数理解析研究所講究録
第 679 巻 1989 年 183-193

On Non–radially Symmetric Bifurcation in the Annulus

SONG–SUN LIN, Department of Applied Mathematics
Hsin–chu, Taiwan, Republic of China

1. Introduction. In this paper we study the multiplicity of radially symmetric positive solutions and the non–radially symmetric bifurcation of these solutions of the following (Gel'fand) equation:

$$\Delta u(x) + 2\lambda e^{u(x)} = 0, \qquad x \in \Omega, \tag{1.1}$$

$$u(x) = 0, \qquad x \in \partial\Omega, \tag{1.2}$$

where Ω is the annulus

$$\Omega = \Omega_a = \{x = (x_1,x_2) \in R^2: a^2 < x_1^2 + x_2^2 < \frac{1}{a^2}\},\ a \in (0,1),\ \lambda > 0.$$ In [3],

Gel'fand found that (1.1) is invariant with respect to the group of transformations

$$u(r,\alpha) \equiv \alpha + u_0(r \exp(\tfrac{\alpha}{2}))$$

i.e., if $u_0(r)$ is a solution of (1.1), then for any $\alpha \in \mathbb{R}^1$, $u(r,\alpha)$ is also a solution of (1.1). Using this property, we can prove that there exists $\lambda^*(a) > 1$ such that there exist exactly two radially symmetric solutions for $\lambda \in (0,\lambda^*(a))$, one for $\lambda = \lambda^*(a)$ and none for $\lambda > \lambda^*(a)$. These solutions can be written explicitly and $\lambda^*(a)$ is computable. Taking the advantage of knowing the explicit formula of radially symmetric solutions $u_\lambda(r)$ (upper branch of solutions), we are able to understand its linearized problem

$$\Delta w(x) + 2\lambda e^{u_\lambda(r)} w(x) = 0, \quad x \in \Omega, \tag{1.3}$$

$$w(x) = 0, \quad x \in \partial\Omega. \tag{1.4}$$

More precisely, we prove that there exists a decreasing sequence $\{\lambda^*(k,a)\}_{k=1}^{\infty}$ with $\lambda^*(k,a) \to 0$ as $k \to \infty$, such that the equation

$$\varphi''(r) + \frac{1}{r}\varphi'(r) + (2\lambda e^{u_\lambda(r)} - \frac{k^2}{r^2})\varphi(r) = 0, \quad r \in (a,\frac{1}{a}), \tag{1.5}$$

$$\varphi(a) = 0 = \varphi(\frac{1}{a}) \tag{1.6}$$

has a non–trivial solution $\varphi_k(r)$ if and only if $\lambda = \lambda^*(k,a)$, $k = 1, 2, 3, \cdots$. For these $\lambda^*(k,a)$, the solution sets of (1.3), (1.4) are spanned by $\varphi_k(r)\cos k\theta$ and $\varphi_k(r)\sin k\theta$. $\varphi_k(r)$ can also be written explicitly.

To obtain the local non–radially symmetric bifurcation results at $\lambda^*(k,a)$, we shall verify a Crandall–Rabinowitz type transversality condition [1].

This paper is organized as follows: In section 2, we study the radially symmetric solutions. In section 3, we study the linearized problem (1.5), (1.6). In section 4, a Crandall–Rabinowitz type transversality condition is verified.

2. Radially symmetric solutions. In this section we shall study the existence and multiplicity problems of (1.1), (1.2) in the class of radially symmetric solutions, i.e., we consider the equation

$$u''(r) + \frac{1}{r}u'(r) + 2\lambda e^{u(r)} = 0, \quad r \in (a,\frac{1}{a}), \tag{2.1}$$

$$u(a) = 0 = u(\frac{1}{a}), \tag{2.2}$$

where $a \in (0,1)$.

By a classical transformation: $x = \log r$ and $v(x) = u(r) + 2\log r$, (2.1), (2.2) are transformed into

$$v''(x) + 2\lambda e^{v(x)} = 0, \quad x \in (-A,A) \tag{2.3}$$

$$v(-A) = -2A \quad \text{and} \quad v(A) = 2A, \tag{2.4}$$

where $A = -\log a > 0$.

The following lemma characterizes the solutions of the problem.

