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Toric ASL Domains

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Abstract. Since ASL(algebra with straightening laws) was axiomatized, several conjectures have been proposed. The purpose of this paper is to show that three conjectures hold true for graded toric ASL domain, namely the normality of ASL domain, the Cohen-Macaulay property of ASL domain, and the Cohen-Macaulay property of poset (partially ordered set) underlying Cohen-Macaulay ASL.

Introduction. Since algebra with straightening laws (an ASL, for short) was axiomatized by [Ei] and [DEP], several conjectures proposed: D.Eisenbud [Ei] conjectured that every ASL domain on a wonderful poset is normal with rational singularities. T.Hibi and K.-i.Watanabe conjectured that every ASL domain is Cohen-Macaulay, which is implicitly written in [HW]. T.Hibi [Hi2] conjectured that every poset underlying Cohen-Macaulay ASL is Cohen-Macaulay.

The purpose of this paper is to verify these three conjectures for graded toric ASL's. A graded toric ASL is a semigroup ring which is a graded ASL on a poset consisting of the system of generators of the semigroup. Some of these rings were previously considered in [Wa2] and [Hi1].

Our main results are the following:

Theorem 4.1. *A graded toric ASL over a normal ring is normal.*

Theorem 5.2. *A poset underlying a graded toric ASL is Cohen-Macaulay.*

Theorem 4.1 can be viewed as a special case of the conjecture concerning the normalities of ASL domains [Ei], although posets underlying graded toric ASL's are not necessarily wonderful. To prove the theorem our assumption "toric" is essential. In fact there exist non-normal Gorenstein homogeneous ASL domains on wonderful posets over a field [HW]. At this symposium the author announced theorem 4.1 as a conjecture, except in the special case(homogeneous and posets of rank≤3), and also gave an approach to this problem. Although the conjecture was verified by M.Miyazaki with a different method after this symposium, we here give a proof of the theorem along the approach.

Theorem 5.2 can be viewed as a special case of a combination of two conjectures on the Cohen-Macaulay properties of ASL domains and the Cohen-Macaulay properties of posets underlying Cohen-Macaulay ASL's [HW][Hi2]. Without our assumption "toric", the corresponding conjectures are still open, as far as the author knows.

The Cohen-Macaulay properties of graded toric ASL's, corresponding to the second conjecture, follows from either of two above theorems as corollary.

To obtain these results, first we give a criterion for a semigroup ring with partially ordered generators of the semigroup to be a graded ASL in terms of configuration of an embedded poset in a vector space over $\mathbb{R}$ (§2). In our toric case the posets are naturally embedded in vector spaces over $\mathbb{R}$. These posets are more relevant to our study than posets represented as Hasse diagrams. Using the criterion, we study the structure of the semigroups generated by posets underlying graded toric ASL's and the combinatorics on embedded posets in vector spaces over $\mathbb{R}$(§3). This allows us to use Hochster's criterion for the normality of semigroup ring (§4) and a well-developed theory of Stanley-Reisner rings (§5).

For lack of space we omit the further investigations of the homogeneous toric ASL domains and their classification in lower dimensional cases. These results will be given elsewhere.

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§ 1. Definition of Graded Algebra with Straightening Laws and Graded Toric ASL

First we introduce a definition of graded algebra with straightening laws.

Definition. Let S be an algebra over a commutative ring k and $\Pi \subset S$ a finite subset with a partial order $\leq$, called *poset* for short. A product of the form $\xi_1 \cdots \xi_m$, $m \in \mathbb{Z}_{>0}$, $\xi_i \in \Pi$ such that $\xi_1 \leq \cdots \leq \xi_m$ is called a *standard monomial*. S is a *graded algebra with straightening laws*, if the following conditions hold:

(H_0) $S = \bigoplus_{i \geq 0} S_i$ is a graded k-algebra such that $S_0 = k$, Π consists of homogeneous elements of positive degree and generates S as a k-algebra.

(H_1) The standard monomials are linearly independent over k.

(H_2) For all incomparable $\xi,\nu \in \pi$, the product $\xi\nu$ has a representation

(*) $\xi\nu = \sum a_\mu \mu$, $a_\mu \in k$, μ standard monomial,

satisfying the following condition: every μ contains a factor $\zeta \in \pi$ such that $\zeta<\xi, \zeta<\nu$ (it is of course allowed that $\xi\nu = 0$, the sum $\sum a_\mu \mu$ being empty).

When $\pi \subset S_1$ we call that S is *homogeneous*. When the right hand sides of relations (*) are 0's, we call S is *discrete*.

