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Almost coinciding families and gaps in $\mathcal{P}(\omega)$

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1. Introduction. In [DSV], Dow, Simon and Vaughan introduced the notion of almost coinciding families and showed the following Proposition 1～3.

Proposition 1 [DSV, Theorem 2.4]. If $\mathfrak{d} = \omega_1$, then there exists a nontrivial almost coinciding family indexed by ${}^{\omega}\omega$.

Proposition 2 [DSV, Theorem 3.1]. The proper forcing axiom (PFA) implies that every almost coinciding family indexed by ${}^{\omega}\omega$ is trivial.

Proposition 3 [DSV, Theorem 4.1, Lemma 4.2, 4.3]. If there exists a nontrivial almost coinciding family indexed by ${}^{\omega}\omega$, then there exists an unfilled $(\mathfrak{b},\mathfrak{b})$-gap in $\mathcal{P}(\omega)$. So, in Kunen's model of "ZFC + Martin's Axiom + $2^{\omega} = \omega_2$ + $\nexists$ unfilled $(\mathfrak{c},\mathfrak{c})$-gap", there doesn't exist a nontrivial almost coinciding family indexed by ${}^{\omega}\omega$.

In this paper, we shall show

Theorem 1. Let P be the poset $\{ p ; \exists x \subset \omega_2 (|x| < \omega \;\&\; p : x \to 2) \}$ adjoining ω_2 Cohen generic reals. Then, in V^P, there doesn't exist a nontrivial almost coinciding family indexed by ${}^{\omega}\omega$.

Theorem 2. Let $\kappa = \kappa^{<\kappa}$ and $\omega_1 < \kappa$. Then, there is a poset P with the ω_1-chain condition such that, in V^P, $2^{\omega} = \kappa$ + Martin's Axiom + $\exists$ unfilled (κ, κ)-gap + there doesn't exist a nontrivial almost coinciding family indexed by ${}^{\omega}\omega$.

Since, in Theorem 1, $\Vdash_P$ " $\mathbf{b} = \omega_1$ + $\mathbf{d} \geqq \omega_2$ ", the assumption $\mathbf{d} = \omega_1$ in Proposition 1 can't be replaced by $\mathbf{b} = \omega_1$.

Question. Is " ZFC + $\mathbf{d} > \omega_1$ + there is a nontrivial almost coinciding family indexed by ${}^{\omega}\omega$ " consistent?

2. Definitions and the proof of Theorem 1. Let ω be the set of natural numbers and ${}^{\omega}\omega$ the set of all functions on ω.
$\forall^{\infty} x$ ($\cdots$ x $\cdots$) means that { x ; not $\cdots$ x $\cdots$ } is finite.
Define the pseudo-ordering $\prec$ on ${}^{\omega}\omega$ by

$f \prec g$ iff $\forall^{\infty} n < \omega$ ($f(n) < g(n)$).

Let F be a subset of ${}^{\omega}\omega$. F is said to be bounded, if there exists a $g \in {}^{\omega}\omega$ such that $\forall f \in F$ ($f \prec g$). F is called a dominating family if, for any $g \in {}^{\omega}\omega$, there exists $f \in F$ such that $g \prec f$.
The cardinals $\mathbf{b}$ and $\mathbf{d}$ are defined by

$\mathbf{b} = \min \{ |F| ; F \text{ is not bounded} \}$,
$\mathbf{d} = \min \{ |F| ; F \text{ is a dominating family} \}$.

For $f \in {}^{\omega}\omega$, L_f denotes the set $\{ (n,m) \in \omega \times \omega ; m \leqq f(n) \}$.
Define the quasi-ordering $\subset^*$ and the equivalence relation $\sim$ by

$X \subset^* Y$ iff $X \setminus Y$ is finite,
$X \sim Y$ iff $X \triangle Y$ is finite.

Let $\mathcal{A}$, $\mathcal{B}$ be subsets of $\mathcal{P}(\omega)$. $\mathcal{A} \perp \mathcal{B}$ means that $A \cap B \sim \phi$, for any $A \in \mathcal{A}$, $B \in \mathcal{B}$. $\mathcal{A} << \mathcal{B}$ means that $A \subset^* B$, for any $A \in \mathcal{A}$, $B \in \mathcal{B}$. $\mathcal{A}$ and $\mathcal{B}$ can be separated, if there is an X such that $\mathcal{A} << \{X\}$ and $\mathcal{B} \perp \{X\}$. A κ-sequence $\langle X_\alpha \mid \alpha < \kappa \rangle$ of subsets of ω is called a κ-tower, if $X_\alpha \subset^* X_\beta$, for any $\alpha < \beta < \kappa$. A κ-sequence $\langle (X_\alpha, Y_\alpha) \mid \alpha < \kappa \rangle$ is called a (κ, κ)-gap, if $\langle X_\alpha \mid \alpha < \kappa \rangle$ and

$\langle Y_\alpha \mid \alpha<\kappa \rangle$ are towers and $\{ X_\alpha ; \alpha<\kappa \} \perp \{ Y_\alpha ; \alpha<\kappa \}$.
A (κ, κ)-gap $\langle (X_\alpha, Y_\alpha) \mid \alpha<\kappa \rangle$ is unfilled, if $\{ X_\alpha ; \alpha<\kappa \}$ and $\{ Y_\alpha ; \alpha<\kappa \}$ can't be separated. Finally, an indexed set $\langle \phi_f \mid f \in F \rangle$ is called an almost coinciding family indexed by F, if

(i) for any $f \in F$, $\phi_f : L_f \to \omega$,

(ii) for any $f, g \in F$ ($\phi_f \restriction (L_f \cap L_g) \sim \phi_g \restriction (L_f \cap L_g)$).

