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Relaxation-oscillations in infinite dimensional dynamical systems

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In this paper, we would like to consider the following reaction- diffusion systems arising in combustion theory:

$$(1)_\varepsilon \quad \begin{cases} \dfrac{\partial \theta}{\partial t} = \Delta\theta + cf(\theta) \\ \dfrac{\partial c}{\partial t} = d\Delta c - \varepsilon c f(\theta) \end{cases} \qquad x \in \Omega, \quad t > 0,$$

where $f(\theta) = exp\{-\frac{H}{1+\theta}\}$ and Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^N$. Here, θ and c are respectively the nondimensionalized temperature and concentration of fuel. d, ε and H are all positive constants. The meaning of these constants is stated in [2] for instance. The initial and boundary conditions for θ and c are

$$(2) \quad \theta(0,x) = \theta_0(x) \ge 0, \quad c(0,x) = c_0(x) \ge 0 \qquad x \in cl\Omega$$

and

$$(3) \quad \theta(t,x) = 0, \quad \frac{\partial c}{\partial \nu} = k_0(c^* - c) \qquad x \in \partial\Omega, \ t > 0$$

respectively, where ν is the outward normal unit vector on $\partial\Omega$. The boundary condition of c indicates that the fuel is supplied through the boundary $\partial\Omega$. Its magnitude is proportional to the difference of c on $\partial\Omega$ and some constant value c^* with the flux rate k_0. To study $(1)_\varepsilon$, (2), (3), we assume here ε to be sufficiently small, which is natural from a chemical view point (see [2], [4] for instance) and k_0 to be $k\varepsilon$ for some k. The latter implies that amounts of the consumption and the supply of fuel would be the same order ε.

Our aim is to study the dependency of c^* on solutions $\left(\theta(t,x), c(t,x)\right)$ of $(1)_\varepsilon$, (2) and (3) with $k_0 = k\varepsilon$ and to show the existence of relaxation oscillations in an appropriate range of c^* ([3]).

First, we analyze the behavior of solutions of $(1)_\varepsilon$, (2) and (3) by formal perturbation argument, so called the "two-timing method". Here, We rewrite $(1)_\varepsilon$ and (3) as

$$(4)_\varepsilon \qquad U_t = A_\varepsilon(U) + \varepsilon F(U),$$

where $U = (\theta, c)$, $F(U) = (0, -cf(\theta))$ and $A_\varepsilon(U) = \left(\Delta\theta + cf(\theta), d\Delta c\right)$ for $U = (\theta, c)$ with $\theta\big|_{\partial\Omega} = 0$ and $\frac{\partial c}{\partial \nu}\Big|_{\partial\Omega} = k\varepsilon(c^* - c)$. We derive the lowest order approximate function by the two- timing method. Introducing two time scales: a *slow* time scale $T = \varepsilon t$ and a *fast* time scale t, we look for solutions of $(4)_\varepsilon$ in the form

(5) $$U(t;\varepsilon) = U^0(t,T,x) + \varepsilon U^1(t,T,x) + O(\varepsilon^2).$$

With the relation $\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \frac{\partial}{\partial T}$, inserting (5) into $(4)_\varepsilon$ and equating coefficients of like powers of ε^0 and ε^1, we obtain

(6) $$U^0_t = A_0(U^0), \quad t > 0, \ T > 0$$

for $U^0 = (\theta^0, c^0)$ satisfying $\theta^0\big|_{\partial\Omega} = 0$ and $\frac{\partial c^0}{\partial \nu}\Big|_{\partial\Omega} = 0$,

(7) $$U^1_t + U^0_T = A_0'(U^0)U^1 + F(U^0), \quad t > 0, \ T > 0$$

for $U^1 = (\theta^1, c^1)$ satisfying $\theta^1\big|_{\partial\Omega} = 0$ and $\frac{\partial c^1}{\partial \nu}\Big|_{\partial\Omega} = k(c^* - c^0)$, respectively, where $A_0(U^0) = \left(\Delta\theta^0 + c^0 f(\theta^0), d\Delta c^0\right)$ and $A_0'(U^0)$ represents the Frechet derivative of $A_0(U^0)$ with respect to U^0. Now, we immediately know the dynamics of $U^0(t,T,x)$ for t from (6). Let us consider the dynamics of $U^0(t,T,x)$ for T. Since ε is sufficiently small, we may assume T to be $O(1)$ for large enough t, so that we put formally $t = \infty$ for any fixed $T > 0$. Consider the asymptotic behavior of $U^0(t,T,x)$ as $t \to \infty$. Since the equation of $c^0(t,T,x)$ for t is $\frac{\partial c^0}{\partial t} = \Delta c^0$ with $\frac{\partial c^0}{\partial \nu}\Big|_{\partial\Omega} = 0$, the spatial average of $c^0(t,T,x)$ is independent of t and the asymptotic behavior of $c^0(t,T,x)$ as $t \to \infty$ is the constant of its spatial average $\frac{1}{|\Omega|}\int_\Omega c^0(t,x,T)dx$, say $\lambda(T)$. So that $\theta^0(t,T,x)$ converges as $t \to \infty$ to the nonnegative stable stationary solution of

$(8)_\lambda \qquad \theta_t = \Delta\theta + \lambda f(\theta),\ x \in \Omega$

with $\theta\big|_{\partial\Omega} = 0$, where $\lambda = \lambda(T)$, which implies that it is important to consider the stationary problem of $(8)_\lambda$:

$(9)_\lambda \qquad \Delta\phi + \lambda f(\phi) = 0$

with $\phi\big|_{\partial\Omega} = 0$ and $\phi \geq 0$ in Ω. This problem has been studied as "Nonlinear eigenvalue problems" by numerous authors. Specially, the problem in the case that Ω is a ball in $\mathbb{R}^N$ has been extensively studied and when Ω is a ball in $\mathbb{R}^N$ with $1 \leq N \leq 2$, the global picture of solutions of $(9)_\lambda$ with respect to λ is S-shaped given as follows mathematically and numerically (Figure 1) (Parks[8], Parter, Stein and Stein[10], Parter[9}, Gidas, Ni and Nirenberg[5], Tam[13], etc.):

(H1) There exist $\underline{\lambda}$ and $\overline{\lambda}$ $(0 < \underline{\lambda} < \overline{\lambda})$ such that only three families of solutions of $(9)_\lambda$, say $\{\phi_1(\cdot;\lambda)\}$, $\{\phi_2(\cdot;\lambda)\}$, $\{\phi_3(\cdot;\lambda)\}$, exist on $0 \leq \lambda \leq \overline{\lambda}$, $\underline{\lambda} \leq \lambda \leq \overline{\lambda}$ and $\lambda \geq \underline{\lambda}$, respectively, and satisfy $\phi_1(x;\lambda) < \phi_2(x;\lambda) < \phi_3(x;\lambda)$ for $x \in \Omega$ and $\underline{\lambda} < \lambda < \overline{\lambda}$ and $\phi_1(x;\overline{\lambda}) = \phi_2(x;\overline{\lambda})$, $\phi_3(x;\underline{\lambda}) = \phi_2(x;\underline{\lambda})$ for $x \in \Omega$.

