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Perturbed solutions of semilinear equations in the singularly perturbed domain

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We deal with some special type of singular deformation of a bounded domain in $\mathbb{R}^n$ and the asymptotic behaviors of the solutions of a semilinear elliptic equation on it with the Neumann boundary condition and we also consider the characterization of the structure of the solutions. There are extremely various singular deformations of the domains and it is very difficult, for the technical reason, to deal with all of them at the same time and then we deal with a very special case of a partial degeneration of the domain, where some moving part of the domain degenerates into a one dimensional set, i.e. it is expressed as follows,

$$\Omega(\zeta) = D_1 \cup D_2 \cup Q(\zeta) \quad (\text{cf. Fig. } 1).$$

Here $Q(\zeta)$ is almost cylindrical and shrinks to a line segment as $\zeta \to 0$. $\Omega(\zeta)$ $(\zeta > 0)$ will be established in §1.

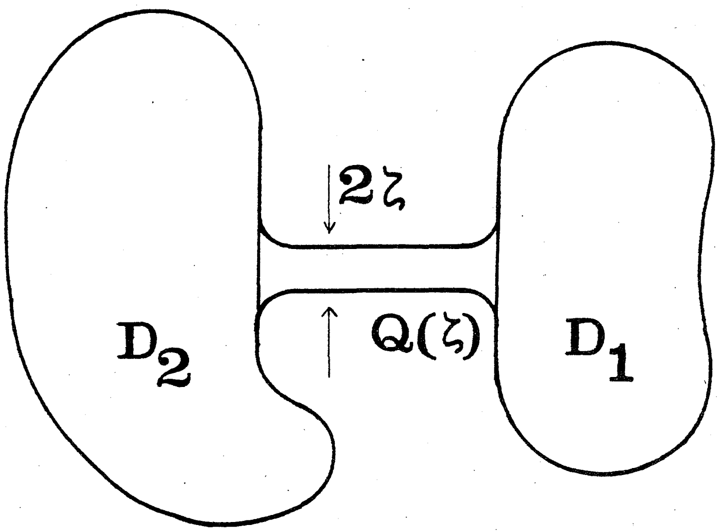

FIGURE 1

We consider the following semilinear elliptic equation for $\Omega = \Omega(\zeta)$ which is a stationary problem of a single reaction diffusion equation and it often appears as a very simple mathematical model in physics, Biology and etc,

$$(1)\begin{cases} \Delta v + f(v) = 0 & \text{in } \Omega, \\ \partial v/\partial \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary and ν is the unit outward normal vector on $\partial\Omega$ and f is a real valued smooth function on $\mathbb{R}$. Such types of domains as $\Omega(\zeta)$ with (1) have been dealt with by Matano [8], Matano and Mimura [9], Hale and Vegas [2], Vegas [10,11]. [8] has first constructed a non-constant stable solutios of (1) on a non-convex domain in contrast with the case of the convex domain. [9] has dealt with a competition system on $\Omega(\zeta)$ and constructed a non-constant stable stationary solution. [10] has obtained a transition of the structure of the solutions of (1) on $\Omega(\zeta)$ as $\zeta \to 0$ for $f(u) = \lambda u - u^p$ when $\lambda > 0$ is sufficiently small, in the framework of the bifurcation theory. [2] has considered (1) on $\Omega(\zeta)$ under some restricted condition on the bound of $|\partial f/\partial u|$ and showed that for a solution w of (1) for $\Omega = D_1 \cup D_2$ such that w is a constant function in each D_i $(i = 1, 2)$, there exists a solution w_ζ of (1) for $\Omega = \Omega(\zeta)$ such that w_ζ approaches w in $D_1 \cup D_2$ as $\zeta \to 0$ and [11] has obtained the similar result to that in [2] without an assumption on the bound of $|\partial f/\partial u|$, by the aid of a topological method. We remark that in the situations of the above papers [2] and [10], $|\partial f/\partial u|$ is imposed to be adequately small around the solutions and we can have an insight that if $|\partial f/\partial u|$ is bounded by a small constant, the structure of the solutions of (1) for $\Omega = \Omega(\zeta)$ ($\zeta > 0$: very small) is almost equivalent to that of (1) for $\Omega = D_1 \cup D_2$. But if f is general in the sense that $|\partial f/\partial u|$ is not imposed to be small, how will the structure of the solutions of (1), be characterized for small $\zeta > 0$? This is our problem. For this purpose, we consider the behaviors of the solutions, in the collapsing part $Q(\zeta)$ as well as in the fixed region $D_1 \cup D_2$ and we present a system of equations on the singular set $D_1 \cup D_2 \cup L$. By this result, we assert that the effect of the infinitesimal part $Q(\zeta)$ on the structure of the solutions does not disappear even if ζ tends to 0 and something described by a certain ordinary differential equation on L, is left behind.