We define some terminology on an abstract poset π.

Definition. A *chain* of π is a totally ordered subset. The *length* of a chain is the cardinality of the chain minus one. The *rank* of the poset π is the maximal length of chains in π. The *height* of an element $\xi \in \pi$ is the maximal length of chains descending from ξ in π. When all maximal chains in π have the same length, we call that the poset π is *pure*. We denote by $K(\pi)$ the abstract simplicial complex consisting of chains of π and by $|K(\pi)|$ the geometric realization of $K(\pi)$.

We summarize some fundamental facts on a graded ASL (S,π). For the proofs of facts below and further discussion of this notion the reader is referred to [BrVe] and [DEP]. Note that our graded ASL is called a *graded ordinal Hodge algebra* in [DEP].

(1.1). When every element of π is not a zero-divisor the poset π has a unique minimal element.

(1.2). A unique minimal element in π is a non-zero-divisor.

(1.3). $\dim S = \dim k + \operatorname{rk} \pi + 1$.

(1.4). There always exists a discrete graded ASL on Π. Hence it is isomorphic to S as k-modules, but has discrete straightening relations (i.e. the right hand sides of straightening relations (*) are zeros). We denote it by S_o.

(1.5). S is reduced if and only if k is reduced.

Now we define graded toric ASL.

Definition. Let A be the Laurent polynomial ring with r-indeterminantes $x_1, \cdots, x_r$ over a commutative ring k. By a *usual Laurent monomial* we shall always mean a term of the form $x_1^{h_1} \cdots x_r^{h_r}$, where $h_1, \cdots, h_r$ are integers. Also by a *Laurent polynomial* we mean a linear combination of usual Laurent monomials. We denote by $\mathcal{M}(r)$ the set of Laurent monomials in A and by $\mathcal{M}^+(r)$ the set of Laurent monomials of positive degree in A. Let $\Pi \subset \mathcal{M}^+(r)$ be a finite poset. Then we set $S = k[\Pi] \subset A$ and call it a *graded toric ring with ordered generators*. When a suitable change of degree of S makes a graded toric ring homogeneous, we call S is *homogeneous*. A graded toric ring with ordered generators (S, Π) is called a *graded toric ASL* if S is a graded ASL over k on Π. And we sometimes call its underlying poset *a graded toric ASL poset*. By definition the commutative ring k might as well be taken to be a field or the ring of integers.

§ 2. Characterization of Graded Toric ASL Posets

Let $k, A, \mathcal{M}(r)$, and $\mathcal{M}^+(r)$ be as in § 1. Under the map log, $\mathcal{M}(r)$ is mapped isomorphically onto the group $\mathbb{Z}^r$ as Abelian groups:

$$\log:\quad \mathcal{M}(r) \longrightarrow \mathbb{Z}^r \subset \mathbb{R}^r$$
$$x_1^{h_1}\cdots x_r^{h_r} \longmapsto \sum_{i=1}^r h_i \log(x_i)$$

where $\log(x_i)$ is the symbol representing $(0,\ldots,0,\overset{i}{1},0,\ldots,0)$. We denote by exp the inverse map of log.

In this section we shall give conditions for a graded (resp. homogeneous) toric ring with orderd generators (S,Π) to be a graded (resp. homogeneous) ASL in terms of configuration of $\log(\Pi)$.

Throughout this paper we assume that $\log(\Pi)$ generates $\mathbb{Z}^r$ as an Abelian group and that $\dim S = r + \dim k$, which are guaranteed by the following lemma:

Lemma 2.1. *Let (S,Π) be an $(n+\dim k)$-dimensional graded toric ring with ordered generators over k of $\dim k<+\infty$. Then the vector subspace V of $\mathbb{R}^r$ spanned by $\log(\Pi)$ has dimension n. Moreover S is homogeneous if and only if there exists an $(n-1)$-dimensional affine subspace P^{n-1} containing $\log(\Pi)$ and not containing 0.*

Proof. Set $S' = k[\{\beta,\beta^{-1}\}_{\beta\in\Pi}]$. Since $\dim S = \dim S' = \dim_{\mathbb{R}} V + \dim k$, we have $\dim V = n$, which proves the first part. By definition S is homogeneous if and only if there exists an affine hyperplane H not containing 0 and containing $\log(\Pi)$. We take P^{n-1} as $H \cap V$ in case S is homogeneous.