(H2) $\phi_1(x;\lambda) \leq \phi_1(x;\lambda')$ for $x \in \Omega$ and $0 \leq \lambda \leq \lambda' \leq \overline{\lambda}$; $\phi_2(x;\lambda) \geq \phi_2(x;\lambda')$ for $x \in \Omega$ and $\underline{\lambda} \leq \lambda \leq \lambda' \leq \overline{\lambda}$; $\phi_3(x;\lambda) \leq \phi_3(x;\lambda')$ for $x \in \Omega$ and $\lambda' \geq \lambda \geq \underline{\lambda}$.

(H3) $\phi_2(x;\lambda)$ is a hyperbolic stationary solution of $(8)_\lambda$ for $\underline{\lambda}$

$< \lambda < \bar{\lambda}$.

Remark 1. i) When Ω is a ball, nonnegative solutions of $(9)_\lambda$ are all symmetric (Gidas, Ni and Nirenberg[5]) and Parks[8], Parter[9], Parter, Stein and Stein[10] investigated symmetric solutions of $(9)_\lambda$ and they proved that there are at least three solutions of $(9)_\lambda$ in a certain range of λ.

ii) When $N \geq 3$, the global picture of solutions of $(9)_\lambda$ with respect to λ is in general not S- shaped and more complicated (see e.g. Bebernes and Eberly[1]). So we don't consider the case in this paper, though we can deal with it in a similar manner.

iii) If (H1) holds, ϕ_1 and ϕ_3 are stable relative to $(8)_\lambda$ for $0 \leq \lambda < \bar{\lambda}$, $\lambda > \underline{\lambda}$, respectively, and ϕ_2 is unstable for $\underline{\lambda} < \lambda < \bar{\lambda}$ (e.g. sattinger [11]). If we assume both (H1) and (H2), then we can show that ϕ_1 and ϕ_3 are stable and ϕ_2 is unstable in a linearized sense (see [3, Lemma A2 in Appendix]).

iv) (H1), (H2) and (H3) hold rigorously in the case that Ω is an interval in $\mathbb{R}^1$, which is shown by [3, Lemma A1 in Appendix].

From now on, we assume (H1), (H2) and (H3) for $(8)_\lambda$ without assuming necessarily that Ω is a ball and $1 \leq N \leq 2$. Suppose $U^0(t,T,x) \to \left(\phi(x;\lambda(T)),\lambda(T)\right)$ as $t \to \infty$, where $\phi(x;\lambda) = \phi_1(x;\lambda)$ or $\phi_3(x;\lambda)$. $\lambda(T)$ is determined as follows: Integrating the equation of the second component of (7) with respect to x, we have

$$(10)\quad \frac{\partial}{\partial t}\int c^1(t,T,x)dx + \frac{\partial}{\partial T}\int c^0(t,T,x)dx = k\int_{\partial\Omega}\left(c^* - c^0(t,T,x)\right)ds$$

$$- \int_\Omega f(\theta^0(t,T,x))dx.$$

Let $t \to \infty$ in (10). Then, noting that $c^0(t,T,x) \to \lambda(T)$, $\theta^0(t,T,x) \to \phi(x;\lambda(T))$ and $\frac{\partial}{\partial t}\int_\Omega c^1(t,T,x)dx \to 0$ as $t \to \infty$, we have

$$(11)\qquad \frac{d\lambda}{dT} = \frac{1}{|\Omega|}\{kd|\partial\Omega|(c^* - \lambda) - \lambda\int_\Omega f(\phi(x;\lambda))dx\},$$

which means that the function $U^0(\infty,T,x)$ moves along $\left(\phi(x;\lambda(T)),\ \lambda(T)\right)$ with the solution $\lambda(T)$ of (11). Since $\phi(x;\lambda) = \phi_1(x;\lambda)$ or $\phi_3(x;\lambda)$, we define $F_i(\lambda) = \frac{1}{|\Omega|}\{kd|\partial\Omega|(c^* - \lambda) - \lambda\int_\Omega f(\phi_i(x;\lambda))dx\}$ and rewrite (11) as

$$(12)_i\qquad \frac{d\lambda}{dT} = F_i(\lambda)$$

($i = 1,3$) in order to clarify the family of solutions of $(9)_\lambda$ to which we pay attention.

It is expected that $U^0(t,T,x)$ approximates well the solution of $(4)_\varepsilon$, so that it is worth to consider the behavior of $U^0(t,T,x)$ in more detail. In order to classify the behavior of $U^0(t,T,x)$ with respect to c^*, we write $F_i(\lambda) = -H_i(\lambda) + ac^*$, where $H_i(\lambda) = \frac{\lambda}{|\Omega|}\{kd|\partial\Omega| + \int_\Omega f(\phi_i(x;\lambda))dx\}$ and $a = \frac{kd|\partial\Omega|}{|\Omega|}$. $H_1(\lambda)$ and $H_3(\lambda)$ are defined for $0 \le \lambda < \overline{\lambda}$ and $\lambda > \underline{\lambda}$, respectively.