§1 Characterization of the solutions

We set the domain $\Omega(\zeta)$ in the following form :

(I) $$\Omega(\zeta) = D_1 \cup D_2 \cup Q(\zeta)$$

where D_i ($i = 1, 2$) and $Q(\zeta)$ are defined in the following conditions and $x' = (x_2, x_3, \cdots, x_n) \in \mathbb{R}^{n-1}$.

(I.1) D_1 and D_2 are bounded domains in $\mathbb{R}^n$ where $\bar{D}_1 \cap \bar{D}_2 = \emptyset$ and each D_i has a smooth boundary ∂D_i and the following conditions hold for some positive constant $\zeta_* > 0$.

$$\bar{D}_1 \cap \{(x_1, x') \in \mathbb{R}^n \mid x_1 \leqq 1 \ , \ |x'| < 3\zeta_*\}$$
$$= \{ (1, x') \in \mathbb{R}^n \mid |x'| < 3\zeta_*\}$$
$$\bar{D}_2 \cap \{(x_1, x') \in \mathbb{R}^n \mid x_1 \geqq -1, \ |x'| < 3\zeta_*\}$$
$$= \{ (-1, x') \in \mathbb{R}^n \mid |x'| < 3\zeta_*\}$$

(I.2) $$Q(\zeta) = R_1(\zeta) \cup R_2(\zeta) \cup \Gamma(\zeta)$$
$$R_1(\zeta) = \{(x_1, x') \in \mathbb{R}^n \mid 1 - 2\zeta < x_1 \leqq 1, \ |x'| < \zeta\rho((x_1 - 1)/\zeta)\}$$
$$R_2(\zeta) = \{(x_1, x') \in \mathbb{R}^n \mid -1 \leqq x_1 < -1 + 2\zeta, \ |x'| < \zeta\rho((-1 - x_1)/\zeta)\}$$
$$\Gamma(\zeta) = \{(x_1, x') \in \mathbb{R}^n \mid -1 + 2\zeta \leqq x_1 \leqq 1 - 2\zeta, \ |x'| < \zeta\}$$

where $\rho \in C^0((-2,0]) \cap C^\infty((-2,0))$ is a positive function such that $\rho(0) = 2$, $\rho(s) = 1$ for $s \in (-2,-1]$, $d\rho/ds > 0$ for $s \in (-1,0)$ and the inverse function $\rho^{-1}: (1,2) \longrightarrow (-1,0)$ satisfies $\lim_{\xi \uparrow 2-0} \frac{d^k \rho^{-1}}{d\xi^k}(\xi) = 0$ holds for any positive integer $k \geqq 1$.

(II) $f \in C^\infty(\mathbb{R})$, $\limsup_{\xi\to+\infty} f(\xi) < 0$, $\liminf_{\xi\to-\infty} f(\xi) > 0$.

Hereafter we put two points $p_1 = (1,0,\cdots,0)$, $p_2 = (-1,0,\cdots,0)$, and the set $L = \{(z,0,\cdots,0) \in \mathbb{R}^n \mid -1 \leqq z \leqq 1\}$. We remark by the conditions in (I) that $\Omega(\zeta)$ is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega(\zeta)$ for $\zeta \in (0,\zeta_*)$ and $\bigcap_{0<\zeta<\zeta_*} \overline{Q(\zeta)} = L$.

Under the above situation, we characterize the behaviors of the solutions of (1) for $\Omega = \Omega(\zeta)$ for small $\zeta > 0$. To obtain a detailed structure of the solutions in $\Omega(\zeta)$, we need to consider the behavior of the solution on $Q(\zeta)$ as well as on $D_1 \cup D_2$. As $\mathrm{Vol}(Q(\zeta))$ tends to 0 as $\zeta \to 0$, we must deal with a solution in the framework of the $L^\infty(\Omega(\zeta))$ - norm or in other words, uniform convergence such as (4) and (5). Now we have the following result.

Theorem 1 ([4]). Assume $n \geqq 3$. For each $\zeta \in (0,\zeta_*)$, let v_ζ be any solution of (1) for $\Omega = \Omega(\zeta)$. Then for any sequence of positive values such that $\lim_{m\to\infty} \zeta_m = 0$, there exist a subsequence $\{\sigma_m\}_{m=1}^\infty \subset \{\zeta_m\}_{m=1}^\infty$ and functions $w_i \in C^\infty(\overline{D_i})$ $(i = 1, 2)$ and $V \in C^\infty([-1,1])$ such that the following conditions are satisfied

$$(2)\begin{cases} \Delta w_i + f(w_i) = 0 & \text{in } D_i, \\ & \qquad (i = 1,\ 2) \\ \partial w_i/\partial\nu = 0 & \text{on } \partial D_i \end{cases}$$

$$(3)\begin{cases} d^2V/dz^2 + f(V) = 0 \quad \text{for } z \in (-1,1), \\ V(1) = w_1(p_1), \quad V(-1) = w_2(p_2), \end{cases}$$

(4) $\lim_{m\to\infty} \sup_{x\in D_1\cup D_2} |v_{\sigma_m}(x) - w(x)| = 0$,

(5) $\lim_{m\to\infty} \sup_{x\in Q(\sigma_m)} |v_{\sigma_m}(x_1,x') - V(x_1)| = 0$.