Lemma 2.1. The solutions of (2.3), (2.4) are given by

$$v_{K,\beta}(x) = \log \frac{\beta^2 \lambda^{-1} K m^{\beta/2} e^{\beta x}}{(1+Km^{\beta/2} e^{\beta x})^2}, \tag{2.5}$$

where $K > 0$, $\beta > 0$ and $v_{K,\beta}$ satisfies

$$\frac{\beta^2 \lambda^{-1} K}{(1+K)^2} = \frac{1}{m} \quad \text{and} \quad \frac{\beta^2 \lambda^{-1} K m^\beta}{(1+Km^\beta)^2} = m, \tag{2.6}$$

and $m = a^{-2}$.

If we set $t = \beta^2 \lambda^{-1}$, then (2.6) are transformed into more compact forms

$$\frac{tK}{(1+K)^2} = \frac{1}{m}, \quad \text{and} \quad \frac{tKm^\beta}{(1+Km^\beta)^2} = m. \tag{2.7}$$

Now, taking $t > 0$ as a parameter, we can solve (2.7) in terms of t as follows:

Lemma 2.2. The solutions of (2.7) are given by two functions $\lambda_\pm(\cdot)$: $[4m,\infty) \to (0,\infty)$,

$$\lambda_\pm(t) = t^{-1}\left(\frac{1}{\log m} \log \frac{P_\pm(t)}{4m}\right)^2,$$

where

$$P_+(t) = P_1(t)P_2(t), \quad P_-(t) = \overline{P}_1(t)P_2(t), \quad P_1(t) = (t-2m) + (t^2-4mt)^{1/2},$$

$$\overline{P}_1(t) = (t-2m) - (t^2-4mt)^{1/2}, \quad P_2(t) = (mt-2) + (m^2t^2-4mt)^{1/2}.$$

Then we can obtain the following result:

Theorem 2.3.

(i) For any $a \in (0,1)$, there exists a number $\lambda^*(a)$ $(= \lambda^*(m))$ such that (2.1) (2.2) has exactly two solutions for $\lambda \in (0,\lambda^*(a))$, exactly one at $\lambda = \lambda^*(a)$ and none for $\lambda > \lambda^*(a)$.

(ii) The solutions are of the form

$$u(r) = \log \frac{\beta^2 \lambda^{-1} K m^{\beta/2} r^\beta}{(1+K m^{\beta/2} r^\beta)^2 r^2},$$

where

$$K = K(t) = \frac{2}{P_2(t)}, \quad \lambda = \lambda_\pm(t) = \beta_\pm^2(t) t^{-1}, \quad \beta = \beta_\pm(t) = \frac{1}{\log m} \log \frac{P_\pm(t)}{4m},$$

$t \geq 4m$ and $P_2(t)$, $P_\pm(t)$ are given in Lemma 2.2.

3. Linearized eigenvalue problems. From last section, we know that for any m $(= \frac{1}{a^2}) > 1$, there are two smooth branches of radially symmetric solutions of (1.1), (1.2) in $(0,\lambda^*(m))$, namely, the upper (maximal) branch u_λ and the lower (minimal) branch $\underline{u}_\lambda$. It is well-known that the minimal branch $\underline{u}_\lambda$ can be obtained by monotone iteration starting from 0 and are stable, and $\underline{u}_\lambda(r) < u_\lambda(r)$ in $(a,\frac{1}{a})$ for any $\lambda \in (0,\lambda^*(m))$.

Let $\mu_1(\lambda)$ be the principal eigenvalue of linearized eigenvalue problem of (1.1), (1.2) at u_λ i.e., $\mu_1(\lambda)$ be the least eigenvalue of

$$\Delta w(x) + 2\lambda e^{u_\lambda(r)} w(x) = -\mu w(x), \quad x \in \Omega, \tag{3.1}$$

$$w(x) = 0, \quad x \in \partial\Omega, \tag{3.2}$$

Due to the convexity of e^u, it has been shown by Crandall and Rabinowitz [2] that

$\mu_1(\lambda) < 0$ for any $\lambda \in (0,\lambda^*(m))$. Therefore it is possible that there is a bifurcation from the upper branch u_λ.

By the method of separation of variables in polar coordinates, (3.1), (3.2) can be reduced to

$$\varphi''(r) + \frac{1}{r}\varphi'(r) + (2\lambda e^{u_\lambda(r)} - \frac{k^2}{r^2})\varphi(r) = -\mu_{k,\ell}(\lambda)\varphi(r),\ r \in (a,\tfrac{1}{a}), \tag{3.3}$$

$$\varphi(a) = 0 = \varphi(\tfrac{1}{a}), \tag{3.4}$$

$k = 0, 1, 2, \cdots, \ell = 1, 2, \cdots$.