Now, we define $H_* = \max_{0\le\lambda<\bar{\lambda}} H_1(\lambda)$ and $H^* = \min_{\lambda>\underline{\lambda}} H_3(\lambda)$. Then, from (H1) and (H2) $H_3(\lambda) > H_1(\lambda)$ holds for $\underline{\lambda} < \lambda < \bar{\lambda}$ and $H_i(\lambda)$ (i = 1,3) are monotone increasing, which implies $H_* = H_1(\bar{\lambda}) < H^* = H_3(\underline{\lambda})$ (Figure 2). Let $S^0(t)\bar{U} = (\theta, c)$ be the solution of (6) with the initial data $\bar{U} = (\bar{\theta}, \bar{c})$, that is, the solution of

$$\begin{cases} \theta_t = \Delta\theta + cf(\theta) \\ c_t = d\Delta c \end{cases}$$

with $\theta\big|_{\partial\Omega} = 0$, $\frac{\partial c}{\partial \nu}\big|_{\partial\Omega} = 0$ and $\big(\theta(0,x), c(0,x)\big) = \bar{U} = \big(\bar{\theta}(x), \bar{c}(x)\big)$.

i) $c^* < H_*/a$

In this case, $F_1(\lambda)$ has only one equilibrium λ_*, which is stable relative to $(12)_1$, and $F_3(\lambda) < 0$ for any $\lambda > \underline{\lambda}$ (Figure 3-1). Suppose that for the initial data $U_0 = (\theta_0, c_0)$, the solution $S^0(t)U_0$ converges to $(\phi_3(x;\lambda_0), \lambda_0)$ as $t \to \infty$, where $\lambda_0 = \frac{1}{|\Omega|}\int_\Omega c_0(x)dx$. We define this orbit $cl\{S^0(t)U_0 \mid t \ge 0\}$ by γ_1. After reaching $\big(\phi_3(x;\lambda_0),\ \lambda_0\big)$, $U^0(t,T,x)$ varies along $\big(\phi_3(x;\lambda(T)), \lambda(T)\big)$, where $\lambda(T)$ is the solution of $(12)_3$ with $\lambda(0) = \lambda_0$. Since $\lambda(T)$ is decreasing for T, $\lambda(T)$ arrives at $\underline{\lambda}$ for a finite time of T and $\phi_3(x;\lambda)$ vanishes by coalescing with $\phi_2(x;\lambda)$. Let the orbit be $\gamma_2 = \{(\phi_3(x;\lambda), \lambda) \mid \underline{\lambda} \le \lambda \le \bar{\lambda}\}$. After $\lambda(T)$ arrives at $\underline{\lambda}$, $U^0(t,T,x)$ is again governed by (6) and converges to $\big(\phi_1(x;\underline{\lambda}), \underline{\lambda}\big)$ as $t \to \infty$. This orbit is given by $\gamma_3 = cl\{(\theta(t,\cdot), \underline{\lambda}) \mid -\infty < t < +\infty\}$, where $\theta(t,x)$ is the solution of

$(8)_{\underline{\lambda}}$ satisfying $\theta(t,x) \to \phi_3(x;\underline{\lambda})$ as $t \to -\infty$ and $\theta(t,x) \to \phi_1(x;\underline{\lambda})$ as $t \to +\infty$. The existence of orbits such as γ_3 is shown by Matano[7]. After reaching $\left(\phi_1(x;\underline{\lambda}),\underline{\lambda}\right)$, $U^0(t,T,x)$ approaches $\left(\phi_1(x;\lambda_*),\lambda_*\right)$ along $\left(\phi_1(x;\lambda(T)),\lambda(T)\right)$, where $\lambda(T)$ is the solution of $(12)_1$ with $\lambda(0) = \underline{\lambda}$. Consequently, defining $\gamma_4 = \left\{(\phi_1(x;\lambda),\lambda) \;\middle|\; \underline{\lambda} \le \lambda \le \lambda_*\right\}$ if $\underline{\lambda} < \lambda_*$ or $\gamma_4 = \left\{(\phi_1(x;\lambda),\lambda) \;\middle|\; \lambda_* \le \lambda \le \underline{\lambda}\right\}$ if $\lambda_* < \underline{\lambda}$, we see that the orbit of $U^0(t,T,x)$ from U_0 to $\left(\phi_1(x;\lambda_*),\lambda_*\right)$ consists of the union of above four orbits $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ (Figure 4-1).

If $S^0(t)U_0 \to \left(\phi_1(x;\lambda_0),\lambda_0\right)$ as $t \to \infty$ for the initial data U_0, then $U^0(t,T,x)$ just approaches $\left(\phi_1(x;\lambda_*),\lambda_*\right)$ along $\left(\phi_1(x;\lambda(T)),\lambda(T)\right)$, where $\lambda(T)$ is the solution of $(12)_1$ with $\lambda(0) = \lambda_0$. In this case, defining $\gamma_1' = cl\left\{S^0(t)U_0 \;\middle|\; t \ge 0\right\}$ and $\gamma_2' = \left\{(\phi_1(x;\lambda),\lambda) \;\middle|\; \lambda_0 \le \lambda \le \lambda^*\right\}$ if $\lambda_0 \le \lambda^*$ or $\gamma_2' = \left\{(\phi_1(x;\lambda),\lambda) \;\middle|\; \lambda^* \le \lambda \le \lambda_0\right\}$ if $\lambda_0 \ge \lambda^*$, the orbit of $U^0(t,T,x)$ from U_0 to $\left(\phi_1(x;\lambda_*),\lambda_*\right)$ is given by $\gamma_1' \cup \gamma_2'$ (Figure 4-1).

Thus, $\left(\phi_1(x;\lambda_*),\lambda_*\right)$ is globally stable and there are mainly two kind of behaviors of $U^0(t,T,x)$, one is the behavior given by the orbit $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, another is the one given by the orbit $\gamma_1' \cup \gamma_2'$, which depends on the initial data U_0.

ii) $H_*/a < c^* < H^*/a$

In this case, $F_1(\lambda) > 0$ for $0 \le \lambda \le \bar{\lambda}$ and $F_3(\lambda) < 0$ for $\lambda \ge \underline{\lambda}$ (Figure 3-2).