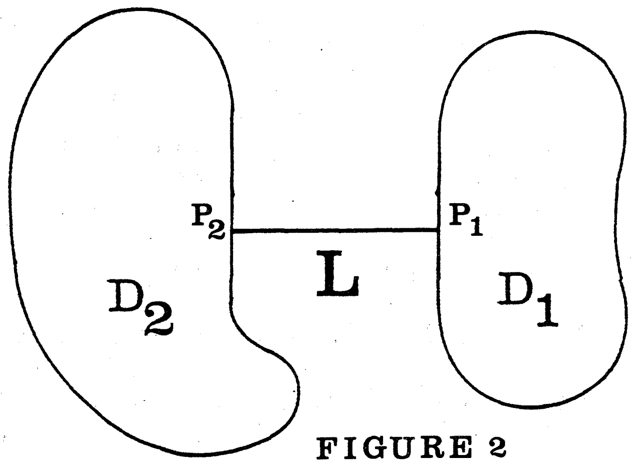

FIGURE 2

Remark. It is easy to show that if $|f'(\xi)| \leqq (\pi/2)^2$ for any $\xi \in (\xi_1,\xi_2)$ where ξ_1 (resp. ξ_2) is the smallest (resp. largest) zero point of the function f, the solution V of (3) is uniquely determined by the values $w(p_1)$ and $w(p_2)$ and then the behavior of v_ζ on $Q(\zeta)$ does not have freedom by itself and it depends on the behavior $v_\zeta|_{D_1\cup D_2}$. Therefore any solution of (1) on $\Omega(\zeta)$ for small $\zeta > 0$ is approximated by some triplet (w_1,w_2,V) in (2) and (3). From this interpretation, we see that our result is not contrary to the insight obtained by [2] and [10]. But, for general f, (3) can have many solutions and then the structure of the solutions of (1) on $\Omega(\zeta)$ ($\zeta > 0$ is small) is much larger than that of (1) on $D_1\cup D_2$.

Now there arises a natural problem which is the inverse problem of the above characterization. That is to say,

(Question) For any given triplet (w_1,w_2,V) in (2) and (3), is there a solution v_ζ of (1) in $\Omega(\zeta)$ such that v_ζ approaches w_i in D_i (i = 1, 2) and V in $Q(\zeta)$ as $\zeta \to 0$.

§2 Inverse problem

In this section, we present our main result concerning the question we stated in §2. To state the theorem we prepare some notations. We assume,

(III) There exists a triplet $(w_1, w_2, V) \in C^\infty(\bar{D}_1)\times C^\infty(\bar{D}_2)\times C^\infty([-1,1])$ which satisfies (2) and (3).

Definition. For the above solutions (w_1, w_2, V) in (III), we denote by $\{\omega_k\}_{k=1}^\infty$ and $\{\lambda_k\}_{k=1}^\infty$, respectively, the system of the eigenvalues arranged in increasing order (counting multiplicity) of the following eigenvalue problems (6) and (7),

$$(6)\begin{cases} \Delta\phi + f'(w)\phi + \omega\phi = 0 & \text{in } D_1 \cup D_2, \\ \partial\phi/\partial\nu = 0 & \text{on } \partial D_1 \cup \partial D_2, \end{cases}$$

where $w(x) = \begin{cases} w_1(x) & \text{for } x \in D_1 \\ w_2(x) & \text{for } x \in D_2 \end{cases}$,

$$(7)\begin{cases} \dfrac{d^2S}{dz^2} + f'(V)S + \lambda S = 0 \quad -1 < z < 1 , \\ S(1) = S(-1) = 0 . \end{cases}$$

Now we present the result.

Theorem 2. Assume $n \geqq 3$ and the following non-degeneracy conditions,

(IV) $\{\omega_k\}_{k=1}^\infty \cup \{\lambda_k\}_{k=1}^\infty \not\ni 0$, $\{\omega_k\}_{k=1}^\infty \cap \{\lambda_k\}_{k=1}^\infty = \emptyset$.

Then, for any $\zeta \in (0,\zeta_*)$, there exists a solution $v_\zeta \in C^\infty(\overline{\Omega(\zeta)})$ of (1) for $\Omega = \Omega(\zeta)$ such that

(8) $\lim_{\zeta\to 0} \sup_{x\in D_1\cup D_2} | v_\zeta(x) - w(x) | = 0$,

(9) $\lim_{\zeta\to 0} \sup_{x\in Q(\zeta)} |v_\zeta(x_1,x') - V(x_1)| = 0$.

