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Asymptotics of Jackson integrals and torus embeddings

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1. We take the elliptic modulus $q = e^{2\pi i\tau}$, Im $\tau > 0$. Let X be an n dimensional integer lattice $\simeq \mathbf{Z}^n$. We put $\overline{X} = X \otimes \mathbf{C}^*$, the n dimensional algebraic torus twisted by q. Let $\chi_1, \chi_2, \ldots, \chi_n$ be a basis of X such that an arbitrary $\chi \in X$ can be uniquely written by $\chi = \sum_{j=1}^n \nu_j \chi_j$, $\nu_j \in \mathbf{Z}$. We may identify $\overline{X}$ isomorphic to $X \otimes (\mathbf{C}/(\frac{2\pi i}{\log q}))$ with the direct product of n pieces of $\mathbf{C}^*$. The inclusion $X \subset \overline{X}$ can be obtained by identifying χ_j with the element $t = (1,\ldots,1,q,1,\ldots,1) \in (\mathbf{C}^*)^n$. We denote by Q_j the shift operator $Q_j f(t) = f(\chi_j \cdot t)$ induced by the displacement $t \to \chi_j \cdot t$ for a function f on $\overline{X}$. We put $Q^\chi = Q_1^{\nu_1} \cdots Q_n^{\nu_n}$. We consider the q-difference equations

$$Q^\chi \Phi(t) = b_\chi(t)\Phi(t), \quad \chi \in X \text{ and } t \in \overline{X}, \tag{1.1}$$

for a set of rational functions $\{b_\chi(t)\}_{\chi\in X}$ on $\overline{X}$, which are not identiacally zero. $\{b_\chi(t)\}_{\chi\in X}$ satisfies the compatibility condition

$$b_{\chi+\chi'}(t) = b_\chi(t)\cdot Q^\chi b_{\chi'}(t), \tag{1.2}$$

so that $\{b_\chi(t)\}_{\chi\in X}$ defines a 1-cocycle on X with values in $R^\times(\overline{X})$ the multiplicative abelian group consisting of non-zero rational functions on $\overline{X}$. We denote by $R(\overline{X})$ the field of rational functions on $\overline{X}$. $\{b_\chi(t)\}_{\chi\in X}$ is a coboundary if and only if $b_\chi(t) = Q^\chi\varphi(t)/\varphi(t)$ for $\varphi \in R(\overline{X})$. We write the corresponding 1-cohomology by $H^1(X,R^\times(X))$.

We put $(x)_\infty = \Pi_{\nu=0}^\infty (1- x q^\nu)$ and $(x)_n = (x)_\infty/(xq^n)_\infty$ for $n \in \mathbf{Z}$. Then the following important result holds.

Lemma 1. An arbitrary cocycle $\{b_\chi(t)\}_{\chi\in X}$ can be expressed by (1.3), where Φ denotes a q-multiplicative function on $\overline{X}$ written by

$$\Phi = \Pi_{j=1}^n t_j^{\alpha_j} \Pi_{j=1}^m \frac{(a_j' t^{\mu_j})_\infty}{(a_j t^{\mu_j})_\infty} \tag{1.3}$$

for some non-negative integer m and α_j, a_j', $a_j \in \mathbf{C}$, where $\mu_j \in \check{X} = \mathrm{Hom}(X,\mathbf{Z})$. t^{μ_j} denotes a monomial $t_1^{\mu_j(\chi_1)} \cdots t_n^{\mu_j(\chi_n)}$. a_j or a_j' may vanish or may not.

This is a q-version of Sato's theorem in [S] and can be proved in a completely similar way.

We shall assume from now on that any of a_j and a_j' don't vanish. If we replace μ_j, a_j and a_j' by $-\mu_j$, $qa_j'^{-1}$ and qa_j^{-1} respectively in the factors of Φ, then

$$\Phi' = t^{(s_j-s_j')\mu_j} \frac{(qa_j^{-1}t^{-\mu_j})_\infty}{(qa_j'^{-1}t^{-\mu_j})_\infty} \Phi \tag{1.4}$$

also satisfies the same equation (1.1). It is convenient to write $\mu_{-j}= -\mu_j$, $a'_{-j}= qa_j^{-1}$, $a_{-j}= qa_j'^{-1}$ for $j \in \{\pm1,\ldots,\pm m\}$.

We denote by $\overline{\omega} = \frac{d_q t_1}{t_1} \wedge \cdots \wedge \frac{d_q t_n}{t_n}$ the canonical invariant n-form on $\overline{X}$. We consider the Jackson integral for a function f on $\overline{X}$ over an orbit $X \cdot \xi$, $\xi \in \overline{X}$ as follows :

$$(1.5) \qquad \tilde{f} = (1-q)^n \sum_{\chi \in X} Q^{\chi} f(\xi),$$

if it is summable.