Suppose that $S^0(t)U_0$ converges to $\left(\phi_3(x;\lambda_0),\lambda_0\right)$ as $t \to \infty$, where $\lambda_0 = \frac{1}{|\Omega|}\int_\Omega c_0(\mathrm{x})d\mathrm{x}$. The orbit of $U^0(t,T,x)$ is quite similar to that in case i) until $U^0(t,T,x)$ reaches $\left(\phi_1(x;\underline{\lambda}),\underline{\lambda}\right)$. The orbit is represented by $\gamma_1 \cup \gamma_2 \cup \gamma_3$ if we use the same symbol in case i). Starting at $\left(\phi_1(x;\underline{\lambda}),\underline{\lambda}\right)$, $U^0(t,T,x)$ moves along $\left(\phi_1(x;\lambda(T)),\lambda(T)\right)$, where $\lambda(T)$ is the solution of $(12)_1$ with $\lambda(0) = \underline{\lambda}$. Since $F_1(\lambda) > 0$ for $0 \le \lambda \le \overline{\lambda}$, $\lambda(T)$ arrives at $\overline{\lambda}$ for a finite time of T. Let the orbit be $\gamma_4 = \{(\phi_1(x;\lambda),\lambda) \mid \underline{\lambda} \le \lambda \le \overline{\lambda}\}$. When $\lambda(T)$ arrives at $\overline{\lambda}$, the dynamics of $U^0(t,T,x)$ is described by (6) and $U^0(t,T,x)$ converges to $\left(\phi_3(x;\overline{\lambda}),\overline{\lambda}\right)$ as $t \to \infty$, after which we can chase the orbit of $U^0(t,T,x)$ by quite a similar manner in case i). Consequently, we see that the orbit of $U^0(t,T,x)$ is asymptotically given by the periodic orbit $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$. Here, $\gamma_1 = cl\{(\theta(t,x),\overline{\lambda}) \mid -\infty < t < +\infty\}$, where $\theta(t,x)$ is the solution of $(8)_{\overline{\lambda}}$ satisfying $\theta(t,x) \to \phi_1(x;\overline{\lambda})$ as $t \to -\infty$ and $\theta(t,x) \to \phi_3(x;\overline{\lambda})$ as $t \to +\infty$; $\gamma_2 = \{(\phi_3(x;\lambda),\lambda) \mid \underline{\lambda} \le \lambda \le \overline{\lambda}\}$; $\gamma_3 = cl\{(\theta(t,x),\underline{\lambda}) \mid -\infty < t < +\infty\}$, where $\theta(t,x)$ is the solution of $(8)_{\underline{\lambda}}$ satisfying $\theta(t,x) \to \phi_3(x;\lambda)$ as $t \to -\infty$ and $\theta(t,x) \to \phi_1(x;\lambda)$ as $t \to +\infty$; γ_4 is as mentioned above (Figure 4-2). Among them, γ_1 and γ_3 are the orbits governed by $(7)_\lambda$ with the fast time scale t and γ_2, γ_4 are those governed by (12) with the slow time scale T. Thus, this periodic orbit γ can be regarded as the "relaxation oscillation in infinite dimensional dynamical systems".

iii) $c^* > H^*/a$

In this case, $F_3(\lambda)$ has only one equilibrium λ^* and $F_1(\lambda) >$

0 for $0 \le \lambda < \bar{\lambda}$ (Figure 3-3). Quite similarly to case i), we find that the orbit of $U^0(t,T,x)$ is given by $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ if $S^0(t)U_0 \to \left(\phi_1(x;\lambda_0),\lambda_0\right)$ or $\gamma' = \gamma_1' \cup \gamma_2'$ if $S^0(t)U_0 \to \left(\phi_3(x;\lambda_0),\lambda_0\right)$, where $\lambda_0 = \frac{1}{|\Omega|}\int_\Omega c_0(x)dx$. Here, $\gamma_1 = cl\{S^0(t)U_0 \mid 0 \le t < +\infty\}$; $\gamma_2 = \{(\phi_1(x;\lambda),\lambda) \mid \underline{\lambda} \le \lambda \le \bar{\lambda}\}$; $\gamma_3 = cl\{(\theta(t,x),\bar{\lambda}) \mid -\infty < t < +\infty\}$, where $\theta(t,x)$ is the solution of $(8)_{\bar{\lambda}}$ satisfying $\theta(t,x) \to \phi_1(x;\bar{\lambda})$ as $t \to -\infty$ and $\theta(t,x) \to \phi_3(x;\bar{\lambda})$ as $t \to +\infty$; $\gamma_4 = \{(\phi_3(x;\lambda),\lambda) \mid \bar{\lambda} \le \lambda \le \lambda^*\}$ if $\lambda^* > \bar{\lambda}$ or $\gamma_4 = \{(\phi_3(x;\lambda),\lambda) \mid \lambda^* \le \lambda \le \bar{\lambda}\}$ if $\lambda^* < \bar{\lambda}$; $\gamma_1' = cl\{S^0(t)U_0 \mid 0 \le t < +\infty\}$; $\gamma_2' = \{(\phi_3(x;\lambda),\lambda) \mid \bar{\lambda} \le \lambda \le \lambda^*\}$ if $\lambda^* > \bar{\lambda}$ or $\gamma_2' = \{(\phi_3(x;\lambda),\lambda) \mid \lambda^* \le \lambda \le \bar{\lambda}\}$ if $\lambda^* < \bar{\lambda}$ (Figure 4-3). $\left(\phi_3(x;\lambda^*),\lambda^*\right)$ is globally stable.

Let us consider the phenomenal meanings of above results. Since it follows from (H1), (H2) and (H3) that $\phi_1(x;\lambda_1) < \phi_2(x;\lambda_2) < \phi_3(x;\lambda_3)$ in Ω for any $0 \le \lambda_1 < \bar{\lambda}$, $\underline{\lambda} < \lambda_2 < \bar{\lambda}$ and $\lambda_3 > \bar{\lambda}$, we can regard the solution families $\{(\phi_1(x;\lambda),\lambda)\}$ and $\{(\phi_3(x;\lambda),\lambda)\}$ as the cold state and the hot state, respectively. The case i) (or iii)) implies that:

If the supply of fuel c^* is below (or beyond) some critical value, that is, $c^* < H_*/a$ (or $c^* > H^*/a$), the state of combustion eventually settles down in the cold state of a low temperature $\left(\phi_1(x;\lambda_*),\lambda_*\right)$ (or the hot state of a high temperature $\left(\phi_3(x;\lambda^*),\lambda^*\right)$). Moreover, the orbit of $U^0(t,T,x)$ describes how the combustion proceeds to the final stage. For example, consider the case i). When the orbit of $U^0(t,T,x)$ is given by $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ as mentioned in the case i), γ_1 means the rapid burn-up to a hot state of a high temperature with the fast time scale t (the explosion) and the combustion proceeds slowly along the hot state γ_2 with the slow time scale T. When the combustion reaches a critical state $\left(\phi_3(x;\underline{\lambda}),\underline{\lambda}\right)$, the combustion rapidly burns down to a cold state of a low temperature along γ_3 with the fast time scale t and proceeds slowly to a final stage $\left(\phi_1(x;\lambda_*),\lambda_*\right)$ along γ_4. On the other hand, the orbit given by $\gamma_1' \cup \gamma_2'$ means no explosion. Thus, whether explosion appears or not depends on the initial data, which is determined by the behavior of $S^0(t)U_0$.