§3 Method of the proof of Theorem 2

The proof consists of two parts. By (II), we assume, without loss of generality, that $\partial f/\partial\xi$ has a compact support in $\mathbb{R}$.

(Part I) Construction of the approximate solution

Lemma 1. There exists a function $A_\zeta \in C^\infty(\overline{\Omega(\zeta)})$ such that

(10) $\lim_{\zeta\to 0} \sup_{x\in D_i} |A_\zeta(x) - w_i(x)| = 0 \quad (i = 1, 2),$

(11) $\lim_{\zeta\to 0} \sup_{x\in Q(\zeta)} |A_\zeta(x_1,x') - V(x_1)| = 0$

(12) $\lim_{\zeta\to 0} \sup_{x\in\Omega(\zeta)} |\Delta A_\zeta(x) + f(A_\zeta(x))| = 0 .$

(Sketch of the proof) It is easy to construct a function $\tilde{A}_\zeta \in C^\infty(\overline{\Omega(\zeta)})$ which satisfies the properties (10) and (11) where A_ζ is replaced by $\tilde{A}_\zeta$. But it is difficult to construct such function so that it also satisfies (12) besides (10) and (11). Our device is to define A_ζ by $\tilde{A}_\zeta$ through the following equation,

$$(13)\begin{cases} \Delta A_\zeta - A_\zeta + \tilde{A}_\zeta + f(\tilde{A}_\zeta) = 0 & \text{in } \Omega(\zeta), \\ \partial A_\zeta/\partial\nu = 0 & \text{on } \partial\Omega(\zeta). \end{cases}$$

We must check the properties (10), (11) and (12). Now we can apply the similar method to prove Theorem 1 (cf. [3], [4]) and we have, for any sequence of positive values $\{\zeta_m\}_{m=1}^\infty$ such that $\lim_{m\to\infty} \zeta_m = 0$, there exist a subsequence $\{\sigma_m\}_{m=1}^\infty$ and functions (b_1, b_2, B) in $C^\infty(\bar{D}_1)\times C^\infty(\bar{D}_2)\times C^\infty([-1,1])$ such that

$$(14)\begin{cases} \Delta b_i - b_i + w_i + f(w_i) = 0 & \text{in } D_i, \\ \partial b_i/\partial\nu = 0 & \text{on } \partial D_i, \end{cases} \qquad (i = 1, 2)$$

$$(15)\begin{cases} d^2B/dz^2 - B + V + f(V) = 0 \quad \text{in } -1 < z < 1, \\ B(1) - b_1(p_1) = B(-1) - b_2(p_2) = 0 . \end{cases}$$

(16) $\lim_{m\to\infty} \sup_{x\in D_i} |A_{\sigma_m}(x) - b_i(x)| = 0 \quad (i = 1,2),$

(17) $\lim_{m\to\infty} \sup_{x\in Q(\sigma_m)} |A_{\sigma_m}(x_1,x') - B(x_1)| = 0 .$

From (2) and (3), we have

$$\Delta(b_i-w_i)-(b_i-w_i) = 0 \text{ in } D_i, \quad \partial(b_i-w_i)/\partial\nu = 0 \text{ on } \partial D_i \ (i = 1,2).$$

By the maximum principle and the Hopf Lemma, we have $b_i \equiv w_i$.

Next, from (3) and (14), we have

$$\frac{d^2}{dz^2}(B-V) - (B-V) = 0 \text{ in } -1<z<1 , \ B(1)-V(1) = B(-1)-V(-1) = 0$$

and we conclude $B \equiv V$ in $[-1,1]$. By the arbitrariness of the choice of the subsequence $\{\zeta_m\}_{m=1}^{\infty}$ and (16), (17), we have (10) and (11). From (13), we have,

$$\Delta A_{\sigma_m} + f(A_{\sigma_m}) = A_{\sigma_m} - \tilde{A}_{\sigma_m} + f(A_{\sigma_m}) - f(\tilde{A}_{\sigma_m}) \quad \text{in} \quad \Omega(\sigma_m)$$

and we obtain, by (10) and (11),

$$\lim_{m\to\infty} \sup_{x\in\Omega(\sigma_m)} |\Delta A_{\sigma_m} + f(A_{\sigma_m})| = 0 .$$

Again, by the arbitrariness of the choice of the sequence $\{\zeta_m\}_{m=1}^{\infty}$, we obtain (12).