We put $\alpha = N\eta + \alpha'$ and study the asymptotic behaviour of Jackson integrals $\tilde{\Phi}$ for $N \to +\infty$, $\eta \in X$ and $\alpha' \in \mathbf{C}^n$ being fixed. Since $\Phi(t) = (t^{\eta})^N \cdot t^{\alpha'} \cdot \Pi_{j=1}^m \frac{(a_j' t^{\mu_j})_\infty}{(a_j t^{\mu_j})_\infty}$, the major part of $|\Phi|$ is played by the absolute value $|t^{\eta}|$ for $N \to +\infty$. $|t^{\eta}|$ attains a maximum if and only if the level function $L_{\eta}(\log_q t) = \mathrm{Re}\,(\eta, \log_q t)$ is a minimum, where (η, λ) denotes $\eta(\lambda)$.

We are going to search for points $t = q^{\lambda}$ in $\overline{X}$ for $\lambda \bmod \frac{2\pi i}{\log_q} X$ satisfying the following 2 properties :

(i) $\log_q a_j' + (\mu_j, \lambda) \equiv 1,2,3,\ldots \bmod \frac{2\pi i}{\log q} \mathbf{Z}$, <u>for</u> $j \in J$, J <u>being a set of</u> n <u>arguments in</u> $\{\pm 1, \pm 2, \ldots, \pm m\}$ <u>such that</u> μ_j, $j \in J$, <u>are linearly independent</u>. <u>We denote by</u> $\overline{X}_J$ <u>the countable set in</u> $\overline{X}$ <u>consisting of these points</u> t.

(ii) $L_{\eta}(\lambda)$ <u>attains a minimum on the set</u> $\overline{X}_J$.

We say that a point $t = q^{\lambda}$ satisfying (i) and (ii) is a <u>critical point with respect to the level function</u> $L_{\eta}(\lambda)$. We denote by Cr_J the set of all critical points in $\overline{X}_J$ and by $Cr(L_{\eta})$ the union $\bigcup_J Cr_J(L_{\eta})$.

Now we make the following assumptions of genericity.

Ass 1. <u>For each</u> J, <u>the set</u> Cr_J <u>is finite or empty</u>. <u>We denote by</u> κ_J <u>its number</u> :

(1.6) $$Cr_J(L_\eta) = \{\xi_J^{(1)}, \ldots, \xi_J^{(\kappa_J)}\}.$$

<u>Assume that</u> $L_\eta(\xi_J^{(r)}) \neq L_\eta(\xi_J^{(s)})$ <u>for every pair</u> r,s, $r \neq s$. <u>Then</u> κ_J <u>turns out equal to</u> $[\mu_{j_1}, \ldots, \mu_{j_n}]^2$ <u>or</u> 0. <u>We say that</u> J <u>is stable if</u> $\kappa_j > 0$.

Ass 2 . <u>Any two critical points</u> $\xi_J^{(r)}$ <u>and</u> $\xi_K^{(s)}$ <u>for</u> $J \neq K$ <u>are</u> X-<u>inequivalent</u>.

From these assumptions we see that, for each $J = \{j_1, \ldots, j_n\}$, $J \subset \{\pm 1, \ldots, \pm m\}$, the only one choice of signs $\{\varepsilon_1 j_1, \ldots, \varepsilon_n j_n\}$ is stable for $\varepsilon_\nu = \pm 1$. This occurs if and only if

(1.7) $$[\eta, \varepsilon_1\mu_{j_1}, \ldots, \varepsilon_{\nu-1}\mu_{j_{\nu-1}}, \varepsilon_{\nu+1}\mu_{j_{\nu+1}}, \ldots, \varepsilon_n\mu_{j_n}](-1)^{\nu-1} > 0$$

for all ν. Hence the total number of critical points $\kappa = \#|C_r(L_\eta)|$ is given by

(1.8) $$\kappa = \sum_J [\mu_{j_1}, \ldots, \mu_{j_n}]^2$$

$$= \det\left(\left(\sum_{j=1}^m \mu_j(\chi_r)\mu_j(\chi_s)\right)\right)_{1 \le r, s \le n}.$$

Under the above 2 assumptions, we deduce the crucial

Lemma 2. $t = q^\lambda$ <u>is critical for</u> $L_\eta(\lambda)$ <u>if and only if</u>

(1.9) $$b^-_\chi(q^{\lambda-\chi}) = 0$$

<u>for any</u> $\chi \in X$ <u>such that</u> $(\eta,\chi) > 0$.

