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Relations between cumulants in a class of estimators

平川文子 東理大(理工)

確率変数 $X_1, X_2, \cdots, X_n$ は互いに独立で，共通の確率密度関数 $f(x, \theta)$，$\theta \in \Theta \subset R$，$\Theta$ は開集合，を持つものとする．X_i's に基づく一致推定量は，$o(n^{-1})$ まで Edgeworth-展開可能で $n^{1/2}(T-\theta)$ の第 i-cumulant $K_i(\theta)$ は

$$
\begin{aligned}
K_1(\theta) &= K_{10}(\theta) + n^{-1/2} K_{11}(\theta) + n^{-1} K_{12}(\theta) + o(n^{-1}) \\
K_2(\theta) &= K_{20}(\theta) + n^{-1/2} K_{21}(\theta) + n^{-1} K_{22}(\theta) + o(n^{-1}) \\
K_3(\theta) &= n^{-1/2} K_{31}(\theta) + n^{-1} K_{32}(\theta) + o(n^{-1}) \\
K_4(\theta) &= n^{-1} K_{42}(\theta) + o(n^{-1}) \\
K_i(\theta) &= o(n^{-1}) \qquad (i \geqq 5)
\end{aligned}
\tag{1}
$$

の形で表わされ，3回連続微分可能であるものとする．$P_\theta(T \leqq \theta) = g(\theta, n^{-1/2})$ に対して，$P_\theta(\theta_n \leqq \theta) = g(\theta, n^{-1/2}) + o(n^{-1})$ を満たし，$o(n^{-1})$ まで Edgeworth-展開可能で，その cumulant に関して T と同様な条件を満たす θ の一致推定量の全体を $C_2(g(\theta, n^{-1/2}))$ とする．このとき $f(x, \theta)$ に対する適当な正則条件の下で，$g(\theta, n^{-1/2})$ によって定まる適当な $b(\theta, t, n^{-1/2})$ に対して，

$$\sum_{i=1}^{n} \log f(x_i, \theta + n^{-1/2} t) - \sum_{i=1}^{n} \log f(x_i, \theta) = b(\theta, t, n^{-1/2})$$

を満たす θ の解を $\theta^*(t)$ とし，$\hat{\theta}(t) = \theta^*(t) + n^{-1/2} t$ とするとき，任意の $\theta_n \in C_2(g(\theta, n^{-1/2}))$ に対して，

$t > 0$ のとき　$P_\theta(n^{1/2}(\theta_n - \theta) \leqq t) \leqq P_\theta(n^{1/2}(\hat{\theta}(t) - \theta) \leqq t) + o(n^{-1})$

$t < 0$ のとき　$P_\theta(n^{1/2}(\theta_n - \theta) \leqq t) + o(n^{-1}) \geqq P_\theta(n^{1/2}(\hat{\theta}(t) - \theta) \leqq t)$

ところで

$$P(n^{1/2}(\tilde{\theta}(t)-\theta)\leqq t)=\Phi(z)$$
$$-n^{-1/2}\phi(z)\left\{I_{1.1}^{1/2}(\theta)K_{11}^{*}(\theta,t)+\frac{1}{2}I_{1.1}(\theta)K_{21}^{*}(\theta,t)H_1(z)+\frac{1}{6}I_{1.1}^{3/2}(\theta)K_{31}^{*}(\theta,t)H_2(z)\right\}$$
$$-n^{-1}\phi(z)\Big[I_{1.1}^{1/2}(\theta)K_{12}^{*}(\theta,t)+\frac{1}{2}I_{1.1}(\theta)K_{22}^{*}(\theta,t)H_1(z)+\frac{1}{6}I_{1.1}^{3/2}(\theta)K_{32}^{*}(\theta,t)H_2(z)$$
$$+\frac{1}{24}I_{1.1}^{2}(\theta)K_{42}^{*}(\theta,t)H_3(z)$$
$$+\frac{1}{2}\Big\{I_{1.1}(\theta)K_{11}^{*2}(\theta,t)H_1(z)+\frac{1}{4}I_{1.1}^{2}(\theta)K_{21}^{*2}(\theta,t)H_4(z)$$
$$+\frac{1}{36}I_{1.1}^{3}(\theta)K_{31}^{*2}(\theta,t)H_6(z)+I_{1.1}^{3/2}(\theta)K_{11}^{*}(\theta,t)K_{21}^{*}(\theta,t)H_3(z)$$
$$+\frac{1}{3}I_{1.1}^{2}(\theta)K_{11}^{*}(\theta,t)K_{31}^{*}(\theta,t)H_4(z)$$
$$+\frac{1}{6}I_{1.1}^{5/2}(\theta)K_{21}^{*}(\theta,t)K_{31}^{*}(\theta,t)H_5(z)\Big\}\Big]+o(n^{-1})$$