(Part II) Reduction to the finite dimensional problem

To seek for the exact solution v_ζ around A_ζ, we need to consider the linearized eigenvalue problem,

$$(18)\begin{cases} \Delta\Phi + f'(A_\zeta)\Phi + \mu\Phi = 0 & \text{in } \Omega(\zeta), \\ \partial\Phi/\partial\nu = 0 & \text{on } \partial\Omega(\zeta). \end{cases}$$

Denote the eigenvalues (counting multiplicity) and the orthonormal system of the eigenfunctions of the above problem (18) by $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ and $\{\Phi_{k,\zeta}\}_{k=1}^{\infty}$ where $(\Phi_{k,\zeta}\cdot\Phi_{m,\zeta})_{L^2(\Omega(\zeta))} = \delta_{k,m}$.

Then we have the following results of this eigenvalue problem under the assumptions of Theorem 2.

Proposition 1 ([5]). $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ can be decomposed as follows,

$$\{\mu_k(\zeta)\}_{k=1}^{\infty} = \{\omega_k(\zeta)\}_{k=1}^{\infty} \cup \{\lambda_k(\zeta)\}_{k=1}^{\infty}$$

where $\lim_{\zeta\to 0} \omega_k(\zeta) = \omega_k$, $\lim_{\zeta\to 0} \lambda_k(\zeta) = \lambda_k$ ($k \geqq 1$).

ω_k and λ_k were defined in §2. Decompose $\{\Phi_{k,\zeta}\}_{k=1}^{\infty}$ according to the decomposition in Proposition 2, as follows,

$$\{\Phi_{k,\zeta}\}_{k=1}^{\infty} = \{\phi_{k,\zeta}\}_{k=1}^{\infty} \cup \{\psi_{k,\zeta}\}_{k=1}^{\infty}$$

and we have,

Proposition 2 ([5]).

$$\limsup_{\zeta\to 0} \|\phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))} < \infty ,$$

$$\lim_{\zeta\to 0} d_{n-1}^{1/2}\zeta^{(n-1)/2}\|\psi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))} = 1 \quad \left(d_{n-1} = \frac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)} \right),$$

$$\|\psi_{k,\zeta}\|_{L^\infty(D_1\cup D_2)} \sim O(\zeta^{-(n-3)/2}) , \quad \|\psi_{k,\zeta}\|_{L^1(\Omega(\zeta))} \sim O(\zeta^{(n-1)/2}),$$

$$\|\psi_{k,\zeta}\|_{L^\infty(D_i\setminus\Sigma_i(\eta))} \sim O(\zeta^{(n-1)/2}) \quad \text{for any fixed } \eta > 0.$$

where $\Sigma_i(\eta) = \{x \in D_i \mid |x-p_i|<\eta \}$ (i = 1, 2).

By using the eigenfunctions, we define some spaces,

$$X(\zeta) = H^1(\Omega(\zeta)), \quad X_1(\zeta) = \text{L.h.}[\{\phi_{k,\zeta}\}_{k=1}^{q}\cup\{\psi_{k,\zeta}\}_{k=1}^{q}]$$

$$X_2(\zeta) = \text{L.h.}[\{\phi_{k,\zeta}\}_{k=q+1}^{\infty} \cup \{\psi_{k,\zeta}\}_{k=q+1}^{\infty}].$$

and the projection operator P_ζ on $L^2(\Omega(\zeta))$ by

$$P_\zeta\Phi = \sum_{k=1}^{q}\left\{(\Phi\cdot\phi_{k,\zeta})_{L^2(\Omega(\zeta))}\phi_{k,\zeta} + (\Phi\cdot\psi_{k,\zeta})_{L^2(\Omega(\zeta))}\psi_{k,\zeta}\right\} .$$

By the aid of Proposition 2, we can easily prove,

Lemma 2. For any $q \geqq 1$, there exists $c(q) > 0$ such that

$$\|P_\zeta\Phi\|_{L^\infty(\Omega(\zeta))} \leqq c(q)\|\Phi\|_{L^\infty(\Omega(\zeta))} \quad \text{for any } \Phi \in L^\infty(\Omega(\zeta)).$$

We seek for the solution v_ζ in the following form $v_\zeta = A_\zeta + \Phi_\zeta^{(1)} + \Phi_\zeta^{(2)}$ where $\Phi_\zeta^{(i)} \in X_i(\zeta)$ (i = 1, 2) and $\|\Phi_\zeta^{(i)}\|_{L^\infty(\Omega(\zeta))}$ is small for small $\zeta > 0$. We parametrize $\Phi_\zeta^{(1)}$

by $T = (\tau_1, \tau_2, \cdots, \tau_{2q}) \in \mathbb{R}^{2q}$ as follows.

$$\Phi^{(1)}_{T,\zeta} = \sum_{k=1}^{q} \left(\tau_k \phi_{k,\zeta} + \tau_{q+k} \bar{\psi}_k\right)$$

where $\bar{\psi}_{k,\zeta}(x) = \psi_{k,\zeta}(x)/\|\psi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}$ and q will be determined later. By using P_ζ, we decompose the original equation (1) for $\Omega = \Omega(\zeta)$ and obtain the equations with variables $T = (\tau_1, \cdots, \tau_{2q})$ and $\Phi^{(2)}_\zeta$ respectively (the Lyapunov-Schmidt method).

