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The cohomology groups of degree 3 of Siegel modular varieties of genus 2

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§0. Introduction.

We discuss two types of modular symbols on Siegel modular varieties of genus 2: One is holomorphic, another totally real. In §1, we give a simple result for the first case. In §2, on the second case we discuss a conjecture on its Hodge type, and explain how to reduce it to another conjectures on L-functions.

<u>Notation.</u> $Sp(2;\mathbb{R})$: the real symplectic group of rank 2. $Sp(2;\mathbb{Z})$ = the integral symplectic subgroup in $Sp(2;\mathbb{R})$. l : natural number ≥ 3. $\Gamma(l)$ = the principal congruence subgroup of $Sp(2;\mathbb{Z})$. $\mathfrak{h}_2$ = the Siegel upper half space of genus 2. $V = \Gamma(l)\backslash\mathfrak{h}_2$ = the quotient of $\mathfrak{h}_2$ by $\Gamma(l)$, which is a smooth algebraic variety over $\mathbb{C}$. $\tilde{V}$ = a smooth toroidal compactification of V.

§1. Modular subvarieties and coniveau filtration.

(1.1) Coniveau filtration.

Let X be a smooth compact algebraic variety over $\mathbb{C}$ with dimension n. Let Y be a subvariety of dimension m in X. Then by restriction, we have a natural map of Hodge structures: $H^k(X, \mathbb{Q}) \longrightarrow H^k(Y, \mathbb{Q})$. If Y is smooth, then by Poincaré duality, we have $H^{2m-k}(Y, \mathbb{Q})(m) \longrightarrow H^{2n-k}(X, \mathbb{Q})(n)$. Or, equivalently, we have $H^i(Y, \mathbb{Q})(-d) \to H^{i+2d}(X, \mathbb{C})$. Here (m) etc. are Tate twists. In general, we have an exact sequence of local cohomology

$$\longrightarrow H^i_Y(X, \mathbb{Q}) \to H^i(X, \mathbb{Q}) \longrightarrow H^i(X-Y, \mathbb{Q}) \to \cdots\cdots .$$

By Deligne, this is an exact sequence of mixed Hodge structures.

When Y is smooth, we have $H^i_Y(X, \mathbb{Q}) \cong H^{i-2d}(Y, \mathbb{Q})(-d)$ by Gysin isomorphism.

Definition-Proposition. The rational sub Hodge structure of $H^i(X, \mathbb{Q})$ defined by

$$F^d H^i(X, \mathbb{Q}) := \sum_{\substack{Y \subset X \\ \text{codimension } d}} \mathrm{Im}\{H^i_Y(X, \mathbb{Q}) \to H^i(X, \mathbb{Q})\}$$

has Hodge type $\{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a+b = i,\ a \geq d,\ b \geq d\}$.

The above filtration is called "coniveau" filtration by Grothendieck. It is assume that that this filtration has a crucial role in the arithmetic of cycles on X.

(1.2) The "boundary part" of Hodge structure $H^3(\tilde{V}, \mathbb{Q})$.

Let $\tilde{V}$ be a smooth toroidal compactification of V along the cusps. Put $D = \tilde{V} - V$ and $D = \bigcup_i D_i$ the decomposition into irreducible components D_i. Since each D_i is smooth in $\tilde{V}$, the restriction map $H^3(\tilde{V}, \mathbb{Q}) \longrightarrow H^3(D_i, \mathbb{Q})$ define a morphism of rational Hodge structures $\rho_{D_i}: H^1(D_i, \mathbb{Q})(-1) \longrightarrow H^3(\tilde{V}, \mathbb{Q})$.

<u>Definition</u>. $H^3(M_\infty, \mathbb{Q}) \underset{\text{defn}}{:=} \sum_i \operatorname{Im} \rho_{D_i}$.

$H^3(M_\infty, \mathbb{Q})$ is a polarized sub-Hodge structure with Hodge type $\{(2,1), (1,2)\}$. If we take the Igusa model for $\tilde{V}$ as [Yamazaki], each D_i is an elliptic modular surface over a modular curve C_i. Then $H^1(D_i, \mathbb{Q}) \cong H^1(C_i, \mathbb{Q})$.