Proposition 2.2. *For a graded toric ring with ordered generators (S,Π) the following conditions are equivalent:*

(a) *(S,Π) satisfies the condition* (H_1).

(b) *The finite poset $\log(\Pi) \subset \mathbb{R}^r$ satisfies the following condition:*

(H_1') *For any two distinct chains $\log(\lambda_1) < \cdots < \log(\lambda_t)$ and*

$\log(\mu_1) < \cdots < \log(\mu_s)$ $(t\geq 1, s\geq 1)$ *in* $\log(\Pi)$, *we have*

$$\text{rel int}\left(\textstyle\sum_{i=1}^{t}\mathbb{R}_{\geq 0}\log(\lambda_i)\right) \cap \text{rel int}\left(\textstyle\sum_{j=1}^{s}\mathbb{R}_{\geq 0}\log(\mu_j)\right) = \phi,$$

where rel int$(\cdot)$ *denotes its relative interior.*

Moreover if S *is homogeneous the condition* (H_1') *is equivalent to the following:*

(H_1'') *For any two distinct chains* $\log(\lambda_1) < \cdots < \log(\lambda_t)$ *and* $\log(\mu_1) < \cdots < \log(\mu_s)$ $(t\geq 1, s\geq 1)$ *in* $\log(\Pi)$, *we have*

$$\text{rel int}(|\log(\lambda_1),\cdots,\log(\lambda_t)|) \cap \text{rel int}(|\log(\mu_1),\cdots,\log(\mu_s)|) = \phi$$

where $|\log(\lambda_1),\cdots,\log(\lambda_t)|$ *denotes the convex hull of* $\log(\lambda_i)$ $(i=1,\cdots,t)$.

Proof. We have only to prove equivalence between (a) and (b) in case S is graded. First note the following lemmas.

Lemma 2.3. *Let* (S,Π) *be a graded toric ring with ordered generators satisfying the condition* (H_1) *and* $\lambda_1<\cdots<\lambda_t$ *be a chain in* Π. *Then* $\log(\lambda_i)$ $(i=1,\cdots,t)$ *are linearly independent over* $\mathbb{R}$.

Proof. If $\log(\lambda_i)$ $(i=1,\cdots,t)$ are linearly dependent, we have a non-trivial relation $\sum_{i=1}^{t}a_i\log(\lambda_i) = 0$, $a_i\in\mathbb{Z}$ because $\log(\lambda_i)$ $(i=1,\cdots,t)$ are contained in $\mathbb{Z}^r$. The relation is mapped by exp to the relation $\prod_{i:a_i>0}\lambda_i^{a_i} = \prod_{j:a_j<0}\lambda_j^{(-a_j)}$, which contradicts (H_1). Hence $\log(\lambda_i)$ $(i=1,\cdots,t)$ are linearly independent.

Lemma 2.4. *Let* (S,Π) *be a graded toric ring with ordered generators satisfying the condition* (H_1') *and* $\lambda_1<\cdots<\lambda_t$ *be a chain in* Π. *Then* $\log(\lambda_i)$ $(i=1,\cdots,t)$ *are linearly independent over* $\mathbb{R}$.

Proof. By induction on t the case of t=1 is obvious. If $\log(\lambda_i)$ $(i=1,\cdots,t)$ are linearly dependent for t>1, we have a non-trivial relation $\sum_{i=1}^{t} a_i \log(\lambda_i) = 0$, $a_i \in \mathbb{Z}$. By induction and the assumption $\pi \subset \mathcal{M}^+(r)$ both $\{\log(\lambda_i)\}_{i:a_i>0}$ and $\{\log(\lambda_j)\}_{j:a_j<0}$ are linearly independent and non-empty. Therefore the relation implies that

$$\text{rel int}\Big(\sum_{i:a_i>0} \mathbb{R}_{\geq 0}\log(\lambda_i)\Big) \cap \text{rel int}\Big(\sum_{j:a_j<0} \mathbb{R}_{\geq 0}\log(\lambda_j)\Big) \neq \phi,$$

which contradicts (H_1'). Hence $\log(\lambda_i)$ $(i=1,\cdots,t)$ are linearly independent over $\mathbb{R}$.

We now go back to the proof of (2.2). Since π consists of usual Laurent monomials in A, which are linearly independent over k, (a) is equivalent to the following:

(c) The forgetful map from the standard monomials to the usual Laurent monomials is injective.

Equivalence between (c) and (b) results from (2.3), (2.4), and the following lemma, which concludes the proof of (2.2).