An almost coinciding family $\langle \phi_f \mid f \in F \rangle$ is trivial, if there exists a $\sigma : \omega \times \omega \to \omega$ such that $\{ \phi_f ; f \in F \} << \sigma$.

To prove Theorem 1, We need the following lemma which is a little modification of Lemma 4.3 in [DSV] and is easily verified by using Fact 2.2. which appears below.

Lemma 2.1. Let $F \subset S \subset {}^\omega\omega$. Suppose that $\langle \phi_f \mid f \in S \rangle$ is a nontrivial almost coinciding family indexed by S and that F is an unbounded subset of ${}^\omega\omega$ which consists strictly increasing functions. Then, $\langle \phi_f \mid f \in F \rangle$ is nontrivial.

Fact 2.2.(well-known/clear) Suppose that F ia an unbounded subset of ${}^\omega\omega$ which consists strictly increasing functions. Then, it holds that, for any infinite subset A of ω,

$$\forall f \in {}^\omega\omega \ \exists g \in F \ \exists^\infty n \in A \ (f(n) < g(n)).$$

Let Q be the poset $\{ q : \exists n <\omega (q : n \to \omega) \}$ and P the poset $\{ p : \exists x \subset \omega_2 (|x| < \omega \ \& \ p : x \to 2) \}$.

Lemma 2.3. Suppose that S is an unbounded subset of ${}^\omega\omega$ which consists strictly increasing functions and $\langle \phi_f \mid f \in S \rangle$ is a nontrivial almost coinciding family indexed by S. Let g be the canonical generic real on Q. Then, in $V^{Q \times P}$, $\langle \phi_f \mid f \in S \rangle$ can not be extended to an almost coinciding family indexed by $S \cup \{g\}$.

Proof. To get a contradiction, suppose that

(1) $(q,p) \in Q\times P$ & ϕ : $Q\times P$-name,

(2) $\Vdash_{Q\times P}$ $\phi : L_g \to \omega$,

(3) $(q,p) \Vdash_{Q\times P} \forall f \in S\ \forall^{\infty} x \in L_f \cap L_g\ (\ \phi(x) = \phi_f(x)\ )$.

Because $Q\times P$ satisfies the ω_1-chain condition, there exists an $A \subset \omega_2$ such that

$|A| \leqq \omega$ & $p \in P\restriction A$ & ϕ is a $Q\times P\restriction A$-name.

By using (3), for each $f \in S$, take an $n_f < \omega$ and (q_f,p_f) in $Q\times P\restriction A$ such that

(4) $\mathrm{dom}(q_f) \subset n_f$ & $(q_f,p_f) \leqq (q,p)$,

(5) $(q_f,p_f) \Vdash_{Q\times P} \forall x \in L_f \cap L_g \setminus (n_f\times\omega)\ (\ \phi(x) = \phi_f(x)\ )$.

Since $|Q\times P\restriction A| \leqq \omega$ and S is unbounded in ${}^{\omega}\omega$, there exist an $n' < \omega$, $(q',p') \in Q\times P\restriction A$ and a subset F of S such that

(6) F is unbounded in ${}^{\omega}\omega$,

(7) $\forall f \in F\ (\ n_f = n'$ & $q_f = q'$ & $p_f = p'\ )$.

By (6) and Lemma 2.1,

(8) $\langle \phi_f \mid f \in F \rangle$ is nontrivial.

Claim 1. $\forall x \in L_f \cap L_h \setminus (n'\times\omega)\ (\ \phi_f(x) = \phi_h(x)\ )$, for any $f, h \in F$.

Proof of Claim 1. Let $f, h \in F$ and $x = (m,k) \in L_f \cap L_h$ and $n' \leqq m$. Take $q'' \in Q$ such that

$q'' \leqq q'$ & $m \in \mathrm{dom}(q'')$ & $q''(m) > k$.

Then, since $(q'',p') \Vdash$ " $x \in L_g \cap L_f \setminus (n'\times\omega)$ ", it holds that

$(q'',p') \Vdash \phi(x) = \phi_f(x)$.

Similary, $(q'',p') \Vdash \phi(x) = \phi_h(x)$. Hence, $\phi_f(x) = \phi_h(x)$. QED.

By Claim 1, it holds that

$\pi = \bigcup\{\ \phi_f\restriction(L_f\setminus(n'\times\omega))\ ;\ f \in F\ \}$ is a function.