ただし

$$z=I_{1.1}^{1/2}(\theta)(t-\tilde{K}_{10}(\theta,t)),$$

$$K_1^{*}(\theta,t,\tfrac{1}{n^{1/2}})=E_\theta\{n^{1/2}(\theta^{*}(t)-\theta)\}$$
$$=C_0(\theta,t)-t+\frac{1}{n^{1/2}}\{-\frac{1}{2}I_{1.1}^{-2}(\theta)(J_{2.1}(\theta)+J_{1.1.1}(\theta))+C_1(\theta,t)\}$$
$$+\frac{1}{n}\{-\frac{1}{2}(C_0(\theta,t)-\frac{t}{2})I_{1.1}^{-3}(\theta)J_{2.1}(\theta)(J_3(\theta)+4J_{2.1}(\theta))$$
$$+\frac{1}{2}(C_0(\theta,t)-\frac{t}{2})I_{1.1}^{-2}(\theta)(K_{2.2}(\theta)-K_{2.1.1}(\theta))$$
$$+\frac{1}{2}C_0'(\theta,t)I_{1.1}^{-2}(\theta)(J_3(\theta)+2J_{2.1}(\theta))$$
$$-(C_0(\theta,t)-\frac{t}{2})+\frac{1}{2}C_0''(\theta,t)I_{1.1}^{-1}(\theta)+C_2(\theta,t)\}+o(n^{-1})$$
$$=K_{10}^{*}(\theta,t)+n^{1/2}K_{11}^{*}(\theta,t)+n^{-1}K_{12}^{*}(\theta,t),\quad(\text{say})$$
$$\tilde{K}_{10}(\theta,t)=K_{10}^{*}(\theta,t)+t=C_0(\theta,t)$$

$$K_2^*(\theta, t, \frac{1}{n^{1/2}}) = E_\theta \{n^{1/2}(\theta^*(t) - \theta) - K_1^*(\theta, t)\}^2$$

$$= I_{1.1}^{-1}(\theta) + \frac{2}{n^{1/2}} I_{1.1}^{-1}(\theta) C_0'(\theta, t)$$

$$+ \frac{1}{n}\{-I_{1.1}^{-3}(\theta)(K_{3.1}(\theta) + 4K_{2.1.1}(\theta) + K_{1.1.1.1}(\theta) + I_{1.1}^{2}(\theta))$$

$$+ \frac{1}{2} I_{1.1}^{-4}(\theta)(7J_{2.1}^{2}(\theta) + 14J_{2.1}(\theta)J_{1.1.1}(\theta) + 5J_{1.1.1}^{2}(\theta))$$

$$+ (C_0(\theta, t) - \frac{t}{2})^2 I_{1.1}^{-3}(\theta)(I_{1.1}(\theta)K_{2.2}(\theta) - I_{1.1}^{3}(\theta) - J_{2.1}^{2}(\theta))$$

$$+ (C_0'(\theta, t))^2 I_{1.1}^{-1}(\theta) + 2C_1'(\theta, t) I_{1.1}^{-1}(\theta)\} + o(n^{-1})$$

$$= K_{20}^*(\theta, t) + n^{-1/2} K_{21}^*(\theta, t) + n^{-1} K_{22}^*(\theta, t) + o(n^{-1}), \quad \text{(say)}$$

$$K_3^*(\theta, t, \frac{1}{n^{1/2}}) = E_\theta \{n^{1/2}(\theta^*(t) - \theta) - K_1^*(\theta, t)\}^3$$

$$= -\frac{1}{n^{1/2}} I_{1.1}^{-3}(\theta)(3J_{2.1}(\theta) + 2J_{1.1.1}(\theta))$$

$$+ \frac{3}{n}\{(C_0(\theta, t) - \frac{t}{2}) I_{1.1}^{-4}(\theta)(I_{1.1}(\theta)K_{2.2}(\theta) - J_{2.1}^{2}(\theta) - I_{1.1}^{3}(\theta))$$

$$- C_0'(\theta, t) I_{1.1}^{-3}(\theta)(3J_{2.1}(\theta) + 2J_{1.1.1}(\theta))$$