Lemma 2.5. *For any two chains in* π, $\lambda_1 < \cdots < \lambda_t$ *and* $\mu_1 < \cdots < \mu_s$ $(t \geq 1, s \geq 1)$ *whose log's are linearly independent respectively, the following conditions are equivalent:*

(1) *There exist positive integers* h_i $(i=1,\cdots,t)$ *and* m_j $(j=1,\cdots,s)$ *such that* $\lambda_1^{h_1}\cdots\lambda_t^{h_t} = \mu_1^{m_1}\cdots\mu_s^{m_s}$ *in* S.

(2) $U := \text{rel int}\big(\sum_{i=1}^{t}\mathbb{R}_{\geq 0}\log(\lambda_i)\big) \cap \text{rel int}\big(\sum_{j=1}^{s}\mathbb{R}_{\geq 0}\log(\mu_j)\big)$ *is not empty.*

Proof. (1)=>(2): Trivial. (2)=>(1): By the rationality of the $\log(\lambda_i)$ and $\log(\mu_j)$, U contains a rational point. So (1) follows.

Proposition 2.6. *For a graded toric ring with ordered generators* (S, π) *satisfying* (H_1), *the following conditions are equivalent:*

(a) (S, π) *satisfies* (H_2).

(b) *The finite poset* $\log(\Pi)$ *satisfies the following:*

(H_2') *For any incomparable* $\log(\mu)$, $\log(\nu)$ *in* $\log(\Pi)$ *there exist* $\log(\xi_1), \ldots, \log(\xi_m)$ *in* $\log(\Pi)$ *such that* $\log(\xi_1) \leq \cdots \leq \log(\xi_m)$, $\log(\mu) > \log(\xi_1)$, $\log(\nu) > \log(\xi_1)$, *and* $\log(\mu) + \log(\nu) = \sum_{i=1}^{m} \log(\xi_i)$.

Moreover if S *is homogeneous,* (H_2') *is equivalent to the following:*

(H_2") *For any incomparable* $\log(\nu)$, $\log(\mu)$ *in* $\log(\Pi)$, *there exist* $\log(\xi_1)$, $\log(\xi_2)$ *in* $\log(\Pi)$ *such that* $\log(\xi_1) \leq \log(\xi_2)$, $\log(\xi_1) < \log(\nu)$, $\log(\xi_1) < \log(\mu)$, *and*
$\frac{1}{2}\log(\nu) + \frac{1}{2}\log(\mu) = \frac{1}{2}\log(\xi_1) + \frac{1}{2}\log(\xi_2)$ *in* P^{r-1}.

Proof. (b)=>(a) is trivial. (a)=>(b) is almost trivial. But note that in the right hand side of each straightening relation in (H_2) exactly one standard monomial appears, under the condition (H_1).

The following examples illustrate (2.2) and (2.6).

Example 2.7. The poset in Fig.1 is a homogeneous toric ASL poset. A homogeneous toric domain with the partially ordered generators in Fig.2 satisfies (H_2) but does not (H_1). A homogeneous toric domain with the partially ordered generators in Fig.3 satisfies (H_1) but does not (H_2).

Posets of rank 2 in $M^+(3)$

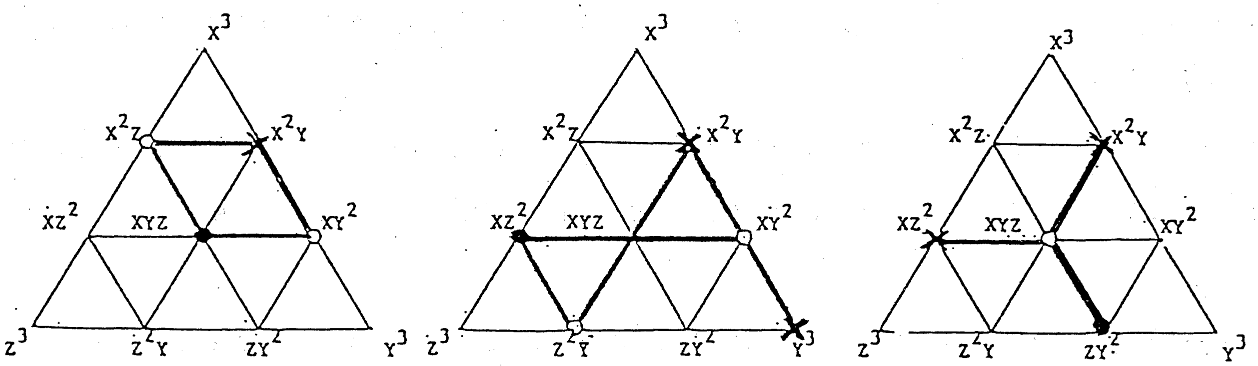

Fig.1 Fig.2 Fig.3

ht ● = 0 ht ○ = 1 ht ✕ = 2

§ 3. Combinatorial Aspects of Graded Toric ASL Posets

Let (S, Π) be a graded toric ASL of dimension r + dim k. In this section we study the combinatorics on $\log(\Pi) \subset \mathbb{R}^{r}$.

