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On Some Elliptic Equations with Nonlocal Nonlinear Terms

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§.1 Introduction and Main Results.

Consider the following problem ; Find u such that

$$(E)_\gamma \quad \begin{cases} -\Delta u = (V * |u|^2)\, u + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω ($\subset R^n$) is a bounded domain with a smooth boundary $\partial\Omega$, $\lambda \in R^1$, $V(x) = 1/|x|^\gamma$ ($0 < \gamma < n$) and

$$(V * |u|^2)(x) = \int_\Omega V(x-y)|u|^2(y)\, dy .$$

In this paper we want to show that the solvability of the problem $(E)_4$ ($n > 4$) is similar to that of the following Yamabe type equation ;

$$(Y) \quad \begin{cases} -\Delta u = u^{(n+2)/(n-2)} + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

One of our motivations for studying the problem $(E)_4$ is the following question.

<u>Question</u> : Is there nonlinear elliptic boundary value problem which has strong dependence on the geometry (or the topology) of the domain Ω ? Is there nonlinear elliptic boundary value problem which

characterizes the geometry (or the topology) of the domain ?

For example, Dirichlet boundary value problem for the minimal surface equation has strong dependence on convexty of the domain. This fact is well known (see [GT]). For the problem (Y) with $\lambda = 0$ some results are known (see [P],[BC]).

(a) If the domain Ω is star-shaped, (Y) has no solution,

(b) If Ω is non-contractible and $n = 3$, (Y) has at least one solution.

From these facts, there is a following conjecture:

"There is a solution of (Y) with $\lambda = 0$."

$\Longleftrightarrow$ " The domain Ω is non-contractible."

For the problem $(E)_4$ we can prove the next statements.

< Effect of the domain (for $(E)_4$) .>

PROPOSITION.1. - Let Ω be an annulus domain in R^n($n > 4$) (i.e. $\Omega = \{ x \in R^n ; 0 < a < |x| < b \}$, a,b > 0 are constants.) and $\lambda < \lambda_1$, where λ_1 is a first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition. Then there is a radially symmetric solution of $(E)_4$.

PROPOSITION.2. - Let Ω be a star-shaped domain in R^n($n > 5$) and $\lambda \leq 0$. Then there is no solution of $(E)_4$.

PROPOSITION.1 can be proved by direct variational method and PROPOSITION.2 is obtained by using Pohozaev's type identity.

REMARK.1. - (a) The condition $\lambda < \lambda_1$ is necessary for the solvability of $(E)_4$ (or (Y)).

(b) Under the conditions $0 < \lambda < n$ and $\lambda \leq 4$, any weak solution of $(E)_4$ is a classical solution.

We do not know whether non-contractibity of the domain implies the existence of solutions for $(E)_4$ or not. In attacking this question,we encounter the following questions.

<u>Question.A</u>. - Is the solution u of (E)' unique ?

$$(E)' \qquad -\Delta u = (1/|x|^4 * |u|^2) u \qquad \text{in } R^n,$$
$$u > 0 \quad (u \in H^1(R^n)).$$

<u>REMARK.2</u>. - Existence of solution for (E)' is proved in this paper.

<u>Question.B</u>. - Is any solution of (E)' radially symmetric ?

To the question B we give next partial result.

<u>PROPOSITION.3</u>. - Let u be a solution of (E)' and $u(x) = 0(1/|x|^{n-2})$ as $|x| \to \infty$. Then u is radially symmetric.

<u>REMARK.3</u>. - For $0 < \gamma < n$, $\gamma \le 4$ symmetry properties of the solution for problem $(E)_\gamma$ can be proved for a symmetric domain by using a slight modification of the argument in Gidas,Ni and Nirenberg [GNN 1]. In particular, we obtain the next theorem.

<u>THEOREM.1</u>. - Let u be a solution of $(E)_\gamma$ ($0 < \gamma < n$, $\gamma \le 4$) and $\Omega = B_R(o)$ ($= \{ x \in R^n ; |x| < R \}$) . Then u is radially symmetric and $u_r(r) < 0$ for all $r \in (0,R]$.

