数理解析研究所講究録
第 739 巻 1991 年 178-185

Reproductive property of the Boussinesq equations

by

Hiroko MORIMOTO

(School of Science and Technology, Meiji University)

Let Ω be a bounded domain in R^n with the boundary $\partial\Omega$ such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \phi$. We consider the following initial boundary value problem:

$$(1)\quad \begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u = -\dfrac{1}{\rho}\nabla p + \nu\Delta u + \beta g\theta, & \\ \operatorname{div} u = 0, & x \in \Omega,\ t > 0, \\ \dfrac{\partial\theta}{\partial t} + (u\cdot\nabla)\theta = \kappa\Delta\theta, & \end{cases}$$

$$(2)\quad \begin{cases} u(x,t) = 0,\ \theta(x,t) = \xi(x,t), & x \in \Gamma_1,\ t > 0, \\ u(x,t) = 0,\ \dfrac{\partial}{\partial n}\theta(x,t) = 0, & x \in \Gamma_2,\ t > 0, \end{cases}$$

$$(3)\quad \begin{cases} u(x,0) = a_0(x), & \\ \theta(x,0) = \tau_0(x), & x \in \Omega, \end{cases}$$

where $u = (u_1, u_2, ..., u_n)$ is the fluid velocity, p is the pressure, θ is the temperature, $u\cdot\nabla = \sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j}$, $\frac{\partial\theta}{\partial n}$ denotes the outer normal derivative of θ at x to $\partial\Omega$, $g(x,t)$ is the gravitational vector function, and ρ(density), ν(kinematic viscosity), β(coefficient of volume expansion), κ(thermal conductivity) are positive constants. $\xi(x,t)$ is a function defined on $\Gamma_1 \times (0,T)$, and $a_0(x)$ (resp. $\tau_0(x)$) is a vector (resp. scalar) function defined on Ω.

This system of equations describes the motion of fluid of heat convection (Boussinesq approximation). The existence of a weak solution of this system and its reproductive property are discussed in this report. For the definition of weak solution, we use the following auxiliary function and solve the system of equation corresponding to it.

Let $\theta_0(x,t)$ be a function defined on $\bar{\Omega} \times [0,T]$ such that

$$\theta_0(x,t) = \xi(x,t), \qquad x \in \Gamma_1,\ t > 0,$$
$$\frac{\partial}{\partial n}\theta_0(x,t) = 0, \qquad x \in \Gamma_2,\ t > 0.$$

We can transform the equations (1),(2), (3) for u and $\tilde{\theta} = \theta - \theta_0$, and we obtain the following:

$$(4) \quad \begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u = -\dfrac{1}{\rho}\nabla p + \nu\Delta u + \beta g\tilde{\theta} + \beta g\theta_0, \\ \mathrm{div}\ u = 0, & x \in \Omega,\ t > 0, \\ \dfrac{\partial\tilde{\theta}}{\partial t} + (u\cdot\nabla)\tilde{\theta} = \kappa\Delta\tilde{\theta} + \kappa\Delta\theta_0 - (u\cdot\nabla)\theta_0 - \dfrac{\partial\theta_0}{\partial t}, \end{cases}$$

$$(5) \quad \begin{cases} u(x,t) = 0,\ \tilde{\theta}(x,t) = 0, & x \in \Gamma_1,\ t > 0, \\ u(x,t) = 0,\ \dfrac{\partial}{\partial n}\tilde{\theta}(x,t) = 0, & x \in \Gamma_2,\ t > 0, \end{cases}$$

$$(6) \quad \begin{cases} u(x,0) = a_0(x) \\ \tilde{\theta}(x,0) = \tau_0(x) - \theta_0(x,0), & x \in \Omega. \end{cases}$$

We suppose boundary $\partial\Omega$ of Ω satisfies the following:

<u>Condition (H)</u>

$\partial\Omega$ is of class C^2 and devided as follows:

$$\partial\Omega = \Gamma_1 \cup \Gamma_2\ ,\ \Gamma_1 \cap \Gamma_2 = \phi,$$
$$\text{measure of } \Gamma_1 \neq 0,$$

and the intersection $\bar{\Gamma}_1 \cap \bar{\Gamma}_2$ is a n-1 dimensional C^1 manifold.