So, $\langle \phi_f \mid f \in F \rangle$ is trivial. This contradicts (8). □

Proof of Theorem 1. To get a contradiction, suppose that

$\Vdash_P$ " $\langle \phi_f \mid f \in {}^{\omega}\omega \rangle$ is a nontrivial almost coinciding family indexed by ${}^{\omega}\omega$ ".

Since $\Vdash_P$" $b = \omega_1$ ", we can take an $A \subset \omega_2$ and a $P \restriction A$-name S such that $|A| \leqq \omega_1$ and $\Vdash_P$" S is an unbounded subset of ${}^\omega\omega$ consisting increasing functions & $|S| = \omega_1$ ". Since P satisfies the ω_1-chain condition, there exists a $B \subset \omega_2$ such that

$$A \subset B \ \& \ |B| \leqq \omega_1 \ \& \ \langle \phi_f \mid f \in S \rangle \text{ is a } P \restriction B\text{-name}.$$

Since $\Vdash_P$ " S is unbounded and consists of increasing functions ", by Lemma 2.1,

$$\Vdash_P \text{"} \langle \phi_f \mid f \in S \rangle \text{ is nontrivial "}.$$

From this and the fact that the formula "x is nontrivial" is Π_1, it holds that

$$\Vdash_{P \restriction B} \text{"} \langle \phi_f \mid f \in S \rangle \text{ is nontrivial "}.$$

Since $P \restriction (\omega_2 \setminus B)$ is isomorphic to P, by replacing a ground model V to $V^{P \restriction B}$, we can assume that S and $\langle \phi_f \mid f \in S \rangle$ are sets in V.

Since ro(P) is isomorphic to ro(Q×P), by Lemma 2.3, in V^P, $\langle \phi_f \mid f \in S \rangle$ can't be extended to an almost coinciding family indexed by ${}^\omega\omega$. But this contradicts the fact that, in V^P, $\langle \phi_f \mid f \in {}^\omega\omega \rangle$ is an almost coinciding family. □

3. The proof of Theorem 2.

Lemma 3.1. The following (a), (b) and (b') are equivalent.

(a) There exists a nontrivial almost coinciding family indexed by ${}^\omega\omega$.

(b) There exist a dominating family $F \subset {}^\omega\omega$ and an indexed set $\langle (A_f, B_f) \mid f \in F \rangle$ such that

(b.1) $\forall f \in F$ ((A_f, B_f) is a partition of L_f),

(b.2) $\{ A_f \ ; \ f \in F \}$ and $\{ B_f \ ; \ f \in F \}$ can't be separated,

(b.3) $\forall f, g \in F$ (if $f \prec g$ then $A_f \subset^* A_g$ & $B_f \subset^* B_g$)

(b') For any dominating family $F \subset {}^{\omega}\omega$, there exists an indexed set $\langle (A_f, B_f) \mid f \in F \rangle$ which satisfy (b.1)~(b.3).

Proof. It is easy to see that (b) and (b') are equivalent to the following (c) and (c'), respectively.

(c) There exists a dominating family $S \subset {}^{\omega}\omega$ and a nontrivial almost coinciding family $\langle \phi_f \mid f \in S \rangle$ such that, for every $f \in S$, $\phi_f : L_f \to 2$.

(c') For any dominating family $S \subset {}^{\omega}\omega$, there exists a nontrivial almost coinciding family $\langle \phi_f \mid f \in S \rangle$ such that, for every $f \in S$, $\phi_f : L_f \to 2$. "

Also, it is easy to see that (c) and (c') are equivalent.

So, it suffices to show that (c) and (a) are equivalent. The implication from (c) to (a) is clear. To show from (a) to (c), let $\langle \phi_f \mid f \in {}^{\omega}\omega \rangle$ be a nontrivial almost coinciding family indexed by ${}^{\omega}\omega$. For each finite sequence $s = \langle a_i \mid i < n \rangle : n \to \omega$, s^* denotes the finite sequence

$$<0, \underbrace{1, \cdots, 1}_{a_1}, 0, 1, \cdots, 0, \underbrace{1, \cdots, 1}_{a_{n-1}}, 0>.$$

For each $g : \omega \to \omega$, $\phi : L_g \to \omega$ and $n < \omega$, let $s_{\phi, n}$ denotes $\langle \phi(n, i) \mid i < g(n) \rangle$. For each $f : \omega \to \omega$, define $f\hat{} : \omega \to \omega$ and $\Psi_{f\hat{}} : L_{f\hat{}} \to 2$ by

$f\hat{}(n)$ = the length of $(s_{\phi_f, n})^*$,

$\Psi_{f\hat{}}$ = the unique $\Phi : L_{f\hat{}} \to 2$ such that, for any $n < \omega$,

$$(s_{\phi_f, n})^* = s_{\Phi, n}.$$

Then, it is easy to see that $\{ f\hat{} ; f \in {}^{\omega}\omega \}$ is a dominating subset of ${}^{\omega}\omega$ and $\langle \Psi_{f\hat{}} \mid f \in {}^{\omega}\omega \rangle$ is a nontrivial almost coinciding family indexed by $\{ f\hat{} ; f \in {}^{\omega}\omega \}$. □

The next lemma is due to Kunen(see [B, p.931 Theorem 4.2]).