Next we consider the effect of lower order perturbations to the problem $(E)_4$. For $0 < \lambda < \lambda_1$, solvability of the problem $(E)_4$ does not depend on the shape of the domain.

< <u>Effect of lower order perturbations</u> >

<u>THEOREM.2</u>. - Let $0 < \lambda < \lambda_1$, $n > 4$ and Ω be an arbitrary bounded domain. Then there is a solution of $(E)_4$.

Next we consider the following lower order perturbation problem:

$$(P)\quad \begin{cases} -\Delta u = (1/|x|^4 * |u|^2) u + \lambda (1/|x|^\gamma * |u|^2) u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < \gamma < 4$ and $0 < \lambda$.

THEOREM.3. - Let $\lambda > 0$. Assume that $1 < \gamma < 4$ if $n = 5$ and $0 < \gamma < 4$ i $n \geq 6$. Then there is a solution of (P).

REMARK.4. - For the equation $(E)_4$ we can obtain analogous results to some interesting results for Yamabe type equation (Y) (e.g. [CFP], [CFS] etc.).

REMARK.5. - Nonlinear term ($V * |u|^2$) u appears ,for example,in Hartree type equation and Choquard type equation (see e.g. [Lio 1], [Lio 3],[GV],[HT]).

In the next section we explain the idea and some technical tools in proving THEOREM.2 and THEOREM.3.

§.2 Sketch of proof of THEOREM.2 and THEOREM.3.

Notations: For $0 < \lambda < \lambda_1$,define $\sigma(\lambda)$ as follows:

$$\sigma (\lambda) = \inf_{\substack{u \in H^1(R^n) \\ u \neq 0}} \frac{\left(\|\nabla u\|^2_{L^2(\Omega)} - \lambda \| u \|^2_{L^2(\Omega)} \right)^2}{J(u)} ,$$

where $J(u) = \iint_{\Omega\times\Omega} \frac{|u(x)|^2 \, |u(y)|^2}{| x - y |^4} dxdy$.

Define $$\Sigma = \inf_{\substack{u \in H^1(R^n) \\ u \neq 0}} \frac{\|\nabla u\|^4_{L^2(\Omega)}}{J_o(u)} ,$$

where $J_o(u) = \iint_{R^n \times R^n} \frac{|u(x)|^2 \ |u(y)|^2}{|x - y|^4} dxdy$.

Note that we can get $\sigma(0) \geq \sum$ easily,but later we prove $\sigma(0) = \sum$. One of the key steps of our argument is the following.

PROPOSITION.4. - There is a minimizer for $\sum$ and the minimizer is (in some sense)uniquely given by the following function $U(x) = \frac{1}{(1 + |x|^2)^{(n-2)/2}}$. (For any minimizer u there are constants c , d and $y \in R^n$ such that $u(x) = c\, U((x - y)/ d)$.)

This is a consequence of E.H.Lieb's theorem [L] and the well known fact for Sobolev's inequality. Note that the inequality :

$$J(u) \leq C \|\nabla u\|^4_{L^2(\Omega)} \quad \text{for any } u \in H^1(R^n),$$

where C is a constant,is obtained by using Hardy-Littlewood-Sobolev's inequality and Sobolev's inequality. The following Brezis and Lieb's type lemma plays an important role throughout our arguments.

LEMMA.1. - Let $u \in H^1_o(\Omega)$ and $v_n \longrightarrow 0$ weakly in $H^1_o(\Omega)$ $(n \longrightarrow \infty)$. Then there is a limit of $[J(u + v_n) - J(v_n)]$ and $\lim_{n \to \infty} [J(u + v_n) - J(v_n)] = J(u)$ holds.

< Sketch of proof of THEOREM.2 >

Our Strategy is on the same line to that of Brezis and Nirenberg [BN]. Using LEMMA.1,we can prove the next proposition.

PROPOSITION.5. - If $\sigma(\lambda) < \sigma(0)$,then $\sigma(\lambda)$ is attained.