$$(19) \quad \Delta\Phi^{(1)}_\zeta + f'(A_\zeta)\Phi^{(1)}_\zeta + P_\zeta\left(f(A_\zeta + \Phi^{(1)}_\zeta + \Phi^{(2)}_\zeta) - f(A_\zeta) - f'(A_\zeta)(\Phi^{(1)}_\zeta + \Phi^{(2)}_\zeta) + g_\zeta\right) = 0 \quad \text{in} \quad \Omega(\zeta)$$

$$(20) \begin{cases} \Delta\Phi^{(2)}_\zeta + f'(A_\zeta)\Phi^{(2)}_\zeta + \\ (I - P_\zeta)\left(f(A_\zeta + \Phi^{(1)}_\zeta + \Phi^{(2)}_\zeta) - f(A_\zeta) - f'(A_\zeta)(\Phi^{(1)}_\zeta + \Phi^{(2)}_\zeta) + g_\zeta\right) = 0 \text{ in } \Omega(\zeta) \\ \partial\Phi^{(2)}_\zeta / \partial\nu = 0 \quad \text{on} \quad \partial\Omega(\zeta). \end{cases}$$

where $g_\zeta = \Delta A_\zeta + f(A_\zeta)$ and $\lim_{\zeta\to 0} \sup_{x\in\Omega(\zeta)} |g_\zeta(x)| = 0$ (cf. Lemma 1). We can find a small solution $\Phi^{(2)}_\zeta$ in (20) for small $\Phi^{(1)}_\zeta$ (or small T).

Lemma 3. Let q be a natural number such that

$$(21) \quad \min(\omega_{q+1}, \lambda_{q+1}) > 4 \sup_{\xi\in\mathbb{R}} |f'(\xi)| + 4 \quad \text{and} \quad \omega_{q+1} > \omega_q .$$

Then there exists a constant $\delta_0 > 0$ such that for any $\Phi^{(1)}_{T,\zeta} \in X_1(\zeta)$ ($|T| \leqq \delta_0$), there exists a unique solution $\Phi^{(2)}_\zeta = \Phi^{(2)}_{T,\zeta} \in X_2(\zeta) \cap C^\infty(\overline{\Omega(\zeta)})$ of (20) with the following property,

$$(22) \quad \lim_{\delta\to 0} \sup_{0<\zeta<\delta, |T|\leqq\delta} \|\Phi^{(2)}_{T,\zeta}\|_{L^\infty(\Omega(\zeta))} = 0 .$$

(Sketch of the proof) We seek for $\Phi^{(2)}_{T,\zeta}$ by the variational method. Define a functional on $X(\zeta)$ by

$$J_\zeta(u) = \int_{\Omega(\zeta)} \left(\frac{1}{2}|\nabla u|^2 - \int_{A_\zeta(x)}^{u(x)} f(\xi)d\xi\right)dx \quad (u \in X(\zeta)).$$

For fixed $\Phi^{(1)}_{T,\zeta} \in X_1(\zeta)$, we minimize $J_\zeta(A_\zeta + \Phi^{(1)}_{T,\zeta} + \Phi^{(2)}_\zeta)$ in $\Phi^{(2)}_\zeta \in$

$X_2(\zeta)$. By a simple calculation, we have, under the condition (21),

$$J_\zeta(A_\zeta+\Phi^{(1)}_{T,\zeta}+\Phi^{(2)}_\zeta) - J_\zeta(A_\zeta+\Phi^{(1)}_{T,\zeta})$$
$$\geqq \|\Phi^{(2)}_\zeta\|^2_{L^2(\Omega(\zeta))} - \|\Delta\Phi^{(1)}_{T,\zeta}+f(A_\zeta+\Phi^{(1)}_{T,\zeta})-f(A_\zeta)+g_\zeta\|^2_{L^2(\Omega(\zeta))} .$$

By the relation between T and $\Phi^{(1)}_{T,\zeta}$ and $\lim_{\zeta\to 0} \sup_{x\in\Omega(\zeta)} |g_\zeta(x)| = 0$, we can obtain $\Phi^{(2)}_{T,\zeta} \in X_2(\zeta)$ for small $T \in \mathbb{R}^{2q}$ with

$$\lim_{\delta\to 0} \sup_{0<\zeta<\delta, |T|\leqq\delta} \|\Phi^{(2)}_{T,\zeta}\|_{L^2(\Omega(\zeta))} = 0 .$$

By the regularity theory of the elliptic equation, $\Phi^{(2)}_{T,\zeta} \in C^\infty(\overline{\Omega(\zeta)})$. To prove $L^\infty(\Omega(\zeta))$-convergence, assume that there exist sequences $\{\zeta_m\}^\infty_{m=1} \subset \mathbb{R}^+$, $\{T_m\}^\infty_{m=1} \subset \mathbb{R}^{2q}$ such that $\lim_{m\to\infty} \zeta_m = 0$, $\lim_{m\to\infty} |T_m| = 0$ and $\|\Phi^{(2)}_{T_m,\zeta_m}\|_{L^\infty(\Omega(\zeta_m))} \geqq c > 0$ ($m \geqq 1$).