Lemma 3.2. Let T is an unfilled (ω_1,ω_1)-gap, then there is a poset P with the ω_1-chain condition such that, $|P| = \omega_1$ and in V^P, T remains unfilled for any generic extension preserving ω_1.

In fact, any finite product of any such posets satisfy the ω_1-chain condition (see Appendix A). So, we get

Lemma 3.3. There is a poset Q such that
(1) Q satisfies the ω_1-chain condition,
(2) $|Q| \leqq 2^{\omega_1}$,
(3) for any unfilled (ω_1,ω_1)-gap T,

$\Vdash_Q$ " T remains unfilled for any generic extension preserving ω_1 ".

The next lemma follows from Lemma 3.3 and the standard forcing arguments.

Lemma 3.4. Let $\omega_1 < \kappa$, $\kappa^{<\kappa} = \kappa$ and $\delta < \kappa$. Suppose that $\mathcal{A} = \langle A_\alpha \mid \alpha<\kappa \rangle$, $\mathcal{B} = \langle B_\xi \mid \xi<\delta \rangle$ are sequences of subsets of $\mathcal{P}(\omega)$ such that $\mathcal{A}$ and $\mathcal{B}$ are towers and $\mathcal{A} \perp \mathcal{B}$.
Then, there exist a poset Q and Q-names f, B such that
(9) Q satisfies the ω_1-chain condition & $|Q| = \kappa$,
(10) $\Vdash$ " $2^\omega = \kappa$ + Martin's Axiom ",
(11) $\Vdash$ " $f \in {}^\omega\omega$ " & $\Vdash$ " $h \prec f$ ", for any $h \in {}^\omega\omega$,
(12) $\Vdash$ " $\mathcal{A}\perp\{B\}$ & $\mathcal{B} << \{B\}$ ",
(13) whenever $X \subset \omega$ and $\mathcal{A}\perp\{X\}$, $\Vdash$ " $B \not\subset^* X$ ",
(14) if T is an unfilled (ω_1,ω_1)-gap, then, in V^Q, T remains unfilled for any generic extension preserving ω_1 ".
(Outline of a proof) Let Q_1 be the poset as in Lemma 3.3. Since $|Q_1| \leqq \kappa$ and Q_1 satisfies the ω_1-chain condition, it holds that $\Vdash_{Q_1}$" $\kappa = \kappa^{<\kappa}$ ". So, in V^{Q_1}, take a poset Q_2 such that Q_2

satisfies the ω_1-chain condition and (10)~(13) except that $\Vdash_{Q_2}$ " Martin's Axiom ". Then, in $V^{Q_1 * Q_2}$, take a poset Q_3 such that Q_3 satisfies the ω_1-chain condition and $\Vdash_{Q_3}$ " $\kappa = \kappa^{<\kappa}$ & Martin's Axiom ". (Such a poset exists under the assumption that $\kappa = \kappa^{<\kappa} > \omega_1$. (see e.g., [B2, Remark after Lemma 3.5, p.16]))
Then, the poset $Q = Q_1 * Q_2 * Q_3$ is as required. □

To prove Theorem 2, assume that $\kappa = \kappa^{<\kappa}$ and $\omega_1 < \kappa$. By replacing the ground model to a certain generic extension, we may assume that there exists a κ-tower $\mathcal{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ in $\mathcal{P}(\omega)$.

By using Lemma 3.4, we can construct a κ-stage finite support iteration P_α, Q_α and P_α-names f_α, B_α (for $\alpha < \kappa$) such that , in V^{P_α},

(9') Q_α satisfies the ω_1-chain condition & $|Q_\alpha| = \kappa$,

(10') $\langle B_\xi \mid \xi < \alpha \rangle$ is a tower & $\mathcal{A} \perp \{ B_\xi ; \xi < \alpha \}$,

(11') for any $X \subset \omega$, if $\mathcal{A} \perp \{X\}$, then $\Vdash_{Q_\alpha}$ " $B_\alpha \not\subset^* X$ ",

(12') Q_α forces " $2^\omega = \kappa$ + Martin's Axiom ",

(13') Q_α forces " $f_\alpha \in {}^\omega\omega$ & $g \prec f_\alpha$ ", for any $g \in {}^\omega\omega$,

(14') if T is an unfilled (ω_1, ω_1)-gap, then Q_α forces that " T remains unfilled for any generic extension preserving ω_1 ".

Set $P = \text{dir lim} (P_\alpha \mid \alpha < \kappa)$. It is easy to see that P satisfies the requirement in Theorem 2 except that

$\Vdash_P$ " there doesn't exist a nontrivial almost coinciding family indexed by ${}^\omega\omega$ ".