(1.3) <u>Hilbert modular surfaces</u>

By modular embedding, or Satake embedding, a Hilbert modular surface S is mapped to V; $S \xrightarrow{f} V$. We can compactify f: $\tilde{S} \xrightarrow{\tilde{f}} \tilde{V}$. Since $b_1(\tilde{S}) = 0$ for any Hilbert modular surface, $H^1(\tilde{S}, \mathbb{Q})(-1) \longrightarrow H^3(\tilde{V}, \mathbb{Q})$ is zero map.

(1.4) <u>"Diagonal" embedding</u>.

Let $(z_1, z_2) \in H \times H \longrightarrow \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \in \mathfrak{H}_2$ be the holomorphic map from the product of the upper half plane H to $\mathfrak{h}_2$. This induces a holomorphic map from a product $E_\alpha = R_1 \times R_2$ of elliptic modular curves R_i $(i=1,2)$ to V : $f_{E_\alpha} : E_\alpha = R_1 \times R_2 \to V$.

By Künneth formula,

$$H^1(\tilde{E}_\alpha, \mathbb{Q}) = H^1(\tilde{R}_1, \mathbb{Q}) \oplus H^1(\tilde{R}_2, \mathbb{Q}).$$

In general, $H^1(\tilde{R}_i, \mathbb{Q}) \neq \{0\}$, hence $H^1(\tilde{E}_\alpha, \mathbb{Q}) \neq \{0\}$.

(1.5) <u>Proposition</u> Consider the Poincaré dual $\rho_{E_\alpha} : H^1(\tilde{E}_\alpha, \mathbb{Q}(-1)) \to H^3(\tilde{V}, \mathbb{Q})$ of the restriction map $H^3(\tilde{V}, \mathbb{Q}) \longrightarrow H^3(\tilde{E}_\alpha, \mathbb{Q})$. Then $\operatorname{Im} \rho_\alpha$ is contained in $H^3(M_\infty, \mathbb{Q})$.

(1.6) <u>Proof of Proposition</u>

Recall that $H^1(\tilde{V}, \mathbb{Q}) = \{0\}$. Hence the intersection form

$$\Phi : H^3(\tilde{V}, \mathbb{Q}) \times H^3(\tilde{V}, \mathbb{Q}) \longrightarrow \mathbb{Q}(-3)$$

is non-degenerate, and its restriction to $H^3(M_\infty, \mathbb{Q})$ is also non-degenerate. Therefore, if $\operatorname{Im} \rho_{E_\alpha} \not\subset H^3(M_\infty, \mathbb{Q})$ is true, then there exists a cohomology class $\xi \neq 0 \in \operatorname{Im} \rho_{E_\alpha}$ such that for any $\eta \in H^3(M_\infty, \mathbb{Q})$ $\Phi(\xi, \eta) = 0$. Thus it suffices to show the following.

(1.7). <u>Lemma</u>. For any $\xi \neq 0$ in $\operatorname{Im} \rho_{E_\alpha}$, there exists $\eta \in H^3(M_\infty, \mathbb{Q})$ such that $\Phi(\xi, \eta) \neq 0$.

In order to prove the above Lemma, it is necessary to write the intersection number $\Phi(\xi, \eta)$ in terms of

$$\xi' \in H^1(\widetilde{E}_\alpha, \mathbb{Q})(-1) \text{ and } \eta' \in H^1(D_i, \mathbb{Q})(-1),$$

if we choose ξ', η' by

$$\rho_{E_\alpha}(\xi') = \xi, \quad \rho_{D_i}(\eta') = \eta.$$

(1.8) Lemma Let Y be the intersection of $\widetilde{E}_\alpha$ and D_i. Y is possibly empty. Let us consider the morphisms by restriction:

$$\begin{cases} r_1 : H^1(\widetilde{E}_\alpha, \mathbb{Q})(-1) \longrightarrow H^1(Y, \mathbb{Q})(-1); \\ r_2 : H^1(D_i, \mathbb{Q})(-1) \longrightarrow H^1(Y, \mathbb{Q})(-1). \end{cases}$$

Then

$$\Phi(\xi, \eta) = \Phi_Y(r_1(\xi'), r_2(\eta')).$$

Here Φ_Y is the intersection form of the curve Y:

$$\Phi_Y : H^1(Y, \mathbb{Q}) \times H^1(Y, \mathbb{Q}) \longrightarrow \mathbb{Q}(-1).$$

The above Lemma is a standard fact in the intersection theory, and may be found in the text book of [Fulton].