We adopt the following notation, to be held throughout this paper. Let α be a unique element of ht 0 in Π (1.1). Let $\{\sigma_1, \cdots, \sigma_m\}$ be the simplices consisting of maximal chains in $\log(\Pi)$ and $K(\log(\Pi))$ be the set of simplices whose vertices consist of elements of chains in $\log(\Pi)$ (2.2). Set $Q = \sigma_1 \cup \cdots \cup \sigma_m$. We denote by $\tilde{\tau}$ the cone joining 0 to all points of $\tau \in K(\log(\Pi))$. The cones $\tilde{\tau}$ are simplicial strongly convex cones by (2.3) and (2.4), where a *strongly convex cone* is a convex cone without linear subspace of positive dimension. Set $\tilde{Q} = \tilde{\sigma}_1 \cup \cdots \cup \tilde{\sigma}_m$. We denote by $\Sigma(K(\log(\Pi)))$ the system of cones $\tilde{\tau}$, $\tau \in K(\log(\Pi))$. Setting $c = \mathrm{l.c.m.}(\{ \deg\beta \}_{\beta\in\Pi})$, we denote by H^{r-1} the affine (r-1)-plane spanned affinely by $\{ (c/\deg\beta)\log(\beta) \}_{\beta\in\Pi}$. Let φ be the projection from 0 to H^{r-1} which maps Q homeomorphically onto its image. Of course we have $H^{r-1} = P^{r-1}$ and $\varphi(Q)=Q$ in case S is homogeneous. For further definitions on convex cones the reader is referred to [Od2].

Lemma 3.0. *The composition of the forgetful map in (c) of the proof of (2.2) and the map* log

$$\begin{array}{ccc} \{ \text{ the standard monomials } \} & \longrightarrow & \tilde{Q} \cap \mathbb{Z}^{r} \\ \xi = \xi_1^{h_1} \cdots \xi_t^{h_t} & \longmapsto & \sum_{j=1}^{t} h_j \log(\xi_j) \\ \xi_1 < \cdots < \xi_t,\ \xi_i \in \Pi & & \end{array}$$

induces two bijections:

$$\{\text{ the standard monomials }\}/\sim \longrightarrow Q \cap \mathbb{Q}^r$$
$$\xi_1^{h_1}\cdots\xi_t^{h_t}\sim \longmapsto \sum_{j=1}^{t}\frac{h_j}{(\Sigma_i h_i)}\log(\xi_j)$$
$$\{\text{ the standard monomials }\}/\sim \longrightarrow \varphi(Q) \cap \mathbb{Q}^r$$
$$\xi_1^{h_1}\cdots\xi_t^{h_t}\sim \longmapsto \sum_{j=1}^{t}\frac{h_j}{(\Sigma_i h_i)}\frac{c}{(\deg \xi_j)}\log(\xi_j)$$

where, for standard monomials ζ *and* ξ, $\zeta \sim \xi$ *denotes that* $\zeta^n = \xi^h$ *for some positive integers* n *and* h.

Proof. Trivial.

Lemma 3.1. *$\varphi(Q)$ is a convex polytope. Moreover $\tilde{Q}$ is a strongly convex cone in $\mathbb{R}^r$.*

Proof. For the density of $\varphi(Q)\cap\mathbb{Q}^r$ in $\varphi(Q)$, we have only to show that $\lambda p+(1-\lambda)q$ belongs to $\varphi(Q)\cap\mathbb{Q}^r$ for p, q $\in \varphi(Q)\cap\mathbb{Q}^r$ and $\lambda\in[0,1]\cap\mathbb{Q}$. It is clear from (3.0) and (H_2'). Since

$\tilde{Q} = \mathbb{R}_{\geq 0}\varphi(Q) := \{ \lambda p : \lambda\in\mathbb{R}_{\geq 0},\ p\in\varphi(Q) \}$, $\tilde{Q}$ is strongly convex.