To show this by a contradiction, assume that $p_0 \in P$ forces the existence of a nontrivial almost coinciding family indexed by ${}^\omega\omega$. Then, by Lemma 3.1, there exist P-names $\langle (X_\alpha, Y_\alpha) \mid \alpha < \kappa \rangle$ such that

(15) $\Vdash$ " (X_α, Y_α) is a partition of L_{f_α} ",

(16) $\Vdash$ " $X_\alpha \subset^* X_\beta$ & $Y_\alpha \subset^* Y_\beta$ ", if $\alpha < \beta < \kappa$,

(17) $p_0 \Vdash$ " $\{ X_\alpha ; \alpha<\kappa \}$, $\{ Y_\alpha ; \alpha<\kappa \}$ can't be separated ".

Set $S = \{ \delta<\kappa$; $\lim\delta$ & $\mathrm{cf}\,\delta = \omega_1$ & X_α, Y_α are Q_δ-names, for any $\alpha<\delta \}$. Since P satisfies the ω_1-chain condition, S is unbounded in κ and ω_1-closed. By (14'),

$$p_0 \Vdash_\delta \text{" } \langle (X_\alpha, Y_\alpha) \mid \alpha<\delta \rangle \text{ is filled ", for any } \delta \in S.$$

By this and the fact that P satisfies the ω_1-chain condition, it holds that, for any $\delta \in S$, there is a $\beta<\delta$ such that

(*) $\exists P_\beta$-name C ($p_0 \Vdash_\delta$ " $\{ X_\alpha ; \alpha<\delta \} << \{C\}$ & $\{ Y_\alpha ; \alpha<\delta \} \perp \{C\}$ ").

So, we can define the function π from S to κ by

$$\pi(\delta) = \text{the least } \beta<\delta \text{ such that (*) holds.}$$

For each $\delta \in S$, take a $P_{\pi(\delta)}$-name C_δ such that

$$p_0 \Vdash_\delta \text{" } \{ X_\alpha ; \alpha<\delta \} << \{C_\delta\} \ \& \ \{ Y_\alpha ; \alpha<\delta \} \perp \{C_\delta\} \text{ ".}$$

Since $\pi : S \to \kappa$ is regressible, there exist a stationary set $S' \subset S$ and $\beta < \kappa$ such that

$$p_0 \in P_\beta \ \& \ \pi(\delta) = \beta, \text{ for any } \delta \in S'.$$

Claim. Let $\delta, \eta \in S'$ and $\beta<\delta<\eta$. Then, it holds that

$$p_0 \Vdash_\beta \text{" } C_\delta \setminus (n \times \omega) = C_\eta \setminus (n \times \omega), \text{ for some } n < \omega \text{ ".}$$

Proof of Claim. To get a contradiction, let $\delta, \eta \in S'$ and $p_1 \leqq p_0$ such that

$$\beta<\delta<\eta \ \& \ p_1 \Vdash_\beta \forall n<\omega (C_\delta \setminus (n \times \omega) \neq C_\eta \setminus (n \times \omega)).$$

Take a P_β-name g such that

$$\Vdash \text{" } g : \omega \to \omega \text{ " } \ \& \ p_1 \Vdash \text{" } L_g \cap (C_\delta \triangle C_\eta) \text{ is infinite "}$$

Since $\Vdash_{\beta+1}$ " $g \prec f_\beta$ ", it holds that

$$p_1 \Vdash_{\beta+1} \text{" } L_{f_\beta} \cap (C_\delta \triangle C_\eta) \text{ is infinite ".}$$

But this contradicts that

$$p_0 \Vdash L_{f_\beta} \cap C_\delta \sim X_\beta \sim L_{f_\beta} \cap C_\eta . \qquad \text{QED. of Claim.}$$

Take $\delta \in S'$ such that $\beta < \delta$. By Claim, since S' is cofinal in κ, it holds that

$$p_0 \Vdash C_\delta \text{ separates } \{ X_\alpha ; \alpha<\kappa \} \text{ and } \{ Y_\alpha ; \alpha<\kappa \}.$$

But, this contradicts (17). □

Appendix A. We start some definitions. Let T = $\langle (a_\alpha, b_\alpha) \mid \alpha < \omega_1 \rangle$ be an (ω_1, ω_1)-gap. For each $\alpha < \omega_1$, set $b'_\alpha = b_\alpha \setminus a_\alpha$. Define the poset P_T by

$$P_T = \{ (s,u) ; \exists n < \omega (s : n \to 2) \ \& \ u \subset \omega_1 \ \& \ |u| < \omega \ \& \ \bigcup_{\alpha \in u} a_\alpha \cap \bigcup_{\alpha \in u} b'_\alpha \subset \mathrm{dom}(s) \},$$

$$(s,u) \leqq (t,v) \text{ iff } t \subset s \ \& \ v \subset u \ \& \ \forall k \in \mathrm{dom}(s \setminus t) \left[\left(k \in \bigcup_{\alpha \in v} a_\alpha \Rightarrow s(k) = 1 \right) \ \& \left(k \in \bigcup_{\alpha \in v} b'_\alpha \Rightarrow s(k) = 0 \right) \right].$$

For each $\alpha < \omega_1$, set $p_\alpha = (\emptyset, \{\alpha\})$. Define the poset Q_T by

$$Q_T = \{ u \subset \omega_1 ; |u| < \omega \ \& \ \{p_\alpha ; \alpha \in u\} \text{ is an antichain of } P_T \}$$

$$u \leqq v \text{ iff } v \subset u.$$

The following theorem is due to Kunen (see [B, p.931 Theorem 4.2]).

