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1-Determinacy of Feasible Sets

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1. Introduction

The purpose of this paper is to present the preparation for the singularity theory on nonlinear programming problems.

In the singularity theory of smooth mappings, the following three problems are fundamental (see Gibson [1]):

(1) Finite determinacy of smooth mappings,

(2) The existence of universal unfolding,

(3) Classification of singular points.

On the one hand, in nonlinear programming problems, to investigate the singularity of the solutions satisfying the Kuhn-Tucker condition is very important for the cases such as studying the bifurcation of the solutions of parametric problems. However, a feasible set as the base space changes together with the objective function, so the singularity theory of smooth mappings can not apply directly to nonlinear programming problems. Therefore, as the preparation to develop the singularity theory of objective functions on feasible sets, we discuss the simplest problem of finite determinacy, i.e., "1-determinacy of feasible sets".

We define 1-determinacy of feasible sets and (strongly) restricted 1-determinacy of map germs to characterize each concept of 1-determinacy by a necessary and sufficient condition and to show all of these concepts are equivalent in some sense. For simplicity, we will deal with feasible sets defined only by inequalities throughout the paper. (See [5] for the general feasible sets defined by equality and inequality constraints.)

2. 1-Determinacy of Feasible Sets.

Let $E_{n,p}$ be the set of all C^1 (continuously differentiable) map germs at 0 of R^n (the n-dimensional Euclidean space) to R^p (see [1] for the definition of map germs), $E^0_{n,p}$ be the subset of $f \in E_{n,p}$ with $f(0) = 0$, Diff_n be the set of all C^1-diffeomorphic germ ϕ's of $R^n \to R^n$ ($\phi(0) = 0$) and Diff^1_n be the subset of Diff_n such that $D\phi(0) = I_n$ where D is the partial differential operator and I_n is the $n \times n$ unit matrix. Define $K_n = \{\Phi_{\sigma,u} \in \mathrm{Diff}_n$: there exist a germ $u \in E_{n,n}$ with $u_i(0) \neq 0$ $(1 \leqq i \leqq n)$ and a permutation σ on $\{1,2,\dots,n\}$ satisfying $\Phi_{\sigma,u}(x) = (u_1 x_{\sigma(1)},\dots,u_n x_{\sigma(n)})$. Define also $K^1_n = K_n \cap \mathrm{Diff}^1_n$. Let $H_{n,p} = K_p \times \mathrm{Diff}_n$ and $H^1_{n,p} = K^1_n \times \mathrm{Diff}^1_n$. Then $H^1_{n,p}$ is the subgroup of $H_{n,p}$. $H_{n,p}$ acts on $E^0_{n,p}$ from left as follows: for any $\alpha = (\beta, \gamma) \in H_{n,p}$ and $f \in E^0_{n,p}$, $\alpha . f := \beta f \gamma^{-1}$. $H^1_{n,p}$ also acts on $E^0_{n,p}$ from left.

Let the 1-jet of $f \in E^0_{n,p}$ at $0 \in R^n$ by $j^1 f_i = Df_i(0)x$ and $J^1_f = \{g \in E^0_{n,p} : j^1 g = j^1 f\}$ If $J^1_f \subset H_{n,p}.f$ (resp. $H^1_{n,p}.f$) holds, then f is called <u>restricted 1-determined</u>

(resp. strongly restricted 1-determined). For a $g \in E^0_{n,p}$, define $M[g] = \{x \in R^n : g_i(x) \geq 0,\ i = 1,\dots,p\}$, where $g_i(x)$ is the i-th component of g. M[g] is called the feasible set defined by g. For $0 \in R^n$, define an equivalence relation on the set $\{M[g] : g \in E^0_{n,p}\}$ by $M[g_1] \sim_0 M[g_2]$ if there exists a neighborhood U of $0 \in R^n$ such that $M[g_1] \cap U = M[g_2] \cap U$. A feasible set germ on R^n is defined as an equivalence class with respect to $\sim_0$ and denoted by $M[g]_0$. We call $M[g]_0$ is 1-determined if $M[g]_0$ is diffeomorphic to $M[h]_0$ by the action of Diff_n whenever $M[j^1 g]_0 = M[j^1 h]_0$.

To characterize the concepts mentioned above, we prepare two definitions and the assumption below.

Definition 1. Let A be a real $p \times n$ matrix. We say that A is of type 1 if

(1) rank A = min (p,n),

(2) $p \leq n+2$,

(3) all the minors of degree n of A are nonzero.

Definition 2. For a $g \in E^0_{n,p}$,

(1) g is irredundant if $Dg_i(0)$ and $Dg_j(0)$ are linearly independent to each other for $i \neq j$.

(2) g is a minimal representative if $M[g] \not\sim_0 M[g^{\langle i \rangle}]$ where $g^{\langle i \rangle}$ is defined by $g^{\langle i \rangle}_j = (1 - \delta^i_j) g_j$. Here δ^i_j is the Kronecker's delta.

<u>Remark</u>. The definition of "irredundant" (resp. "a minimal representative") corresponds to that of "smoothness condition" (resp. "a conical basis") in [3].

If cusp points are in $M[g]$, then the situation becomes rather complicated. So it is natural to assume some condition to exclude the existence of cusps. Our assumption is then:

<u>Assumption 1</u>. (Mangasarian-Fromovitz [2]) The map germ $g \in E^0_{n,p}$ satisfies <u>Mangasarian-Fromovitz condition at 0</u>, i.e., there is a $w \in R^n$ such that $Dg(0)w > 0$.

<u>Remark</u>. In [4], it is shown that the above assumption is equivalent to the condition that $C = R_+\{Dg_j(0)\}_{j=1,\ldots,p}$ is a strictly convex cone, i.e., C is convex and $C \cap (-C) = \{0\}$, where $R_+ = \{x \in R : x \geqq 0\}$ and $R_+T = \{$all finite sums $\Sigma r_i t_i$ for $r_i \in R_+$ and $t_i \in T\}$ for any subset T of R^n. Furthermore, [4] presents a new characterization of Mangasarian-Fromovitz condition.

The theorem below gives a necessary and sufficient condition for $M[g]_0$ to be 1-determined:

<u>Theorem 1</u>. Let $n \geqq 2$ and $g \in E^0_{n,p}$ be irredundant, a minimal representative and satisfy Assumption 1. Then $M[g]_0$ is 1-determined if and only if $Dg(0)$ is of type 1.

Proof) See Theorem 3.1 and Lemma 3.4 of [3].

The next theorem characterizes the restricted 1-determinacy:

<u>Theorem 2</u>. Let $n \geqq 2$ and $g \in E^0_{n,p}$. Suppose g satisfies Assumption 1. Then g is restricted 1-determined if and only if Dg(0) is of type 1.

Proof) See Theorem 2.15 of [5].

The following result shows that the concepts of 1-determinacy mentioned above are equivalent:

<u>Theorem 3</u>. Let $n \geqq 2$ and $g \in E^0_{n,p}$ be irredundant, a minimal representative and satisfy Assumption 1. Then the next three statements are equivalent:

(1) g is restricted 1-determined,

(2) g is strongly restricted 1-determined,

(3) $M[g]_0$ is 1-determined.

Proof) See Corollary 2.16 of [5].

<u>References</u>

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