Lemma 3.2. *For any simplices $\tau_1, \tau_2 \in K(\log(\Pi))$ such that $\tau_1\cap \tau_2 \neq \phi$, $\tau_1\cap\tau_2$ is a face of τ_1 and τ_2. Therefore $K(\log(\Pi))$ is a simplicial complex with $|K(\log(\Pi))|=Q$ and $(\mathbb{R}^r, \Sigma(K(\log(\Pi))))$ is a finite rational partial polyhedral decomposition.*

Proof. Set Ver $\tau_1 = \{\log(\xi_1),\cdots,\log(\xi_t)\}$,

Ver $\tau_2 = \{\log(\mu_1),\cdots,\log(\mu_s)\}$, and

Ver $\tau_1 \cap$ Ver $\tau_2 = \{\log(\xi_{i_1}),\cdots,\log(\xi_{i_h})\} = \{\log(\mu_{j_1}),\cdots,\log(\mu_{j_h})\}$,

where Ver$(\cdot)$ denotes the set of its vertices. Then we claim that $|\log(\xi_{i_1}),\cdots,\log(\xi_{i_h})|$ coincides with $\tau_1\cap\tau_2$. If we have $\tau_1\cap\tau_2 \supsetneqq |\log(\xi_{i_1}),\cdots,\log(\xi_{i_h})|$ there exists a rational point

$p \in (\tau_1 \cap \tau_2 \setminus |\log(\xi_{i_1}),\cdots,\log(\xi_{i_h})|) \cap \mathbb{Q}^r$, because of the rationality of Ver τ_1 and Ver τ_2. Then p has two representations

$p = \sum_{i\in I_p} \lambda_i \log(\xi_i)$, $\quad I_p \subseteqq \{1,\cdots,t\}$, $\lambda_i \in \mathbb{Q}_{>0}$ for all $i\in I_p$,

$p = \sum_{j\in J_p} \chi_j \log(\mu_j)$, $\quad J_p \subseteqq \{1,\cdots,s\}$, $\chi_j \in \mathbb{Q}_{>0}$ for all $j\in J_p$,

which imply rel int$(\sum_{i\in I_p} \mathbb{R}_{\geq 0}\log(\xi_i)) \cap$ rel int$(\sum_{j\in J_p} \mathbb{R}_{\geq 0}\log(\mu_j))$ is non-empty. By (H_1'), we have $\{\log(\xi_i)\}_{i\in I_p} = \{\log(\mu_j)\}_{j\in J_p}$, which contradicts the choice of p.

Proposition 3.3. *For a simplex σ_i whose vertices consist of a maximal chain in* $\log(\Pi)$, *we have* $\dim \sigma_i = r-1$. *Hence a graded toric ASL poset is pure.*

Proof. If $\dim \sigma_i$ is smaller than $r-1$ then there exists an element $\log(\xi)$ in $\log(\Pi)$ not belonging to the vector space spanned by σ_i, because at least one $(r-1)$-dimensional simplex exists by (1.3). Fix a rational point p of rel int$(\varphi(\sigma_i))$. Since rational points densely exist on the segment $\overline{p\varphi(\log(\xi))}$, we have a sequence of rational points converging to p on the segment. Let $\varphi(\sigma_j)$ be a simplex containing infinitely many points of the sequence. It contains p by the closedness. By (H_1'), $\varphi(\sigma_i)$ must be a proper face of $\varphi(\sigma_j)$, which contradicts the maximality of the chain ver σ_i.

Lemma 3.4. *Let σ_i be a simplex of maximal dimension* $|\log(\alpha)=\log(\xi_1),\cdots,\log(\xi_r)|$ *in* $\log(\Pi)$ *and let* L *be the vector subspace of* $\mathbb{R}^r$ *spanned by* $\tau=|\log(\xi_2),\cdots,\log(\xi_r)|$. *Then* $L \cap \tilde{Q}$ *is an $(r-1)$-dimensional face of* $\tilde{Q}$.

Proof. We claim that L is a supporting hyperplane of the convex cone

$\tilde{Q}$. In fact if there exists an element of $\log(\Pi)$ on the opposite side of $\log(\alpha)$ with respect to L, using the same argument as in (3.3), we have a simplex σ_j (of maximal dimension) adjacent to σ_i with τ as a common face. This contradicts uniqueness of the element of height zero in (1.1). Hence L is a supporting hyperplane carried by $L \cap \tilde{Q}$.