Theorem A. Let T be an (ω_1, ω_1)-gap. Set $P = P_T$ and $Q = Q_T$.

(a) If T is filled, then P satisfies the countable chain condition.

(b) If T is unfilled, then

(b.1) $q \Vdash_Q$ " P has an uncountable antichain ", for some $q \in Q$,

(b.2) Q satisfies the countable chain condition.

We shall show

Theorem B. Let $n < \omega$ and T_i be an unfilled (ω_1, ω_1)-gap, for each $i < n$. Then, the product of $\langle Q_{T_i} \mid i<n \rangle$ satisfies the countable chain condition.

Remark. Let T be an unfilled (ω_1, ω_1)-gap. Then, under the assumption of MA+$\neg$CH, Theorem B is a trivial consequence of Theorem A , because any poset which satisfies the countable chain condition also satisfies Knaster's condition. The next theorem claims that the assumption of MA+$\neg$CH (or some assumption as this) is necessary to show that Q_T satisfies Knaster's condition.

Theorem C. There are a poset R and an R-name X such that
(1) R satisfies the countable chain condition and $|R| = \omega_1$,
(2) $\Vdash_R$ " X is an unfilled (ω_1, ω_1)-gap and Q_X doesn't satisfy Knaster's condition. "

Theorems B, C shall be proved in Appendix B, C (respectively). The rest of this appendix is

Proof of Lemma 3.3. For each unfilled (ω_1, ω_1)-gap T, by using Theorem A (b.1), take a $q_T \in Q_T$ such that

$$q_T \Vdash \text{" } P_T \text{ has an uncountalbe antichain "},$$

and set $Q'_T = \{ q \in Q_T ; q \leqq q_T \}$. Set Q = the finite support product of $\langle Q'_T \mid T \text{ is an unfilled } (\omega_1, \omega_1)\text{-gap} \rangle$. Then, by theorem B, Q is as required. □

Appendix B. We first show the following combinatorial lemma.

Lemma B.1. Let $n < \omega$ and $\langle (a^i_\alpha, b^i_\alpha) \mid \alpha < \omega_1 \rangle$ be an unfilled (ω_1, ω_1)-gap, for each $i < n$. Then, there are $\alpha, \beta < \omega_1$ such that

$$a^i_\alpha \cap b^i_\beta \neq \phi \text{ , for all } i < n.$$

To show Lemma B.1, we need the following definition.

Definition. For each $\mathcal{A} = \langle a_\alpha \mid \alpha < \omega_1 \rangle$ and $U \subset \omega_1$, set

$$\lim_U \mathcal{A} = \bigcap_{\alpha < \omega_1} \bigcup_{\beta \in U \setminus \alpha} a_\beta .$$

Sublemma. Let $\mathcal{A}$ be an ω_1-tower and U a cofinal subset of ω_1. Then, it holds that $\mathcal{A} \ll \{ \lim_U \mathcal{A} \}$.

Proof. Let $\mathcal{A} = \langle a_\alpha \mid \alpha < \omega_1 \rangle$ be an ω_1-tower and U a cofinal subset of ω_1. Set $x = \lim_U \mathcal{A}$. To get a contradiction, assume that $a_\alpha \setminus x$ is infinite, for some $\alpha < \omega_1$. Since

$$\forall n \in \omega \setminus x \ \exists \beta < \omega_1 (n \notin \bigcup_{\gamma \in U \setminus \beta} a_\gamma),$$

take a $\beta < \omega_1$ such that

$$\alpha < \beta \ \ \& \ \ (a_\alpha \setminus x) \cap (\bigcup_{\gamma \in U \setminus \beta} a_\gamma) = \phi .$$

Since U is cofinal in ω_1, take a $\gamma \in U \setminus \beta$. Then, it holds that $(a_\alpha \setminus x) \cap a_\gamma = \phi$. But, this contradicts that

$$a_\alpha \subset^* a_\gamma \ \text{ and } \ a_\alpha \setminus x \ \text{ is infinite.}$$ □

Proof of Lemma B.1. Let $n < \omega$ and $T_i = \langle (a^i_\alpha, b^i_\alpha) \mid \alpha < \omega_1 \rangle$ an unfilled (ω_1, ω_1)-gap, for each $i < n$. Set $Seq = \bigcup_{i<n} {}^i\omega$ and $Seq^* = \bigcup_{i \leqq n} {}^i\omega$. Define $U_s \subset \omega_1$ (for $s \in Seq^*$) and $x_s, y_s \subset \omega$, $\gamma_s < \omega_1$ (for $s \in Seq$) by induction on length(s) as follows:

Set $U_\phi = \omega_1$.