By several changes of variables, we can bring (3.3), (3.4) into a more desirable form. First, set $x = \log r$, $\psi(x) = \varphi(r)$ and $\bar{v}(x) = u(r)$. Next, set $y = \beta x$, $\Psi(y) = \psi(x)$, $c = \frac{-k}{\beta(t)}$, $R(y) = 2K_1e^y(1+K_1e^y)^{-2}$ and $\tilde{\Psi}(y) = e^{-cy}\Psi(y)$ where $K_1 = Km^{\beta/2}$. Finally, set $X = K_1e^y$ and $\Phi(X) = \tilde{\Psi}(y)$. Then (3.3), (3.4) with $\mu_{k,\ell} = 0$ are transformed into

$$\Phi''(X) + \frac{1+2c}{X}\Phi'(X) + \frac{2}{X(1+X)^2}\Phi(X) = 0,\ X \in (L,R), \tag{3.5}$$

$$\Phi(L) = 0 = \Phi(R), \tag{3.6}$$

where $L = K$ and $R = Km^\beta$.

Set $s = -2c$, $X_s = \frac{1-s}{1+s}$, $s \in (0,1)$. Then it can be proved that

$$\Phi_1(X) = \frac{X-X_s}{X+1} \text{ and } \Phi_2(X) = \frac{X^s}{SX_s(1+X)}(XX_s-1)$$

are two linearly independent solutions of (3.5). Then we have:

Lemma 3.1. (3.6) has a non–trivial solution if and only if

$$(L-X_s)(RX_s-1)R^s - (R-X_s)(LX_s-1)L^s = 0. \tag{3.7}$$

Furthermore, (3.7) is equivalent to the following system

$$H(t,s,k) = 0, \text{ and } s\beta(t) = 2k, \tag{3.8}$$

where

$$H(t,s,k) = (L-X_s)(RX_s-1)m^{2k} - (R-X_s)(LX_s-1).$$

The corresponding eigenfunction can be taken as

$$\Phi(X) = \frac{1}{sX_s(L+1)(X+1)} \left\{(L-X_s)(XX_s-1)X^s-(X-X_s)(LX_s-1)L^s\right\}. \tag{3.9}$$

Lemma 3.2. For any $k > 0$, there exists a unique solution $(t(k), s(k), k)$ of (3.8). Furthermore, $t(k)$ and $s(k)$ are smooth in k and $\lim_{k\to\infty} t(k) = \infty$.

From Lemma 3.1 and 3.2, we obtain

Theorem 3.3. For any $k \in (0,\infty)$ there exists a unique $\lambda^*(k) > 0$ such that $\mu_{k,1}(\lambda^*(k)) = 0$. The function $\lambda^*(\cdot) : (0,\infty) \to (0,\lambda^*(m))$ is smooth and has the following properties:

$$\text{(i) } \lim_{k\to 0} \lambda^*(k) = \lambda^*(m), \quad \text{(ii) } \lim_{k\to\infty} \lambda^*(k) = 0.$$

Theorem 3.4. The linearized problems

$$\Delta w + 2\lambda e^{u_\lambda(r)} w = 0, \text{ in } \Omega,$$

$$w = 0, \qquad \text{on } \Omega,$$

have a non–trivial solution if and only if $\lambda = \lambda^*(k)$, $k = 0, 1, 2, \cdots$. Furthermore, for each $k \geq 1$, the corresponding eigenspace is spanned by $\varphi_k(r)\cos k\theta$ and $\varphi_k(r)\sin k\theta$, where $\varphi_k(r) = \Phi_k(X)$ and $\Phi_k(X)$ is given in (3.9) with $X = Km^{\beta/2}r^\beta$.

4. Symmetry Breaking. In this section we shall prove that there are non–radially symmetric solutions which bifurcate from the upper branch u_λ at every $\lambda^*(k)$, k = 1, 2, ···. We shall apply a bifurcation theorem of Crandall and Rabinowitz [1].

Theorem 4.1. Let X, Y be Banach spaces, V a neighborhood of 0 in X, $\bar{\lambda}$, ϵ in $\mathbb{R}^1$ and

$$F : (\bar{\lambda}-\epsilon, \bar{\lambda}+\epsilon) \times V \longrightarrow Y$$

have the properties

(a) $F(\lambda,0) = 0$ for $\lambda \in (\bar{\lambda}-\epsilon, \bar{\lambda}+\epsilon)$,

(b) the partial derivatives F_λ, F_u, $F_{\lambda u}$ exist and are continuous,

(c) $N(F_u(\bar{\lambda},0))$ and $Y/R(F_u(\bar{\lambda},0))$ are one–dimensional,

(d) $F_{\lambda u}(\bar{\lambda},0)w_0 \notin R(F_u(\bar{\lambda},0))$, where $N(F_u(\bar{\lambda},0)) = \mathrm{span}\{w_0\}$.