Lemma 3.5. *Let* F^j *be a j-dimensional face of* $\tilde{Q}$. *Then the subset* $F^j \cap \log(\Pi)$ *of* $\log(\Pi)$ *with the induced order from* $\log(\Pi)$ *satisfies the conditions* (H_1') *and* (H_2'). *Hence* $F^j \cap \log(\Pi)$ *is a graded toric ASL poset of rank j-1.*

Proof. By induction on the dimension of F^j, we have only to prove it in the case of dimension r-1. Clearly $F^{r-1} \cap \log(\Pi)$ satisfies (H_1'). Let L be the supporting hyperplane carried by F^{r-1}. Since all elements of $\log(\Pi)$ exist only on the same side of L or on L, for any incomparable pair $\log(\nu)$ and $\log(\mu)$ on L, $\log(\nu)+\log(\mu)$ is "straightened" by some elements of $\log(\Pi)$ contained in L. Hence $F^{r-1} \cap \log(\Pi)$ satisfies (H_2').

§ 4. Normality of Graded Toric ASL Domains

In this section we shall prove the normality of a graded toric ASL domain (S,Π) over a field k (or a normal ring k). As we have already assumed in §2, $\log(\Pi)$ generates the free Abelian group $\mathbb{Z}^r = \log(\mathcal{M}(r))$ as a group.

Definition. Let $M \subset \mathbb{Z}^r$ be a finitely generated semigroup generating $\mathbb{Z}^r$ as a group. We say that M is *saturated* (or *normal*) if the condition $jm \in M$, where j is an integer ≥ 1 and m an element of $\mathbb{Z}^r$, implies $m \in M$.

Theorem 4.1. *Let* (S,Π) *be a graded toric ASL domain over a field* k *(or a normal ring* k*). Then the semigroup* M *generated by* $\log(\Pi)$ *in* $\mathbb{Z}^r = \log(\mathcal{M}(r))$ *is saturated and* S *is normal.*

Proof. First we prove the saturation of M. Let $m \in \mathbb{Z}^r$ and $jm \in M$ for some positive integer j. Since $jm \in M \subset \tilde{Q}$, m belongs to an (r-1)-dimensional cone $\tilde{\sigma}_t = |\log(\mu_1),\cdots,\log(\mu_r)|^{\sim} \in \Sigma(K(\log(\Pi)))$. On the other hand, m also belongs to $\mathbb{Z}^r = M + (-M)$. So we have

$$m = \sum_{\beta\in\Pi} a_\beta \log(\beta) - \sum_{\beta\in\Pi} b_\beta \log(\beta)$$

where $a_\beta \in \mathbb{Z}_{\geq 0}$ and $b_\beta \in \mathbb{Z}_{\geq 0}$. Since $\sum_{i=1}^r \log(\mu_i) \in \text{rel int}(\tilde{\sigma}_t)$, $\sum_{\beta\in\Pi} a_\beta \log(\beta) + n\cdot\sum_{i=1}^r \log(\mu_i)$ and $\sum_{\beta\in\Pi} b_\beta \log(\beta) + n\cdot\sum_{i=1}^r \log(\mu_i)$ are contained in $\tilde{\sigma}_t$ for a sufficiently large positive integer n. By (3.0) and (3.1) we have

$$\begin{aligned} m &= \{ \sum_{\beta\in\Pi} a_\beta \log(\beta) + n\cdot\sum_{i=1}^r \log(\mu_i)\} - \{ \sum_{\beta\in\Pi} b_\beta \log(\beta) + n\cdot\sum_{i=1}^r \log(\mu_i)\} \\ &= \sum_{i=1}^r h_i \log(\mu_i) - \sum_{i=1}^r m_i \log(\mu_i) \end{aligned}$$

for some positive integers h_i and m_i. Since $jm \in M \cap \tilde{\sigma}_t$ and $\tilde{\sigma}_t$ is simplicial, $j(h_i - m_i)$ $(i=1,\cdots,r)$ must be non-negative integers, hence $h_i - m_i \geq 0$ for $i=1,\cdots,r$. Therefore we have

$$m = \sum_{i=1}^r (h_i - m_i)\log(\mu_i) \in M \cap \tilde{\sigma}_t \subset M.$$

By the well-known fact on semigroup rings (c.f.[Ho], or [Od 1,2]), the semigroup M is saturated if and only if the semigroup ring k[Π] is normal. Therefore S is normal.

Remark 4.2. To prove the saturation of M in (4.1) we needs (3.0), (3.1), and (H_1') only, which correspond in our toric case to the condition (ASL-1) in another definition of ASL appeared in [Ei]. For another proof of (4.1) in this direction, see [Mi] for details.