Asumme that $s \in Seq$ and U_s is defined. Set i = the length of s.

⟨the definition of x_s, y_s and γ_s⟩

Case 1. U_s is not cofinal in ω_1.

Set $x_s = y_s = \phi$ and $\gamma_s = 0$.

Case 2. otherwise.

Set $x_s = \lim_{U_s} \langle a^i_\alpha \mid \alpha < \omega_1 \rangle$. Since $\{ a^i_\alpha \ ; \ \alpha < \omega_1 \} << \{x_s\}$, take a $\gamma_s < \omega_1$ such that

$$\gamma_t < \gamma_s, \text{ for any } t \subset s \ (t \neq s) \ \ \& \ \ x_s \cap b^i_{\gamma_s} \text{ is infinite.}$$

Set $y_s = x_s \cap b^i_{\gamma_s}$.

⟨the definition of $U_{s^\frown\langle k \rangle}$ (for $k < \omega$)⟩

Set $U_{s^\frown\langle k \rangle} = \{ \alpha \in U_s \ ; \ k \in a^i_\alpha \}$.

Set $\beta = \sup \{ \gamma_t \ ; \ t \in Seq \}$.

Claim. There are $k_j < \omega$ (for $j < n$) such that

(1) $k_j \in b^j_\beta \cap y_{<k_0,\cdots,k_{j-1}>}$, for each $j < n$.

Proof of Claim. By induction on $j < n$. Suppose that $j < n$ and k_m (for $m < j$) are chosen which satisfy (1). Set $t = <k_0,\cdots,k_{j-1}>$. Then, it holds that U_t is cofinal in ω_1. So, y_t is infinite. From this and the fact that $y_t \subset b^j_{\gamma_t} \subset^* b^j_\beta$, we can take a k_j which satisfies (1). QED of Claim.

Let $s = <k_0,\cdots,k_{n-1}>$ be as in Claim. Since U_s is cofinal in ω_1, take an $\alpha \in U_s$ such that $\beta < \alpha$. Then, for each $i < n$, since $\alpha \in U_{<k_0,\cdots,k_i>}$, it holds that $k_i \in a^i_\alpha$.

So, $k_i \in a^i_\alpha \cap b^i_\beta$, for each $i < n$. □

Now we are ready to prove Theoem B. The proof is similar to the proof of Theorem A (b.2) (in [B, p.932]) except we need Lemma B.1. Let $n < \omega$ and $T_i = \langle (a^i_\alpha, b^i_\alpha) \mid \alpha < \omega_1 \rangle$ an unfilled (ω_1,ω_1)-gap, for $i < n$. Set Q = the product of $\langle Q_{T_i} \mid i < n \rangle$. To get a contradiction, suppose that $\langle w_\alpha \mid \alpha < \omega_1 \rangle$ is an antichain of Q. For each $\alpha < \omega_1$, let $w_\alpha = (w^0_\alpha,\cdots,w^{n-1}_\alpha)$. By using Δ-system argment, we may assume that there are $k_0,\cdots,k_{n-1} \in \omega \setminus \{0\}$ such that , for each $i < n$,

(2) $|w^i_\alpha| = k_i$, for each $\alpha < \omega_1$,

(3) if $\alpha < \beta$, then $w^i_\alpha \cap w^i_\beta = \phi$ and $\max(w^i_\alpha) < \min(w^i_\beta)$.

For each $i < n$ and $\alpha < \omega_1$, take $m_{i,\alpha} < \omega$ such that

$$a^i_\xi \setminus m_{i,\alpha} \subset a^i_\eta \setminus m_{i,\alpha} \quad \text{and} \quad b^i_\xi \setminus m_{i,\alpha} \subset b^i_\eta \setminus m_{i,\alpha},$$

if $\xi, \eta \in w^i_\alpha$ and $\xi < \eta$. Again without loss of generality, we may assume that $m_{i,\alpha} = m$, for all $i < n$ and all $\alpha < \omega_1$. For each $i < n$ and $\alpha < \omega_1$, set

$$c^i_\alpha = a^i_\xi \setminus m \quad \text{and} \quad d^i_\alpha = b^i_\xi \setminus m \text{ , where } \xi = \min(w^i_\alpha).$$

Then, it holds that

$$\langle (c^i_\alpha, d^i_\alpha) \mid \alpha < \omega_1 \rangle \text{ an unfilled } (\omega_1, \omega_1)\text{-gap, for } i < n.$$

So, by Lemma B.1, there are α, $\beta < \omega_1$ such that

$$c^i_\alpha \cap d^i_\beta \neq \emptyset, \text{ for all } i < n.$$

So, w_α and w_β are compatible, a contradiction. □

Appendix C. A poset P is said to satisfy Knaster's condition if for any uncountable $X \subset P$ there is an uncountable $Y \subset X$ such that any two members of Y are compatible. The following facts are well-known.

(1) If P satisfies Knaster's condition, then P satisfies the countable chain condition,

(2) If P satisfies Knaster's condition and Q satisfies the countable chain condition, then $P \times Q$ satisfies the countable chain condition.

