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Sharp estimates for some integral operators of convex functions of order alpha

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Introduction

For $0 \leq \alpha < 1$, denote by $C(\alpha)$ the class of normalised univalent convex functions f of order α, defined in the open unit disc $D = \{z : |z| < 1\}$. Thus $f \in C(\alpha)$, if and only if, $f(0) = 0, f'(0) = 1$ and

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$$

for $z \in D$. The class $C(\alpha)$ has been extensively studied. In [**1**] Bernardi gave a series of non-sharp lower bounds for the real part of certain weighted integral operators of $f \in C(0)$. The object of this paper is to give sharp versions of some of Bernardi's results for $f \in C(\alpha)$. We also extend a classical result of Strohhäcker [**3**] to obtain sharp estimates for the real part of some iterated integral operators in $C(\alpha)$. Our methods are quite elementary.

Results

Theorem 1. *Let $f \in C(\alpha)$ and $z = re^{i\theta} \in D$. For $n \geq 2$, set $n!A_n(\alpha) = \prod_{k=1}^{\infty} (k - 2\alpha)$ and $A_1(\alpha) = 1$. Then*

(i) For a real and $a \neq -1, -2, ...,$

$$\operatorname{Re}\left(z^{-(1+a)} \int_0^z t^{a-1} f(t)dt\right) \geq \sum_{n=1}^{\infty} \frac{(-r)^{n-1} A_n(\alpha)}{(n+a)},$$

(ii) For $c_1, c_2 \neq -1, -2, ..$ and $c_2 > c_1$,

$$\mathrm{Re}\left(z^{-2}\int_0^z f(t)\left[(t/z)^{c_1-1}-(t/z)^{c_2-1}\right]dt\right)\geq$$
$$(c_2-c_1)\sum_{n=1}^{\infty}\frac{(-r)^{n-1}A_n(\alpha)}{(n+c_1)(n+c_2)},$$

(iii) For a, c real and $a \neq 0, -1, -2, .., c \neq -1, -2, ..,$

$$\mathrm{Re}\left(z^{-(1+c)}\int_0^z f(t)t^{c-1}\left(\log(z/t)\right)^{a-1}dt\right)\geq\Gamma(a)\sum_{n=1}^{\infty}\frac{(-r)^{n-1}A_n(\alpha)}{(n+c)^a},$$

where Γ is the Gamma function.

(iv) For c real and $c \neq 0, -1, -2, ...,$

$$\mathrm{Re}\left(z^{-(1+c)}\int_0^z f(t)(z-t)^{c-1}dt\right)\geq\sum_{n=1}^{\infty}(-r)^{n-1}B(c,n+1)A_n(\alpha),$$

where B is the Beta function.
In all cases, equality occurs for the function $f_0 \in C(\alpha)$, where

$$f_0(z)=\sum_{n=1}^{\infty}(-1)^{n-1}A_n(\alpha)z^n$$
$$=\begin{cases}\dfrac{1-(1+z)^{2\alpha-1}}{(1-2\alpha)}, & \text{for } \alpha\neq 1/2\\ \log(1+z), & \text{for } \alpha=1/2.\end{cases}$$

THEOREM 2. *Let $f \in C(\alpha)$ and $z = re^{i\theta} \in D$. For $n = 1, 2, ...,$ define*

$$I_n(z)=\frac{1}{z}\int_0^z I_{n-1}(t)dt,$$

where $I_0(z) = f(z)/z$. Then for $n \geq 0$,

$$\mathrm{Re}\, I_n(z)\geq\gamma_n(r),$$

where

$$\frac{1}{2} \le \gamma_n(r) = \sum_{k=1}^{\infty} \frac{(-r)^{k-1} A_k(\alpha)}{k^n} < 1.$$

The result is sharp for f_0 as given in Theorem 1.

We note that when $n = 0$, we obtain the following result of Brickman et al. [2] which we shall use in the proofs of Theorem 1 and 2.

LEMMA. *Let $f \in C(\alpha)$ and $z = re^{i\theta}$. Then for $0 \le \alpha < 1$,*

$$\operatorname{Re}\left(\frac{f(z)}{z}\right) \ge \begin{cases} \dfrac{1-(1+r)^{2\alpha-1}}{(1-2\alpha)r}, & \text{for } \alpha \neq 1/2 \\ \dfrac{\log(1+r)}{r}, & \text{for } \alpha = 1/2. \end{cases}$$

The results are sharp for the function f_0 given above.

PROOF OF THEOREM 1:

In each case, we will give the proof when $\alpha \neq \frac{1}{2}$. When $\alpha = \frac{1}{2}$, the proofs are similar. Write $t = \rho e^{i\theta}$, then applying the Lemma in each of the following, we have

(i)

$$\begin{aligned} \operatorname{Re}\left(\frac{1}{z^{1+a}} \int_0^z f(t) t^{a-1} dt\right) &= r^{-(1+a)} \int_0^r \rho^a \operatorname{Re}\left(f(\rho e^{i\theta})/\rho e^{i\theta}\right) d\rho \\ &\ge \frac{r^{-(1+a)}}{(1-2\alpha)} \int_0^r \rho^{a-1}\left(1-(1+\rho)^{2\alpha-1}\right) d\rho \\ &= r^{-(1+a)} \sum_{n=1}^{\infty} (-1)^{n-1} A_n(\alpha) \int_0^r \rho^{n+a-1} d\rho, \end{aligned}$$

The result now follows at once.