If Z is any complement of $N(F_u(\bar{\lambda},0))$ in X, then there is a neighborhood U of $(\bar{\lambda},0)$ in $\mathbb{R} \times X$, an interval $(-\delta,\delta)$, and continuous functions

$$\varphi : (-\delta,\delta) \to \mathbb{R}^1, \quad \psi : (-\delta,\delta) \to Z$$

such that $\varphi(0) = 0$, $\psi(0) = 0$ and

$$F^{-1}(0) \cap U = \{(\varphi(\alpha), \alpha w_0 + \alpha\psi(\alpha)) : |\alpha|<\delta\} \cup \{(\lambda,0) : (\lambda,0) \in U\}.$$

To apply Theorem 4.1, we need to rewrite (1.1), (1.2) as a nonlinear operator equation on an appropriate function space. We shall work on Hölder spaces.

Denoted by $C_0^{1+\gamma}(\bar{\Omega})$ the set of continuously differentiable functions on $\bar{\Omega}$ which vanish on $\partial\Omega$ and whose first order derivatives are Hölder continuous in $\bar{\Omega}$ with exponent $\gamma \in (0,1)$.

Then (1.1), (1.2) is equivalent to $F(\lambda,u) = 0$, where $F(\lambda,u) : (0,\lambda^*(m)) \times$

$C_0^{1+\gamma}(\overline{\Omega}) \to C_0^{1+\gamma}(\overline{\Omega})$ is defined by $F(\lambda,u) = u + u_\lambda + 2\lambda G f(u+u_\lambda)$ with $G = \Delta^{-1}$ and $f(u) = e^u$. The linearized operator $F_u(\lambda,0)$: $C_0^{1+\gamma}(\overline{\Omega}) \to C_0^{1+\gamma}(\overline{\Omega})$ is given by $F_u(\lambda,0)w = w + 2\lambda G(e^{u_\lambda}w)$ and the mixed derivative $F_{\lambda u}(\lambda,0) : \mathbb{R}^1 \times C_0^{1+\gamma}(\overline{\Omega}) \to C_0^{1+\gamma}(\overline{\Omega})$ is given by $F_{\lambda u}(\lambda,0)w = G\left\{\frac{\partial}{\partial\lambda}(2\lambda e^{u_\lambda})w\right\}$.

Let $\tilde{C}_0^{1+\gamma}(\overline{\Omega}) = \left\{u \in C_0^{1+\gamma}(\overline{\Omega}) : u(-x_1,x_2) = u(x_1,x_2)\right\}$. By Theorem 3.4, the kernel of $F_u(\lambda,0)$ is non–trivial if and only if $\lambda = \lambda^*(k)$, for some $k \geq 1$. If we restrict (1.1), (1.2) on $\tilde{C}_0^{1+\gamma}(\overline{\Omega})$ then for any $k \geq 1$,

$$\operatorname{Ker} F_u(\lambda^*(k),0) \cap \tilde{C}_0^{1+\gamma}(\overline{\Omega}) = \begin{cases} \text{span }\{\varphi_k(r)\cos k\theta\} & \text{if } k \text{ is even} \\ \text{span }\{\varphi_k(r)\sin k\theta\} & \text{if } k \text{ is odd}. \end{cases}$$

Therefore, with this setting the conditions (a), (b), (c) of Theorem 4.1 are satisfied and (d) is

$$\int_a^{\frac{1}{a}} r\varphi_k^2(r)\frac{\partial}{\partial\lambda}\left\{\lambda e^{u_\lambda(r)}\right\}\Big|_{\lambda=\lambda^*(k)}\, dr \neq 0. \tag{4.6}$$

Lemma 4.2. For $k > 0$, we have

(i) $\dfrac{d\lambda^*(k)}{dk} < 0$,

(ii) $\displaystyle\int_a^{\frac{1}{a}} r\varphi_k^2(r)\frac{\partial}{\partial\lambda}\left\{\lambda e^{u_\lambda(r)}\right\}\Big|_{\lambda=\lambda^*(k)}\, dr < 0.$