(3) MA + $\neg$CH implies the reverse implication of (1).

There are several examples of a poset which satisfies the countalbe chain condition but does not satisfy Knaster's condition, under some set theoretical assumption (see e.g., [W] section 3). Theorem C gives another such example.

We turn to a proof of Theorem C.

Lemma C.1. Let R be a poset and X an R-name such that

(c.1) R satisfies the countable chain condition and $|R| = \omega_1$,

(c.2) $V^R \models$ " X is an unfilled (ω_1, ω_1)-gap. "

Suppose that there exists an R-name Y such that, in V^R,

(c.3) Y is a poset and satisfies the countalbe chain condition,

(c.4) $\Vdash_Y$ " X is filled ".

Then, it holds that, in V^R, Q_Y doesn't satisfy Knaster's condition.

So, R and X satisfy (1) and (2) in Theorem C.

Proof. Set $W = V^R$ and $W^* = W^Y$. By (c.4) and Theorem A (a), it holds that

$W^* \models P_X$ satisfies the countalbe chain condition.

Since $\omega_1^{W^*} = \omega_1^W$, it holds that

(c.5) $W \models P_X$ satisfies the countalbe chain condition.

Since

$W \models \exists q \in Q_X (q \Vdash_{Q_X}$ " P_X has an uncountable antichain "),

it holds that

(c.6) $W \models Q_X \times P_X$ doesn't satisfy the countalbe chain condition.

By (c.5) and (c.6),

$W \models Q_X$ doesn't satisfy Knaster's condition. □

We shall construct a poset R and R-names X and Y which satisfy (c.1)~(c.4). The method for doing this is due to Hechler [H] and Dordal [D]. Hechler used it for adjoining a tower in a generic extension and later Dordal generalized it for adjoining an arbitrary partially order type of $\mathcal{P}(\omega)$/finite in a generic extension.

Definition (Hechler and Dordal). Let $A = (A, <_A)$ be a partial order type. Define the poset P(A) by

$$P(A) = \{ p ; \exists u \subset A \exists n < \omega (|u| < \omega \ \& \ p : u \times n \to 2) \},$$

and for any p, q $\in$ P(A) such that $p : u \times n \to 2$ and $q : v \times m \to 2$,

$$p \leqq q$$

$$\text{iff} \quad q \subset p \ \& \ \forall a,b \in v \ \forall k \in [m,n) \ (a <_A b \Rightarrow p(a,k) \leqq p(b,k))$$

For each $a \in A$, define P(A)-name H_a by

$$\Vdash \text{" } H_a \subset \omega \text{ "},$$

$$\| n \in H_a \| = \{ p \in P(A) ; p(a,n)=1 \}, \quad \text{for each } n < \omega.$$

The following lemma is due to P. Dordal ([D, Lemma 5.4, p. 45]).

Lemma C.2. Let $A = (A, <_A)$ be a linear order type and B is a sub-order type of A.

(1) P(A) satisfies the countalbe chain condition.

(2) If G is V-generic on P(A), then $G \cap P(B)$ is V-generic on P(B).

(3) If x is a P(A)-name such that $\Vdash$ "$x \subset \omega$", then there exists a countable subset C of A such that x is a P(C)-name.

(4) For any $a, b \in A$, $a <_A b$ if and only if $\Vdash$ "$H_a \subset^* H_b$". (i.e., $\langle H_a \mid a \in A \rangle$ is a chain of $\mathcal{P}(\omega)$/finite.)

Let Q denote the set of rationals. Set $A = Q \times \omega_1 \times 2$ and $B = A \cup \{0\}$. Define the linear ordering $<_B$ on B by

$(q, \alpha, 0) <_B 0 <_B (q, \alpha, 1)$, for any $q \in Q$ and any $\alpha < \omega_1$,

$(q, \alpha, 0) <_B (r, \beta, 0)$, if $\alpha < \beta$ or ($\alpha = \beta$ and $q < r$).

$(q, \alpha, 1) <_B (r, \beta, 1)$, if $\alpha > \beta$ or ($\alpha = \beta$ and $q < r$).

We regard B as the linear order type $(B, <_B)$ and A it's sub-order type. Set the poset $R = P(A)$. Define R-names a_α, b_α (for $\alpha < \omega_1$) by

$$a_\alpha = H_{(0,\alpha,0)} \text{ and } b_\alpha = \omega \setminus H_{(0,\alpha,1)}, \text{ for each } \alpha < \omega_1.$$

Set $W = V^R$. In W, set $X = \langle (a_\alpha, b_\alpha) \mid \alpha < \omega_1 \rangle$ and take the poset Y such that $W^Y = V^{P(B)}$. Then, by Lemma C.2 (4), it holds that

$$W \models \text{" X is an } (\omega_1, \omega_1)\text{-gap " and } W^Y \models \text{" X is filled ".}$$

So, the next lemma completes a proof of Theorem C. The lemma is proved by the same way in the proof of Theorem 5.3 in [D]. So, we omit a proof.

Lemma C.3. $W \models$ " X is unfilled. "

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