The case ii) implies that: If the supply of fuel c^* is in the appropriate range, that is, in the range $H_*/a < c^* < H^*/a$, the state of combustion varies periodically in time. Its orbit $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ given in the case ii) shows that the cold state of a low temperature and the hot state of a high temperature appear alternatively by repeating burn- up and burn- down.

Thus, the combustion varies from the cold state to the hot state by way of the periodic state as c^* increases. The global picture of combustion with respect to c^* is drawn in Figure 5.

Finally, we give the validity of above discussions ([3], [6]).

In addition to assumptions (H1), (H2) and (H3), we impose the following assumption on Ω: (H4) There exist a smooth function $g(x)$ for $x \in \Omega$ and positive constants r_0, R_0 $(r_0 \leq R_0)$

so that $r_0 \le \Delta g(x) \le R_0$ for $x \in \Omega$ and $\frac{\partial g}{\partial \nu} = 1$ for $x \in \partial\Omega$.

Remark 2. Such a function $g(x)$ really exists when Ω is a ball.

Let $F_i(\lambda)$, $H_i(\lambda)$ $(i = 1,3)$ and constants H_*, H^* and a be those given above. $U_\varepsilon(t)U_0$ denotes the solution of $(4)_\varepsilon$ with $U_\varepsilon(0)U_0 = U_0$. Let B_i $(i = 1,2)$ be the Banach space $L^p(\Omega)$ for $p > N$ with the usual norm and B_i^α be the domain of A_i^α with the graph norm $\|\cdot\|_\alpha$, where $A_1 = \Delta$ (Laplace operator in $\mathbb{R}^N$) with the domain $D(A_1) = \{\theta \in W^{2,p}(\Omega) \mid \theta = 0 \text{ on } \partial\Omega\}$ and $A_2 = d\Delta$ with the domain $D(A_2) = \{c \in W^{2,p}(\Omega) \mid \frac{\partial c}{\partial \nu} = 0 \text{ on } \partial\Omega\}$. When $N = 1$, we put $p = 2$. We define $B = B_1 \times B_2$ with the norm $\|U\| = \|\theta\|_{L^p(\Omega)} + \|c\|_{L^p(\Omega)}$ for $U = (\theta, c) \in B$ and $B^\alpha = B_1^\alpha \times B_2^\alpha$ with the norm $\|U\|_\alpha = \|\theta\|_\alpha + \|c\|_\alpha$. Hereafter, we fix $\alpha \in \left(\frac{p+N}{2p}, 1\right)$ so that $B^\alpha \subset C^1(\Omega)\times C^1(\Omega)$ with the continuous imbedding. Moreover, we define the norm of $L^q(\Omega)$ for $q \ge 1$ by $\|\cdot\|_{L^q}$ and define projections $Pc = \frac{1}{|\Omega|}\int_\Omega c(x)dx$ and $Qc(x) = c(x) - Pc$ for $c \in L^p(\Omega)$.

Theorem 1. (Point dissipativeness) There exist $\varepsilon_0 > 0$, $M_0 > 0$ and $c_* > 0$, $\underline{\theta}(x)$ such that a compact set K_ε in B^α exists for $0 < \varepsilon \le \varepsilon_0$ so that $K_\varepsilon \subset \{U = (\theta, c) \in B^\alpha \mid \underline{\theta}(x) \le \theta(x),\ c_* \le c(x) \le c^*$ for $x \in \Omega$, $\|U\|_\alpha \le M_0$ and $\|Qc\|_\alpha \le \varepsilon M_0\}$, where $\underline{\theta}(x)$ is a nonnegative and nontrivial function on Ω with $\underline{\theta}\big|_{\partial\Omega} = 0$, and that for any $U_0 = (\theta_0, c_0) \in B$ with $\theta_0(x) \ge 0$ and $c_0(x) \ge 0$ for $x \in \Omega$, the solution $U_\varepsilon(t)U_0$ eventually enters K_ε as $t \to \infty$.

Theorem 2. Suppose $0 < c^* < H_*/a$ (or $c^* > H^*/a$) and let λ_* (or λ^*) be the equilibrium of $(12)_1$ (or $(12)_3$). If $\frac{dF_1}{d\lambda}(\lambda_*) < 0$ (or $\frac{dF_3}{d\lambda}(\lambda^*) < 0$), then there exist $\varepsilon_0 > 0$ such that $(4)_\varepsilon$ has a unique stationary solution $\left(\bar{\theta}(x;\varepsilon),\bar{c}(x;\varepsilon)\right)$ for $0 < \varepsilon \le \varepsilon_0$, which satisfies $\left(\bar{\theta}(\cdot;\varepsilon),\bar{c}(\cdot;\varepsilon)\right) \in C\left((0,\ \varepsilon_0];B\right)$ and $\lim_{\varepsilon\downarrow 0}\left(\bar{\theta}(\cdot;\varepsilon),\bar{c}(\cdot;\varepsilon)\right) = \left(\phi_1(\cdot;\lambda_*),\lambda_*\right)$ (or $= \left(\phi_3(\cdot;\lambda^*),\lambda^*\right)$). Moreover, $\left(\bar{\theta}(\cdot;\varepsilon),\bar{c}(\cdot;\varepsilon)\right)$ is globally stable.

Let $\gamma = \gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4$ be the orbit mentioned in the case ii) and $Y_\delta = \left\{(\theta,c) \in B \;\middle|\; dist_{B^\alpha}\{\gamma,(\theta,c)\} < \delta\right\}$.

