality and the assertion for the branch $\mathcal{S}$ in λ-u plane follows. //

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