This condition is used in order to extend the function $\xi(x,t)$(Lemma 2).

Now we define the solenoidal function spaces:

D_σ = {vector function $\varphi\in C^\infty(\Omega)$ | supp $\varphi \subset \Omega$, div $\varphi=0$ in Ω},

H = completion of D_σ under the $L_2(\Omega)$-norm ,

V = completion of D_σ under the $H^1(\Omega)$-norm .

D_0 = { scalar function $\varphi \in C^\infty(\bar{\Omega})$ |

$\varphi \equiv 0$ in a neighborhood of Γ_1} ,

W = completion of D_0 under the $H^1(\Omega)$-norm .

Let $\tilde{V}$ be the completion of D_σ under the norm $\|u\|_{L^n(\Omega)}+ \|u\|_V$, and $\tilde{W}$ the completion of D_0 under the norm $\|\theta\|_{L^n(\Omega)}+ \|\theta\|_W$. For $2 \le n \le 4$, $\tilde{V} = V$ and $\tilde{W} = W$ because of Sobolev's imbedding theorem(e.g.Adams[1]).

First we study the auxiliary problem (4),(5),(6). We take the inner product of $v \in \tilde{V}$ (resp.$\tau\in \tilde{W}$) and the first equation of (4) (resp. the third equation of (4)) and we obtain:

$$(7)\quad \begin{cases} \dfrac{d}{dt}(u,v) + ((u\cdot\nabla)u,v) \\ \qquad = -\nu(\nabla u,\nabla v) + (\beta g\tilde{\theta},v) + (\beta g\theta_0,v), \qquad v \in \tilde{V}, \\ \dfrac{d}{dt}(\tilde{\theta},\tau) + ((u\cdot\nabla)\tilde{\theta},\tau) \\ \qquad = -\kappa(\nabla\tilde{\theta},\nabla\tau) - ((u\cdot\nabla)\theta_0,\tau) - \dfrac{d}{dt}(\theta_0,\tau) -\kappa(\nabla\theta_0,\nabla\tau), \\ \qquad\qquad \tau \in \tilde{W}. \end{cases}$$

Definition 1.

A pair of functions $\{u,\tilde{\theta}\}$ is called a weak solution of (4),(5), if $u \in L^2(0,T:V)$ and $\tilde{\theta} \in L^2(0,T:W)$ and they satisfy the equation (7) in distribution sense $\mathcal{D}'(0,T)$.

If we suppose merely $u \in L^2(0,T:V)$ and $\tilde{\theta} \in L^2(0,T:W)$, the condition (6) doesn't necessarily make sense but we have:

Lemma 1.

Suppose

$$g \in L^{\infty}(\Omega \times (0,T)),$$

$$\theta_0 \in C^1(\bar{\Omega} \times [0,T]),$$

$$u \in L^2(0,T:V),$$

$$\tilde{\theta} \in L^2(0,T:W)$$

and $\{u,\tilde{\theta}\}$ satisfy (7). Then u (resp. $\tilde{\theta}$) is equal to an absolutely continuous function from $[0,T]$ into $\tilde{V}'$(resp. $\tilde{W}'$), where $\tilde{V}'$(resp. $\tilde{W}'$) is the dual space of $\tilde{V}$(resp. $\tilde{W}$).

Therefore conditions

$$u(x,0) = a_0(x), \quad \tilde{\theta}(x,0) = \tau_0(x) - \theta_0(x,0)$$

make sense.

Now we define the weak solution of (1),(2).

Definition 2

A pair of functions $\{u,\theta\}$ is called a weak solution of (1),(2) if there exists a function $\theta_0(x,t) \in C^1(\bar{\Omega} \times [0,T])$ such that

$$u \in L^2(0,T:V),$$

$$\theta - \theta_0 \in L^2(0,T:W),$$

$$\theta_0(x,t) = \xi(x,t), \qquad x \in \Gamma_1,\ t \in (0,T),$$

$$\frac{\partial}{\partial n}\theta_0(x,t) = 0, \qquad x \in \Gamma_2,\ t \in (0,T),$$

and $\{u,\tilde{\theta}\}$ ($\tilde{\theta} = \theta - \theta_0$) is a weak solution of (4),(5).