Thus the proof of Lemma (1.7) is reduced to the following

(1.9) Lemma For any $\xi \neq 0$ in $\mathrm{Im}\, \rho_{E_\alpha}$, there exists an element $\eta' \in H^1(D_i, \mathbb{Q})$ of some irreducible component D_i of D such that for $Y = D_i \cap \widetilde{E}_\alpha$, $\Phi_Y(r_1(\xi'), r_2(\eta')) \neq 0$.

This Lemma is immediate from the following result of [Yamazaki].

(1.10) Proposition. For $\check{E}_\alpha = \tilde{R}_1 \times \tilde{R}_2 \longrightarrow \tilde{V}$, there exists an irreducible component D_i of D such that for $Y = D_i \cap \tilde{E}_\alpha$,

$$H^1(D_i, \mathbb{Q}) \cong H^1(Y, \mathbb{Q}).$$

This completes the proof of Proposition (1.5).

§2. Conjectures on totally real modular symbols

(2.1) In the construction of L-functions associated to Siegel modular forms of genus 2, Andrianov considered a kind of totally real embedding of hyperbolic three spaces into the Siegel upper half space of degree 2.

K is an imaginary quadratic field, $SL_2(O_K)$ is the special linear group with entries in the ring of integers O_K of K, which is a discrete subgroup of $SL_2(\mathbb{C})$. For some congruence subgroup Γ' of $SL_2(O_K)$, there exists a map

$$m: \Gamma' \backslash SL_2(\mathbb{C})/K' \longrightarrow \Gamma(\ell) \backslash Sp(2; \mathbb{R})/K$$

induced from an injective homomorphism $SL_2(\mathbb{C}) \longrightarrow Sp(2; \mathbb{R})$. Here K' and K are maximal compact subgroups of $SL_2(\mathbb{C})$ and $Sp(2; \mathbb{R})$, respectively.

Let $\overline{V}$ be the Satake compactification of $V = \Gamma(\ell)\backslash \mathfrak{H}_2$. Then the canonical compactification of m defines an element in $H_3(\overline{V}; \mathbb{Z})$. More careful investigation show the following.

(2.2) Lemma $\overline{m}$ defines an element $[\overline{m}]$ in $IH_3(\overline{V}; \mathbb{Z})$.

Here $IH_3(\overline{V}; \mathbb{Z})$ is the intersection homology group of degree 3 of $\overline{V}$ with middle perversity.

We denote by $[\overline{m}]^* \in IH^3(\overline{V}; \mathbb{C})$ the Poincaré dual of the image of the fundamental class $[\overline{m}]$ of $\Gamma'\backslash SL_2(O_K)/K'$.

The cohomology group $IH^3(\overline{V}; \mathbb{C})$ has a Hodge structure defined by Morihiko Saito.

(2.3) Conjecture. The cycle $[\overline{m}]^* \in IH^3(\overline{V}, \mathbb{C})$ has Hodge type $\{(3,0); (0,3)\}$.

The above conjecture is reduced to the following conjectures for L functions for harmonic forms on V.

(2.4) Conjecture Let ω be a L^2-harmonic form on V of type (2.1). Assume that ω is an common eigen-form of all Hecke operators. Then,

(i) Let $L(s,\omega)$ be the L-function for Spinor representation associated to ω. Then $L(s,\omega)$ has a possibly simple pole at $s=2$, &

$$\mathrm{Res}_{s=2}\ L(s,\omega)=\int_{\gamma}\omega,$$

with γ is a linear combination of cycles $[m]\in IH_3(\bar{V};\mathbb{Z})$.

(ii) $L(s,\omega)$ is an entire function if it is multiplied with the Γ-factor $\Gamma(s)\Gamma(s-1)$.

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