Theorem 3. Suppose $H_*/a < c^* < H^*/a$. Then for sufficiently small $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that $(4)_\varepsilon$ has a periodic solution $\Pi_p(t,x;\varepsilon) = \left(\theta_p(t,x;\varepsilon),\ c_p(t,x;\varepsilon)\right)$ with the period $p(\varepsilon)$ for $0 < \varepsilon \le \varepsilon_\delta$, which satisfies $\Pi_p(t,\cdot;\varepsilon) \in Y_\delta$ for $0 \le t \le \mathrm{p}(\varepsilon)$ and $0 < \varepsilon \le \varepsilon_\delta$.

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Caption

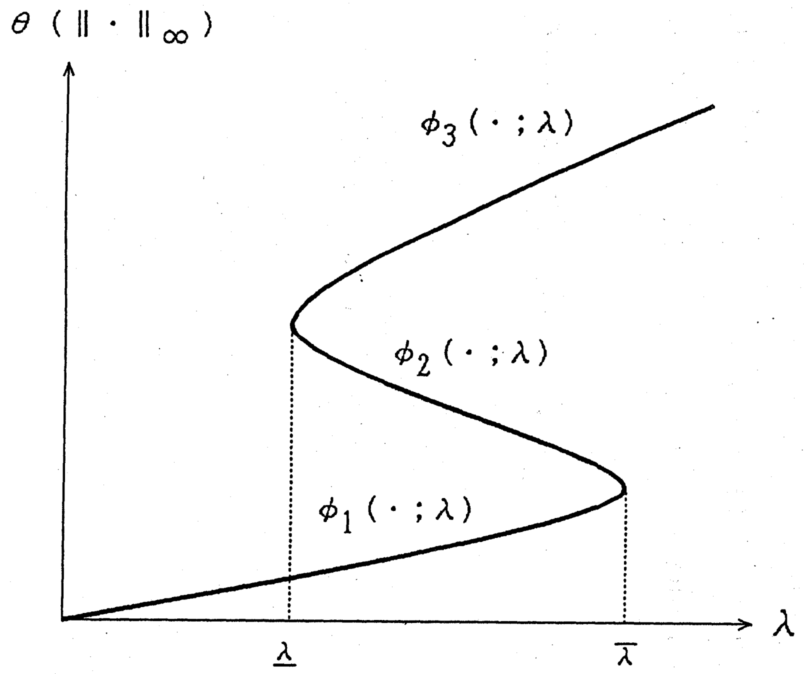
Fig. 1. Global diagram of stationary solutions of $(9)_\lambda$ with respect to λ.

Fig. 2. The graph of $H_i(\lambda)$ (i = 1,3).

Fig. 3. The graph of $F_i(\lambda)$ (i = 1,3) in the case that: i) $0 < c^* < H_*/a$; ii) $H_*/a < c^* < H^*/a$; iii) $c^* > H^*/a$.

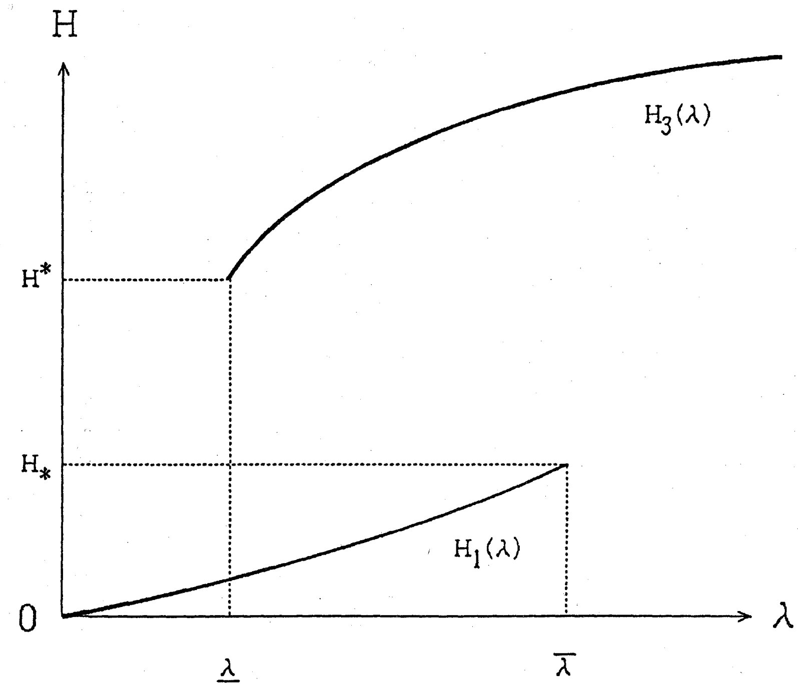
Fig. 4. Orbits of solutions of $(4)_\varepsilon$ in the case that: i) $0 < c^* < H_*/a$; ii) $H_*/a < c^* < H^*/a$; iii) $c^* > H^*/a$.

Fig. 5. Global structure of dynamics of $(4)_\varepsilon$ with respect to c^*.



θ ($\|\cdot\|_\infty$)
$\phi_3(\cdot;\lambda)$
$\phi_2(\cdot;\lambda)$
$\phi_1(\cdot;\lambda)$
λ
$\underline{\lambda}$
$\overline{\lambda}$

Fig. 1



H
$H_3(\lambda)$
H^*
H_*
$H_1(\lambda)$
0
λ
$\underline{\lambda}$
$\overline{\lambda}$