As for the boundary condition, we extend the function $\xi(x,t)$ defined on $\bar{\Gamma}_1 \times [0,T]$, onto $\bar{\Omega} \times [0,T]$ satisfying certain smallness condition.

<u>Lemma 2</u> (Whitney)

Suppose Ω satisfies Condition (H) and $\xi(x,t) \in C^1(\overline{\Gamma}_1 \times [0,T])$. Then for every $\varepsilon > 0$ and $p > 1$, there exists a function $\theta_0(x,t)$ such that

$$\theta_0 \in C^1(\overline{\Omega} \times [0,T]),$$

$$\theta_0(x,t) = \xi(x,t), \qquad x \in \Gamma_1, \quad t > 0 ,$$

$$\frac{\partial}{\partial n}\theta_0(x,t) = 0, \qquad x \in \Gamma_2, \quad t > 0,$$

$$\sup_{0 \le t \le T} \|\theta_0(t)\|_{L^p(\Omega)} < \varepsilon .$$

The proof is similar to that in §4 of Morimoto[7]. See also [6]. Using this result, the existence of a weak solution of (1),(2) satisfying (3) is proved for $2 \le n \le 4$, in a similar way to J.L.Lions [4],[5], R.Temam [9]. The argument is based on the construction of approximate solutions by the Galerkin method and a passage to the limit where we use an a priori estimate on a fractional derivative in time of the approximate solutions and a compactness theorem (Morimoto [8]).

Our results are the following theorems.

<u>Theorem 1</u> (Existence of weak solutions)

Let n be an integer $2 \le n \le 4$, and Ω a bounded domain in R^n with C^2 boundary satisfying Condition(H). If the function $g(x,t)$ is in $L^\infty(\Omega \times (0,T))$, then for any ξ in $C^1(\overline{\Gamma}_1 \times [0,T])$, a_0 in H , τ_0 in $L^2(\Omega)$, there exists a weak solution $\{u,\theta\}$ of (1),(2) satisfying the initial condition (3).

Furthermore

$$u \in L^\infty(0,T:H) , \quad \theta \in L^\infty(0,T:L^2(\Omega)).$$

Theorem 2 (Uniqueness for n = 2)

Let n = 2. The weak solution $\{u,\theta\}$ of (1),(2) satisfying the initial condition (3) is unique if

$$u \in L^{\infty}(0,T:H),\ \theta \in L^{\infty}(0,T:L^2(\Omega)).$$

Theorem 3 (Uniqueness for $n \geq 3$)

Let $n \geq 3$. The weak solution $\{u,\theta\}$ of (1),(2) satisfying the initial condition (3) is unique on condition that :

$$u \in L^2(0,T:V) \cap L^{\infty}(0,T:H)$$

$$\theta \in L^2(0,T:H^1(\Omega)) \cap L^{\infty}(0,T:L^2(\Omega))$$

and

$$u \in L^s(0,T:L^r(\Omega)) \text{ and } \theta \in L^s(0,T:L^r(\Omega)),$$

hold for some $r > n$, $s = 2r/(r-n)$.

Let $\{u,\theta\}$ be a weak solution of (1),(2). If they satisfy the following condition:

$$(8)\quad \begin{cases} u(x,0) = u(x,T), \\ \theta(x,0) = \theta(x,T), \end{cases}$$

then we say they have reproductive property(Kaniel-Shinbrot [3]). Under some conditions on ν,κ,β,g and Ω, we can show the existence of solutions with reproductive property.

Theorem 4

Let $2 \leq n \leq 4$, and Ω be a bounded domain in R^n with C^2 boundary satisfying Condition (H). Let $g(x,t)$ be in $L^{\infty}(\Omega \times (0,T))$ and ξ in $C^1(\overline{\Gamma}_1 \times [0,T])$. Set $g_{\infty} = \|g\|_{L^{\infty}(\Omega\times(0,T))}$.

If $\frac{\beta g_\infty}{\sqrt{\nu\kappa}}$ is sufficiently small, then there exists a weak solution of (1),(2) satisfying (8). Furthermore

$$u \in L^\infty(0,T:H),$$

$$\theta \in L^\infty(0,T:L^2(\Omega)).$$

For the proof, we use Leray-Schauder's fixed point theorem ([2]), and show there exist approximate solutions with reproductive property. Passage to the limite is similar to the nonstationary case.

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