Next we shall prove ,

PROPOSITION.6. - Under the assumption of THEOREM.2,the inequality

$\sigma(\lambda) < \sigma(0)$ holds.

We need some calculations for the proof of PROPOSITION.6. Let

$$Q_\lambda(v) = \frac{\left(\|\nabla v\|^2_{L^2(\Omega)} - \lambda \|v\|^2_{L^2(\Omega)}\right)^2}{J(u)}.$$

Note that there is a positive constant c_o such that $U_o(x) = c_o U(x)$ is a solution of

$$-\Delta U_o = \left(\frac{1}{|x|^4} * |U_o|^2\right) U_o \quad \text{in } R^n.$$

We may assume that $o \in \Omega$. Let $\phi_\varepsilon(x) = \xi(x)\, U_{o,\varepsilon}(x)$, where $\varepsilon > 0$,

$$U_{o,\varepsilon}(x) = \frac{1}{\varepsilon^{(n-2)/2}} U_o\left(\frac{x}{\varepsilon}\right)$$

and $\xi \in C_o^\infty(\Omega)$ is a function such that $0 \le \xi \le 1$, $\xi = 1$ on $B_\rho(o)$ and $\xi = 0$ on ${}^cB_{2\rho}(o)$. The constant $\rho > 0$ shoud be taken sufficiently small in order to $\phi_\varepsilon \in H^1_o(\Omega)$.

<u>LEMMA.2.</u> - (1) $Q_\lambda(\phi_\varepsilon) \longrightarrow \Sigma$ as $\varepsilon \longrightarrow 0$.

(2) $J(\phi_\varepsilon) = J(U_o) + 0(\varepsilon^{n-2})$ (as $\varepsilon \longrightarrow 0$).

(3) $\|\nabla \phi_\varepsilon\|^2_{L^2(\Omega)} = \|\nabla U_o\|^2_{L^2(R^n)} + 0(\varepsilon^{n-2})$ (as $\varepsilon \longrightarrow 0$).

(4) There are constants $C_1, C_2 > 0$ such that

$$C_1\varepsilon^2 \le \|\phi_\varepsilon\|^2_{L^2(\Omega)} \le C_2\varepsilon^2.$$

(5) There are positive constant δ and $C \in R^1$ such that

$$Q_\lambda(\phi_\varepsilon) = \Sigma\left(1 - \delta\varepsilon^2 + C\varepsilon^{n-2} + o(\varepsilon^2)\right) \quad \text{as } \varepsilon \longrightarrow 0.$$

(Note that $n > 4$ implies $\varepsilon^{n-2} = o(\varepsilon^2)$.)

Hence (5) implies $Q_\lambda(\phi_\varepsilon) < \Sigma$ for sufficientry small $\varepsilon > 0$. Therefore we get $\sigma(\lambda) < \Sigma$. □

< <u>Sketch of proof of THEOREM.3</u> >

Let
$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{4}\int_\Omega \left(\frac{1}{|x|^4} * |u|^2\right)|u|^2\,dx - \frac{\lambda}{4}\int_\Omega \left(\frac{1}{|x|^\gamma} * |u|^2\right)|u|^2\,dx.$$

We claim that " F satisfies (PS)-condition for $(-\infty, \frac{1}{4}\Sigma)$."

PROPOSITION.6. - Assume $n \geq 5$, $0 < \gamma < 4$ and $\lambda > 0$. Then for any sequence $\{ u_j \} \subset H_0^1(\Omega)$ satisfying

(a) $F(u_j) \longrightarrow c$, (b) $F'(u_j) \longrightarrow 0$ in $H^{-1}(\Omega)$ and

(c) $c < \frac{1}{4} \Sigma$, there is a subsequence which converges in $H_0^1(\Omega)$ strongly.

Using the function U_0 we can prove

PROPOSITION.7. - Under the assumptions in THEOREM.3 ,there is a $v_0 \in H_0^1(\Omega)$ such that $\sup_{t \geq 0} F(t v_0) < \frac{1}{4} \Sigma$.

Using PROPOSITION.6 and POROSITION.7 ,THEOREM.3 can be proved by Mountain Pass Lemma. □

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