Fig. 2

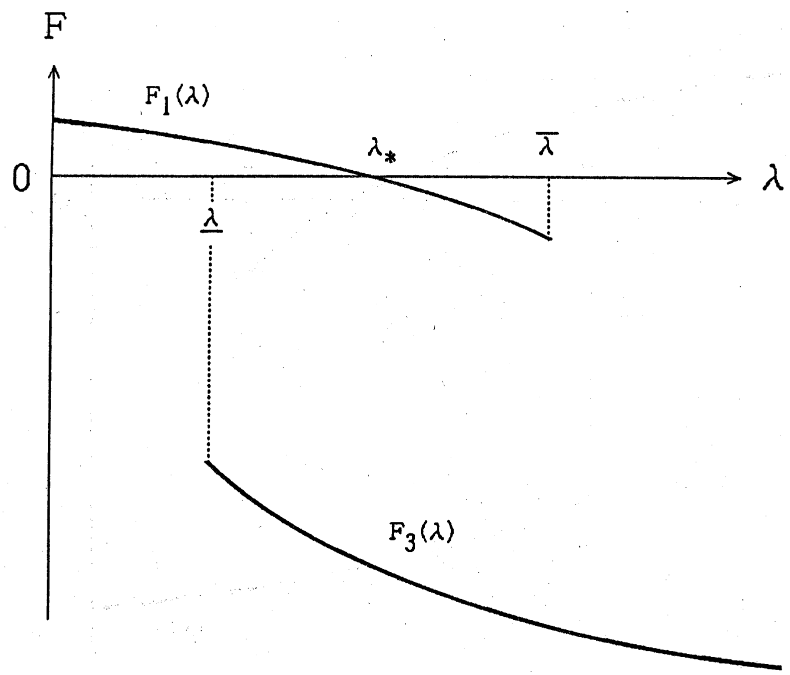

F
$F_1(\lambda)$
λ_*
$\overline{\lambda}$
0
λ
$\underline{\lambda}$
$F_3(\lambda)$

Fig. 3－1

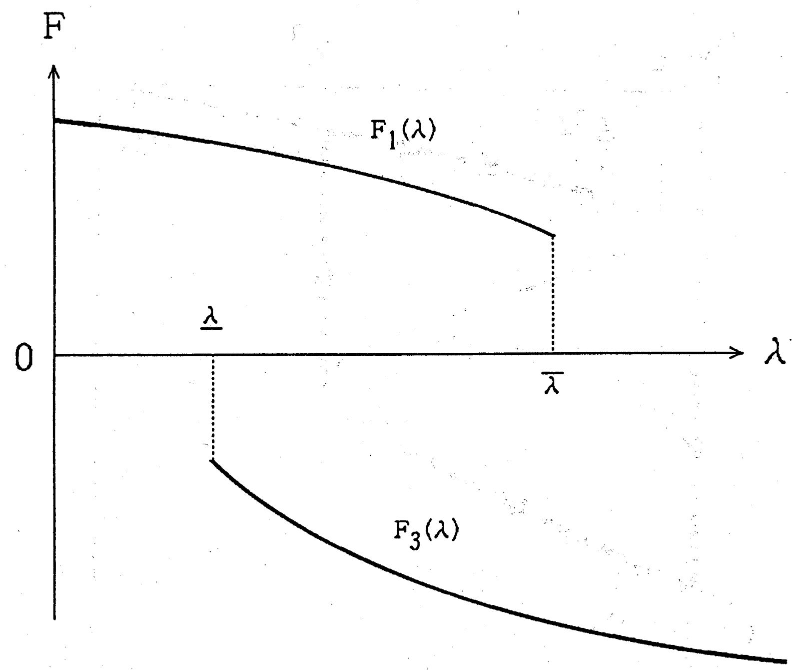

F
$F_1(\lambda)$
$\underline{\lambda}$
0
λ
$\overline{\lambda}$
$F_3(\lambda)$

Fig. 3－2

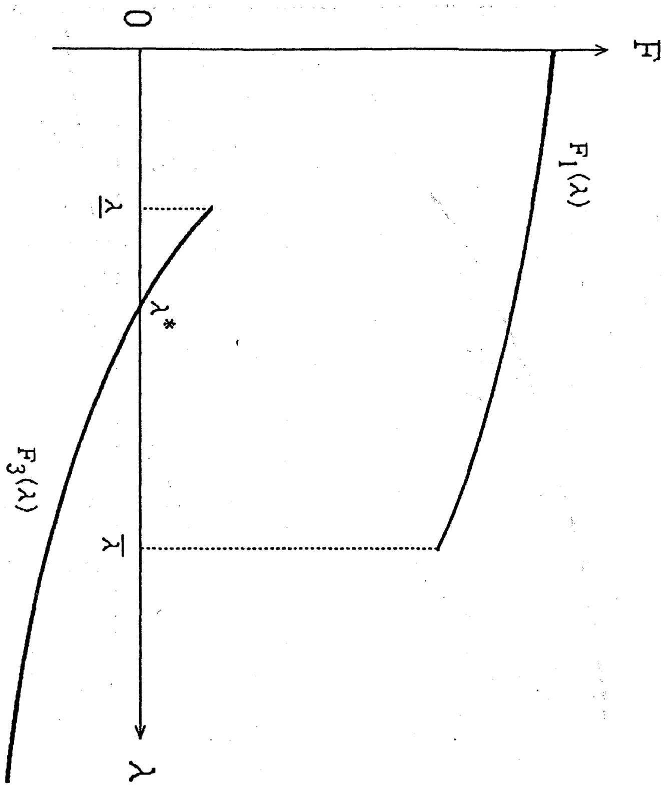

F
$F_1(\lambda)$
0
$\underline{\lambda}$
λ^*
$F_3(\lambda)$
$\overline{\lambda}$
λ

Fig. 3−3

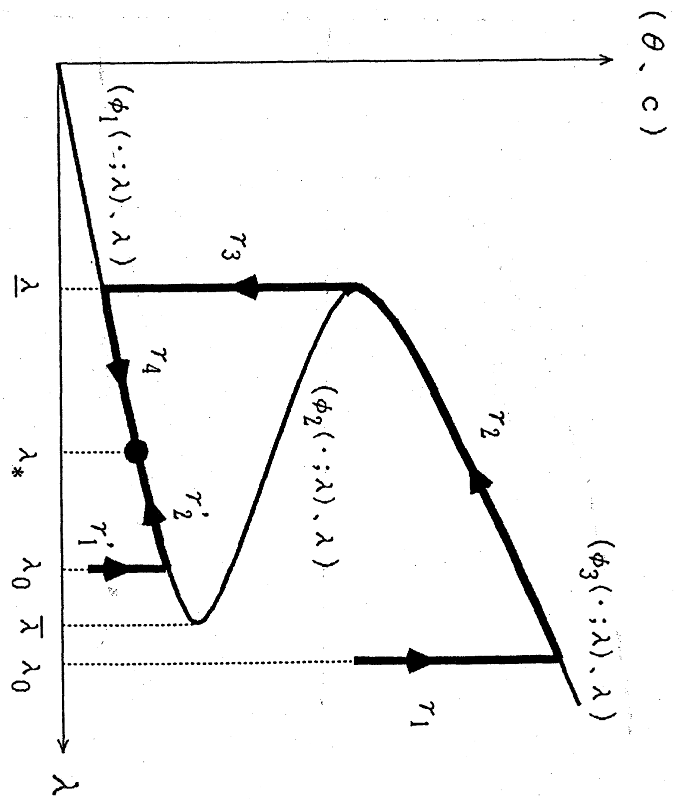

(θ, c)
$(\phi_1(\cdot;\lambda), \lambda)$
$(\phi_2(\cdot;\lambda), \lambda)$
$(\phi_3(\cdot;\lambda), \lambda)$
τ_1
τ_2
τ_3
τ_4
τ_1'
τ_2'
$\underline{\lambda}$
λ_*
λ_0
$\overline{\lambda}$
λ_0
λ