Proof. For $\lambda \in (0,\lambda^*(m))$ and $k \in (0,\infty)$, let $\mu(\lambda,k)$ and $\varphi(r,\lambda,k)$ be the principal eigenvalue and principle eigenfunction of linearized eigenvalue problem

$$\varphi''(r) + \frac{1}{r}\varphi'(r) + (2\lambda e^{u_\lambda(r)} - \frac{k^2}{r^2})\varphi(r) = -\mu\varphi(r),\ r \in (a,\tfrac{1}{a}),$$

$$\varphi(a) = 0 = \varphi(\tfrac{1}{a}),$$

where $\varphi(r,\lambda,k)$ is normalized by $\int_a^{\frac{1}{a}} r\varphi^2(r,\lambda,k)\,dr = 1.$

Denote by $W(r,\lambda,k) = \frac{\partial\varphi}{\partial\lambda}(r,\lambda,k)$ and $V(r,\lambda,k) = \frac{\partial\varphi}{\partial k}(r,\lambda,k)$.

Then W and V satisfy second order ordinary differential equations and (2.2). Then, we have

$$\frac{\partial\mu}{\partial\lambda}(\lambda,k) = -\int_a^{\frac{1}{a}} r\varphi^2(r,\lambda,k)\frac{\partial}{\partial\lambda}\left\{2\lambda e^{u_\lambda(r)}\right\}dr \text{ and } \frac{\partial\mu}{\partial k}(\lambda,k) = 2k\int_a^{\frac{1}{a}}\frac{1}{r}\varphi^2(r,\lambda,k)\,dr > 0.$$

Since $\mu(\lambda^*(k),k) = 0$, using Theorem 3.3, we have

$$\frac{\partial\mu}{\partial\lambda}(\lambda^*(k),k)\frac{d\lambda^*(k)}{dk} + \frac{\partial\mu}{\partial k}(\lambda^*(k),k) = 0,$$

and the results follow.

Using Theorem 4.1 and Lemmas 4.2, we obtain the following theorem:

Theorem 4.3. The upper branch u_λ of radially symmetric solutions of (1.1), (1.2) has a non–radially symmetric bifurcation at each $\lambda^*(k)$, $k = 1, 2, \cdots$. Furthermore, in a neighborhood of $(\lambda^*(k), u_{\lambda^*(k)})$, the dimension of the set of bifurcating asymmetric solutions is two.

Remark 4.4. By using the global bifurcation theorem of Rabinowitz [7], we can obtain the following global results:

Denote by S the solution set of (1.1), (1.2) and R the set of radially symmetric solutions of (1.1), (1.2). Let C be the closure of $\{(0,\lambda^*(m)) \times \tilde{C}_0^{1+\gamma}(\overline{\Omega})\} \cap (S\backslash R)$.

Then, for any $k > 1$, the connected component C_k of $C \cup \{(\lambda^*(k), u_{\lambda^*(k)})\}$ to which $(\lambda^*(k), u_{\lambda^*(k)})$ belongs is either unbounded or meets $(\lambda^*(\ell), u_{\lambda^*(\ell)})$ for some positive integer $\ell \neq k$.

REFERENCES

1. M.G. CRANDALL and P.H. RABINOWITZ, Bifurcation form simple eigenvalues, J. Functional Analysis, 8(1971), 321–340.
2. M.G. CRANDALL and P.H. RABINOWITZ, Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems, Arch. Rational Mech. Anal. 58(1975), 207–218.
3. I.M. GELFAND, Some problems in the theory of quasilinear equaitons. Amer. Math. Soc. Translations. 1(2) 29, (1963), 295–381.
4. B. GIDAS, W.M. NI and L. NIRENBERG, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68(1979), 209–243.
5. H.B. KELLER and D.S. COHEN, Some positone problems suggested by nonlinear heat generation. J. Math. Mech. 16(1967), 1361–1376.
6. J. LIOUVILLE, Sur l'équation aux dérivées partielles $\frac{\partial^2 \log\lambda}{\partial u \partial v} \pm 2\lambda a^2 = 0$. J. de Math. 18(1853), 71–72.
7. P.H. RABINOWITZ, Some global results for nonlinear eigenvalue problems. J. Functional Analysis, 7(1971), 487–513.
8. J. SMOLLER and A. WASSERMAN, Symmetry–breaking for positive solutions of semilinear elliptic equations., Arch. Rational Mech. Anal. 95(1986) 217–225.

9. J. SMOLLER and A. WASSERMAN, Symmetry–breaking for solutions of semilinear elliptic equaitons with general boundary conditions. Commun. Math. Phys. 105(1986), 415–441.

10. SUZUKI, T. and NAGASAKI, K.: On the nonlinear eigenvalue problem $\Delta u+\lambda e^{u} = 0$. to appear in Trans. Amer. Math. Soc., see also this Proceeding.