Remark 4.3. In the proof of (4.1) we have also shown that any maximal chain $\log(\mu_1)<\cdots<\log(\mu_r)$ in $\log(\Pi)$ is a basis of $\mathbb{Z}^r = M + (-M)$.

Corollary 4.4. *A graded toric ASL over a field* k *(or a Cohen-Macaulay ring) is Cohen-Macaulay.*

Proof. This follows from theorem 1 in [Ho].

§ 5. Cohen-Macaulay and Gorenstein Properties on Graded Toric ASL Posets

In this section we shall show that a graded toric ASL poset is Cohen-Macaulay. And we shall give a sufficient condition for a graded toric ASL poset to be Gorenstein. To do them we use the theory of Stanley-Reisner rings.

Definition. Let k be a field. Let Δ be a finite simplicial complex and $\mathrm{Ver}(\Delta)=\{x_1,\cdots,x_n\}$ be the set of vertices of Δ. Set $k[\Delta]:= k[x_1,\cdots,x_n] / I_\Delta$, where x_i $(i=1,\cdots,n)$ are considered as indeterminantes over k and

$$I_\Delta = \left(x_{i_1}\cdots x_{i_s} : i_1<\cdots<i_s,\ |x_{i_1},\cdots,x_{i_s}| \text{ not belonging to } \Delta \right).$$

We call this ring a *Stanley-Reisner ring* of complex Δ.

If (S,Π) be a graded toric ASL domain over a field k, by (1.4) and (3.2), S_o is a Stanley-Reisner ring of complex $K(\log(\Pi))\simeq K(\varphi(\log(\Pi)))$ which realizes geometrically an

$(r-1)$-dimensional combinatorial manifold $Q \simeq \varphi(Q)$. Concerning Q and a unique element $\log(\alpha)$ of height zero, we have two cases by (3.2) and (3.4):

Case I: $\log(\alpha) \in \mathrm{int}(Q)$ and $\log(\Pi-\{\alpha\}) \subset \partial Q$.

Case II: $\log(\alpha) \in \partial Q$, i.e. $\log(\Pi) \subset \partial Q$,

where int and ∂ denote the interior and the boundary of a topological space. Since the star of $|\log(\alpha)|$ in $K(\log(\Pi))$ is just $K(\log(\Pi))$, the link of $|\log(\alpha)|$ in $K(\log(\Pi))$ coincides with the simplicial subcomplex Δ of $K(\log(\Pi))$ consisting of the simplices without $\log(\alpha)$ as their vertices. By the definition of combinatorial manifold we have:

Lemma 5.1. (a) *In the case I the geometric realization* $|\Delta| \subset \mathbb{R}^r$ *is homeomorphic to an* $(r-2)$*-dimensional sphere* S^{r-2}.

(b) *In the case II the geometric realization* $|\Delta| \subset \mathbb{R}^r$ *is homeomorphic to a closed* $(r-2)$*-dimensional ball* B^{r-2}

Theorem 5.2. *Let* (S, Π) *be a graded toric ASL domain over a field* k.

(a) *In the case I the poset* Π *is Gorenstein (i.e.* S_o *is Gorenstein). Moreover* S *is Gorenstein.*

(b) *In the case II the poset* Π *is Cohen-Macaulay (i.e.* S_o *is Cohen-Macaulay). Moreover* S *is Cohen-Macaulay.*

Hence a graded toric ASL poset is Cohen-Macaulay.

Proof.(a) Since a unique minimal element α is a non-zero-divisor, S_o is Gorenstein if and only if $S_o/\alpha S_o$ is Gorenstein. Since the discrete ASL $S_o/\alpha S_o \simeq (S/\alpha S)_o$ is isomorphic to the Stanley-Reisner ring $k[\Delta]$, we have reduced to showing that $k[\Delta]$ is Gorenstein. It follows from (a) of (5.1) and corollary 5.2 in [St1] that $k[\Delta]$ is Gorenstein.

This implies, using corollary 7.2 in [DEP], that S itself is Gorenstein.

(b) By the same argument as in (a), S_o is Cohen-Macaulay if and only if $k[\Delta]$ is Cohen-Macaulay. It follows from (b) of (5.1) and corollary 3.2 and theorem 3.1 in [Mu] that $k[\Delta]$ is Cohen-Macaulay. Therefore S_o is Cohen-Macaulay. This implies that S itself is Cohen-Macaulay by corollary 7.2 in [DEP].

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