Putting $\tilde{\Phi}^{(2)}_{T,\zeta} = \Phi^{(2)}_{T,\zeta}/\|\Phi^{(2)}_{T,\zeta}\|_{L^\infty(\Omega(\zeta))}$, we investigate $\tilde{\Phi}^{(2)}_{T_m,\zeta_m}$ which satisfies the following equation,

$$(23)\begin{cases} \Delta\tilde{\Phi}^{(2)}_\zeta + f'(A_\zeta)\tilde{\Phi}^{(2)}_\zeta + \\ (I-P_\zeta)\Big((f(A_\zeta+\Phi^{(1)}_\zeta+\Phi^{(2)}_\zeta)-f(A_\zeta))/a_{T,\zeta}-f'(A_\zeta)(\frac{\Phi^{(1)}_{T,\zeta}}{a_{T,\zeta}}+\tilde{\Phi}^{(2)}_\zeta)+\frac{g_\zeta}{a_{T,\zeta}}\Big) = 0 \\ \qquad\qquad \text{in } \Omega(\zeta) \\ \partial\Phi^{(2)}/\partial\nu = 0 \quad \text{on } \partial\Omega(\zeta). \end{cases}$$

where $a_{T,\zeta} = \|\Phi^{(2)}_{T,\zeta}\|_{L^\infty(\Omega(\zeta))}$.

By applying the similar method to prove Theorem 1 ([3], [4]), we can prove that there exist a sequence $\{m(j)\}^\infty_{j=1}$ and $\Phi \in C^\infty(\overline{D_1 \cup D_2})$, $\bar{\Phi} \in C^\infty([-1,1])$ such that

$$(24) \quad \lim_{j\to\infty} \sup_{x\in D_1\cup D_2} |\tilde{\Phi}^{(2)}_{T_{m(j)},\zeta_{m(j)}}(x) - \Phi(x)| = 0 ,$$

$$(25) \quad \lim_{j\to\infty} \sup_{x\in Q(\zeta_{m(j)})} |\tilde{\Phi}^{(2)}_{T_{m(j)},\zeta_{m(j)}}(x_1,x') - \bar{\Phi}(x_1)| = 0 .$$

From $\lim_{m\to\infty} \|\Phi^{(2)}_{T_m,\zeta_m}\|_{L^2(\Omega(\zeta_m))} = 0$, we have $\Phi \equiv 0$ in $D_1 \cup D_2$ and then $\sup_{z\in[-1,1]} |\bar{\Phi}(z)| = 1$ follows from $\|\tilde{\Phi}^{(2)}_{T,\zeta}\|_{L^\infty(\Omega(\zeta))} = 1$ and (25).

On the other hand, by the investigation of (23) by (24) and (25) by the aid of the detailed property of P_ζ (cf. [5]), we can prove,

$$(26)\begin{cases} d^2\bar{\Phi}/dz^2 + (f(V+\bar{\Phi})-f(V)) = 0 \quad \text{in} \quad (-1,1), \\ \bar{\Phi}(1) = \bar{\Phi}(-1) = 0 , \end{cases}$$

and $(\bar{\Phi}\cdot S_k)_{L^2((-1,1))} = 0 \quad (1 \leqq k \leqq q)$ where $S_k(z) = \sin \frac{k\pi}{2}(z+1)$.

Therefore from (21), we see that $\bar{\Phi} \equiv 0$ in $(-1,1)$ by the above properties. But it contradicts to the fact $\sup_{z\in[-1,1]} |\bar{\Phi}(z)| = 1$.

Along the same line as Lemma 3, we can prove the following.