Fig. 4−1

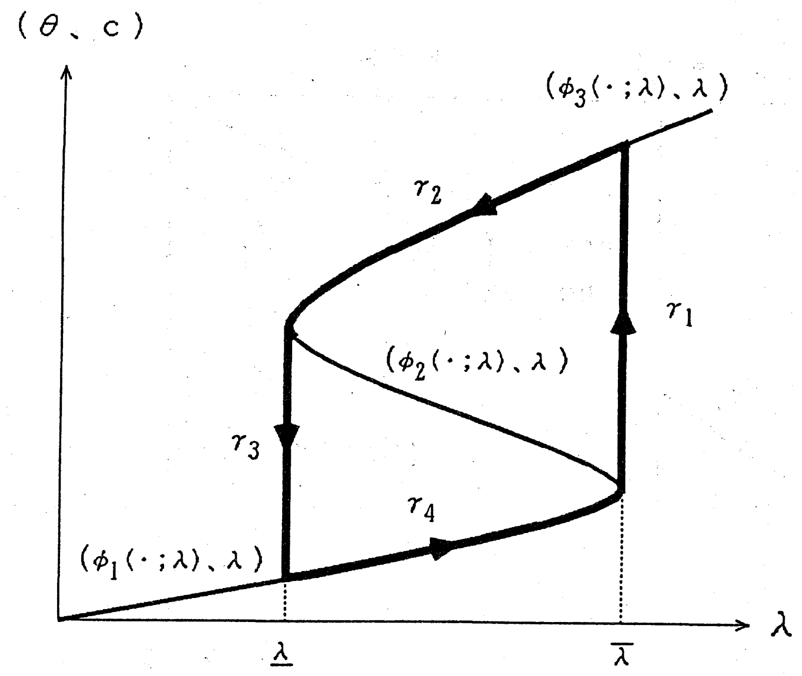


Fig. 4−2

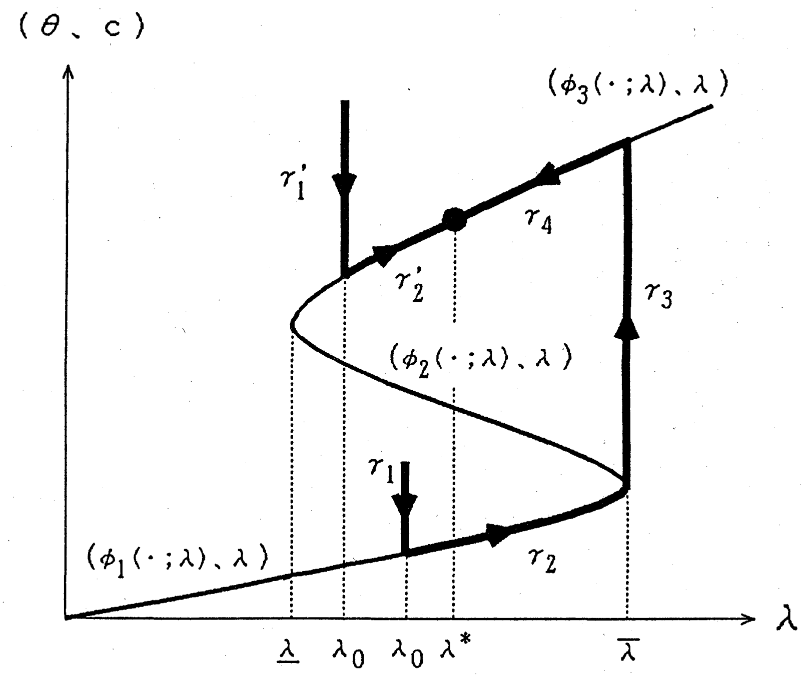


Fig. 4−3

19

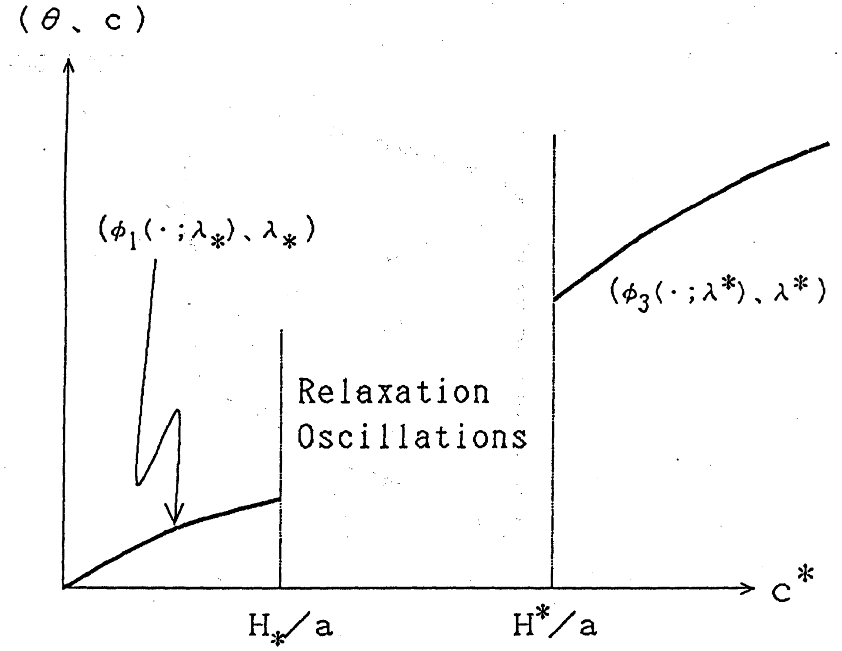


Fig. 5