(ii)

$$\begin{aligned}
&\operatorname{Re}\left(\frac{1}{z^2}\int_0^z f(t)\left[(t/z)^{c_1-1}-(t/z)^{c_2-1}\right]dt\right)\\
&\quad=\frac{1}{r^2}\int_0^r \rho\left[(\rho/r)^{c_1-1}-(\rho/r)^{c_2-1}\right]\operatorname{Re}\left(f(\rho e^{i\theta})/\rho e^{i\theta}\right)d\rho\\
&\quad\geq\frac{1}{r^2}\int_0^r \left[(\rho/r)^{c_1-1}-(\rho/r)^{c_2-1}\right]\sum_{n=1}^{\infty}(-1)^{n-1}A_n(\alpha)\rho^n d\rho\\
&\quad=\sum_{n=1}^{\infty}(-r)^{n-1}A_n(\alpha)\int_0^1 x^n\left(x^{c_1-1}-x^{c_2-1}\right)dx\\
&\quad=(c_2-c_1)\sum_{n=1}^{\infty}\frac{(-r)^{n-1}A_n(\alpha)}{(n+c_1)(n+c_2)},
\end{aligned}$$

for $c_2 > c_1$ and $c_1, c_2 \neq -1, -2, \ldots.$

(iii)

$$\begin{aligned}
&\operatorname{Re}\left(\frac{1}{z^{1+c}}\int_0^z f(t)t^{c-1}\left(\log(z/t)\right)^{a-1}dt\right)\\
&\quad=\frac{1}{r^{1+c}}\int_0^r \rho^c\left(\log(r/\rho)\right)^{a-1}\operatorname{Re}\left(f(\rho e^{i\theta})/\rho e^{i\theta}\right)d\rho\\
&\quad\geq\frac{1}{r^2}\int_0^r (\rho/r)^{c-1}\left(\log(r/\rho)\right)^{a-1}\sum_{n=1}^{\infty}(-1)^{n-1}A_n(\alpha)\rho^n d\rho\\
&\quad=\sum_{n=1}^{\infty}(-r)^{n-1}A_n(\alpha)\int_0^1 x^{n+c-1}\left(\log(1/x)\right)^{a-1}dx\\
&\quad=\Gamma(a)\sum_{n=1}^{\infty}\frac{(-r)^{n-1}A_n(\alpha)}{(n+c)^a},
\end{aligned}$$

for $a \neq 0, -1, -2, .., c \neq -1, -2, \ldots$

(iv)

$$\begin{aligned}
\operatorname{Re}\left(\frac{1}{z^{1+c}}\int_0^z f(t)(z-t)^{c-1}dt\right)
&= \frac{1}{r^{1+c}}\int_0^r \rho(r-\rho)^{c-1}\operatorname{Re}\left(f(\rho e^{i\theta})/\rho e^{i\theta}\right)d\rho\\
&\geq \frac{1}{r^2}\int_0^r \left(1-\frac{\rho}{r}\right)^{c-1}\sum_{n=1}^{\infty}(-1)^{n-1}A_n(\alpha)\rho^n d\rho\\
&= \sum_{n=1}^{\infty}(-r)^{n-1}A_n(\alpha)\int_0^1 (1-x)^{c-1}x^n dx\\
&= \sum_{n=1}^{\infty}(-r)^{n-1}B(c,n+1)A_n(\alpha), \qquad \text{for } c \neq 0,-1,-2,..
\end{aligned}$$

This completes the proof of Theorem 1.

PROOF OF THEOREM 2:
It follows easily from the Lemma that for $0 \leq \alpha < 1$,

$$\operatorname{Re} I_0(z) \geq \sum_{k=1}^{\infty}(-r)^{k-1}A_k(\alpha) = \gamma_0(r).$$

Next, writing $t = \rho e^{i\theta}$ we have,

$$\begin{aligned}
\operatorname{Re} I_n(z) &= \operatorname{Re}\frac{1}{z}\int_0^z I_{n-1}(t)dt\\
&\geq \frac{1}{r}\int_0^r \sum_{k=1}^{\infty}\frac{(-\rho)^{k-1}A_k(\alpha)}{k^{n-1}}d\rho\\
&= \sum_{k=1}^{\infty}\frac{(-r)^{k-1}A_k(\alpha)}{k^n} = \gamma_n(r),
\end{aligned}$$

where we have used induction. For $n \geq 0$ and $0 \leq \alpha < 1$, $\gamma_n(r)$ is absolutely convergent for $0 \leq r < 1$ and hence rearranging the terms appropriately shows that $\frac{1}{2} < \gamma_n(r) < 1$.

REFERENCES

1. S.D.Bernardi, *Convex and Starlike Functions*, Trans.Amer.Math.Soc. **135** (1969), 429-446.
2. L.Brickman,D.J.Hallenbeck,T.H.Macgregor and D.R.Wilken, *Convex hulls and Extreme Points of Families of Starlike and Convex Mappings*, Trans.Amer.Math.Soc. **185** (1973), 413-428.
3. E.Strohhäcker, *Beiträge zur Theorie der schlichten Funktionen*, Math. Z., **37** (1933), 356-380.

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