Lemma 4.

$$\lim_{\zeta\to 0} \sup_{0<\zeta<\delta,|T|\leqq\delta} \left\| \frac{\partial \Phi^{(2)}_{T,\zeta}}{\partial \tau_k} \right\|_{L^\infty(\Omega(\zeta))} = 0 \quad (1 \leqq k \leqq 2q).$$

Now expressing (19) by the variable $T = (\tau_1,\tau_2,\cdots,\tau_{2q})$, we have

$$(27) \quad F_{k,\zeta}(T) = 0 \quad (1 \leqq k \leqq 2q),$$

where we have put, for $k = 1,2,\cdots,2q$,

$$F_{k,\zeta}(T) = -\omega_k(\zeta)\tau_k + \int_{\Omega(\zeta)} \phi_{k,\zeta}(x)G_{T,\zeta}(x)dx$$

$$F_{q+k,\zeta}(T) = -\lambda_k(\zeta)\tau_{q+k} + \int_{\Omega(\zeta)} \|\psi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}\psi_{k,\zeta}(x)G_{T,\zeta}(x)dx$$

$$G_{T,\zeta} = f(A_\zeta+\Phi^{(1)}_{T,\zeta}+\Phi^{(2)}_{T,\zeta}) - f(A_\zeta) - f'(A_\zeta)(\Phi^{(1)}_{T,\zeta}+\Phi^{(2)}_{T,\zeta}) + g_\zeta .$$

It is easy to see, from Proposition 2 and Lemma 3,

$$\lim_{\delta\to 0} \sup_{0<\zeta<\delta,|T|\leqq\delta} \sup_{x\in\Omega(\zeta)} |G_{T,\zeta}(x)| = 0 .$$

(Part III) <u>Construction of the solution</u>

We seek for the solution of the finite dimensional equation (27). First we easily see the following property.

Lemma 5.

$$\lim_{\zeta\to 0} |F_{k,\zeta}(0)| = 0 \quad (1 \leqq k \leqq 2q) .$$

Next we investigate the behavior of the Jacobian matrix of $F_{k,\zeta}$ at $T = 0$. We calculate $\partial F_{i,\zeta}/\partial\tau_j$ as follows,

$$\frac{\partial F_{i,\zeta}}{\partial\tau_j}(T) = -\delta_{i,j}\omega_i(\zeta) + \int_{\Omega(\zeta)}\phi_{k,\zeta}(x)\frac{\partial G_{T,\zeta}}{\partial\tau_j}(x)\,dx$$

$$\frac{\partial F_{q+i,\zeta}}{\partial\tau_j}(T) = -\delta_{q+i,j}\lambda_i(\zeta) + \int_{\Omega(\zeta)}\|\psi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}\psi_{k,\zeta}(x)\frac{\partial G_{T,\zeta}}{\partial\tau_j}(x)dx$$

$$\frac{\partial G_{T,\zeta}}{\partial\tau_j} = (f'(A_\zeta+\Phi^{(1)}_{T,\zeta}+\Phi^{(2)}_{T,\zeta})-f'(A_\zeta))(\phi_{j,\zeta} + \frac{\partial\Phi^{(2)}_{T,\zeta}}{\partial\tau_j}) \quad (1 \leqq j \leqq q)$$

$$\frac{\partial G_{T,\zeta}}{\partial\tau_{q+j}} = (f'(A_\zeta+\Phi^{(1)}_{T,\zeta}+\Phi^{(2)}_{T,\zeta})-f'(A_\zeta))(\tilde\psi_{j,\zeta} + \frac{\partial\Phi^{(2)}_{T,\zeta}}{\partial\tau_{q+j}}) \quad (1 \leqq j \leqq q)$$

Then we have,

$$\left\|\left(\frac{\partial F_{i,\zeta}}{\partial\tau_j}(0)\right)_{1\leqq i,j\leqq 2q} + \begin{pmatrix} \omega_1(\zeta)\,\omega_2(\zeta)\cdots\omega_q(\zeta) & & \bigcirc \\ & \lambda_1(\zeta)\,. & \\ \bigcirc & & \ddots\ \lambda_q(\zeta)\end{pmatrix}\right\|$$

$$\leqq c\sup_{x\in\Omega(\zeta)}\left|\frac{\partial G_{T,\zeta}}{\partial\tau_j}\right| \max_{1\leqq k\leqq q}\left(\|\phi_{k,\zeta}\|_{L^1(\Omega(\zeta))},\ \|\psi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}\|\psi_{k,\zeta}\|_{L^1(\Omega(\zeta))}\right)$$

Applying Proposition 1 and Proposition 2 to the above inequality, we have the following.

Lemma 6.

$$\left(\frac{\partial F_{i,\zeta}}{\partial\tau_j}(0)\right)_{i,j} \sim -\begin{pmatrix} \omega_1\ \omega_2\cdots\omega_q & & \bigcirc \\ & \lambda_1\,. & \\ \bigcirc & & \ddots\ \lambda_q\end{pmatrix} \quad \text{as} \quad \zeta \to 0 .$$

The right hand side of the above expression is non-singular from the assumption (IV) and then applying the standard implicit function theorem, we obtain the unique solution T_ζ such that $\lim_{\zeta\to 0}|T_\zeta| = 0$. We obtain the desired solution $v_\zeta = A_\zeta + \Phi^{(1)}_{T_\zeta,\zeta} + \Phi^{(2)}_{T_\zeta,\zeta}$.

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