<u>Def</u> 1. We denote by $c(\xi)$ the set of all $t = \chi\cdot\xi$, $\chi \in X$, such that $L_\eta(\log_q t) \geq L_\eta(\log_q \xi)$ so that ξ is a minimum point in $c(\xi)$. We call such a $c(\xi)$ <u>stable cycle</u> if ξ is critical. There are κ stable cycles say $c(\xi^{(1)}),\dots,c(\xi^{(\kappa)})$.

We fix $\eta \in X$ under Ass 1 ~ 2. $\tilde{\Phi}$ is meromorphic in $u \in (\mathbf{C}^*)^n$ and satisfies the system of linear q-difference equations $(\mathcal{E})$:

(1.10) $$(b^-_\chi(\tilde{Q})u^{-\chi} - b^+_\chi(\tilde{Q}))\tilde{\Phi} = 0 , \quad \text{for } \chi \in X.$$

These are equivalent to the subsystem $(\mathcal{E}^+)$:

(1.11) $$(b^-_\chi(\tilde{Q})u^{-\chi} - b^+_\chi(\tilde{Q}))\tilde{\Phi} = 0 , \quad \text{for } \chi \in X,$$

such that $(\eta,\chi) > 0$.

If $\tilde{\Phi}$ has an asymptotic behaviour

(1.12) $$\tilde{\Phi} \sim u_1^{\lambda_1}\cdots u_n^{\lambda_n} (1 + O(\tfrac{1}{N})),$$

for $\alpha = \eta N + \alpha'$, $N \to +\infty$, then q^λ must satisfy (1.9), i.e. $t = q^\lambda$ coincides with a critical point of the function $L_\eta(\lambda)$. (1.8) shows that the number of such points is equal to κ, whence the number of asymptotic solutions of $(\mathcal{E})$ is also equal to κ. The corresponding solutions are given by the Jackson integrals of Φ over the q-cycles

$c(\xi^{(s)})$ for $\xi^{(s)} \in C_r(L_\eta)$.

Theorem. Under Ass 1 and 2, $(\mathcal{E}^+)$ has κ linearly independent solutions which have asymptotic behaviours (1.12) satisfying (1.11) in a generic direction $\eta \in X$. These solutions are given by the Jackson integrals over the κ stable cycles defined in Def 1.

For arbitrary n-1 linearily independent $\mu_{j_1},\dots,\mu_{j_{n-1}}$ the equation

$$\det[\eta,\mu_{j_1},\dots,\mu_{j_{n-1}}] = 0 \tag{1.13}$$

defines a rational hyperplane H_J in $\check{X}_R = \check{X} \otimes R$. Each connected component of the complement $X_R - \cup_J H_J$ defines a finite rational polyhedral cone σ so that $X_R - \cup_J H_J$ gives a fan F in the sense of torus embedding. According to the theory of torus embedding, to this fan F there corresponds to a toric variety $T_{emb}(F)$. There also exists a subdivision $\hat{F}$ of F such that for each cone σ in $\hat{F}$ and $\eta \in \sigma$, there exist κ asymptotics of the form (1.12) which do not depend on the choice of $\eta \in \sigma$.

2. Example.

Let $\Phi = \Pi_{j=1}^{n} t_j^{\alpha_j} \prod_{0 \le i < j \le n} \frac{(a'_{i,j} t_j/t_i)_\infty}{(a_{i,j} t_j/t_i)_\infty}$, for $t_0 = 1$ and $m = \binom{n+1}{2}$. $\mu_j(\chi) = \nu_k - \nu_\ell$ for $k \ne \ell$ (we put $\nu_0 = 0$). $[\mu_{j_1},\dots,\mu_{j_n}] = \pm 1$, or 0. $\sum_{j=1}^{m} \mu_j(\chi_r)\mu_j(\chi_s)$ equal n or -1 according as r = s or r ≠ s. κ is then equal to $(n+1)^{n-1}$. This case has been investigated in more detail in [A]. (1.9) is given explicitely by

$$(2.1) \qquad H_J : \quad \eta_{j_1} + \cdots + \eta_{j_r} = 0 ,$$

for $J = \{j_1, \dots, j_r\} \subset \{1,2,\dots,n\}$ and $n \geq r \geq 1$. $\hat{F}$ coincides with F consisting of connected components of the complement $\check{X}_R - \cup_J H_J$. Hence the number of different asymptotic directions $\psi(n)$ is equal to the one of connected components $\check{X}_R - \cup_J H_J$. For $n = 1, 2, 3, 4$ they are equal to 2, 6, 32, 370 respectively but generally unknown to me.

It is an interesting problem to find out the connection formula among asymptotic solutions of (1.10) along $\psi(n)$ different directions.

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