$$+ C_0''(\theta, t) I_{1.1}^{-2}(\theta)\} + o(n^{-1})$$

$$= n^{-1/2} K_{31}^*(\theta, t) + n^{-1} K_{32}^*(\theta, t) + o(n^{-1}), \quad \text{(say)}$$

$$K_4^*(\theta, t, \frac{1}{n^{1/2}}) = E_\theta \{n^{1/2}(\theta^*(t) - \theta) - K_1^*(\theta, t, \frac{1}{n^{1/2}})\}^4 - 3(K_2^*(\theta, t, \frac{1}{n^{1/2}}))^2$$

$$= \frac{1}{n}\{-I_{1.1}^{-4}(\theta)(4K_{3.1}(\theta) + 12K_{2.1.1}(\theta) + 3K_{1.1.1.1}(\theta) + 3I_{1.1}^{2}(\theta))$$

$$+ 12 I_{1.1}^{-5}(\theta)(2J_{2.1}(\theta) + J_{1.1.1}(\theta))(J_{2.1}(\theta) + J_{1.1.1}(\theta))\} + o(n^{-1}),$$

$$= n^{-1} K_{42}^*(\theta, t) + o(n^{-1}) \quad \text{(say)}$$

以後 $o(n^{-k/2})$ まで Edgeworth-展開可能で，$o(n^{-k/2})$ まで (1) と同様な形の cumulant を持ち，$K_{ij}(\theta)$ $(i = 1, 2, \cdots, 5;\ j = 0, 1, 2, \cdots, k)$ が $(k+1)$ 回微分可能である θ の推定量の全体を $C_E(k)$ で表わそう．
また $T^* \in C_E(k)$ に対して次の (a), (b) を満たす $\theta_n \in C_E(k)$ が存在しないとき，T^* は $(k+1)$th order admissible であると言うことにしよう．

(a) すべての $x_1, x_2 \geqq 0$ およびすべての $\theta \in \Theta$ に対して,

$$P_\theta(-x_1 \leqq n^{1/2}(T^* - \theta) \leqq x_2) \leqq P_\theta(-x_1 \leqq n^{1/2}(\theta_n - \theta) \leqq x_2) + o(n^{-1}),$$

(b) ある $x_1', x_2' \geqq 0$ およびある $\theta' \in \Theta$ に対して,

$$P_{\theta'}(-x_1 \leqq n^{1/2}(T^* - \theta) \leqq x_2) < P_{\theta'}(-x_1 \leqq n^{1/2}(\theta_n - \theta) \leqq x_2) + o(n^{-1}$$

ところで $\tilde{K}_{10}(\theta, t)$, $K^*_{12}(\theta, t)$, $K^*_{20}(\theta, t)$, $K^*_{21}(\theta, t)$, $K^*_{31}(\theta, t)$ は t と無関係であるから, 任意の t, x に対して

$$\tilde{\theta}(t) = \tilde{\theta}(x) + o(n^{-1}) \qquad (\text{in law}).$$

従って任意の $x_1, x_2 \geqq 0$ および任意の $\theta \in \Theta$ に対して

$$P_\theta(-x_1 \leqq n^{1/2}(T - \theta) \leqq x_2) \leqq P_\theta(-x_1 \leqq n^{1/2}(\tilde{\theta}(t) - \theta) \leqq x_2) + o(n^{-1/2})$$

このことから任意の $T \in C_E(2)$ は次の (I), (II) のいずれか1つを満たす.

任意の $\theta \in \Theta$ に対して,

(I) $K_{20}(\theta) > I_{1.1}^{-1}(\theta)$

(II) $K_{20}(\theta) = I_{1.1}^{-1}(\theta)$, $K_{21}(\theta) \geqq 2\, I_{1.1}^{-1}(\theta) K_{10}'(\theta)$

次に任意の $x_1, x_2 \geqq 0$ に対して

$$P_\theta(-x_1 \leqq n^{1/2}(T - \theta) \leqq x_2) = P_\theta(-x_1 \leqq n^{1/2}(\tilde{\theta}(t) - \theta) \leqq x_2) + o(n^{-1})$$

を満たす場合を考えよう. すなわち任意の t に対して

$$n^{1/2}(T - \theta) = n^{1/2}(\tilde{\theta}(t) - \theta) + o(n^{-1/2}) \qquad (\text{in law})$$

が成り立つとする. このとき

$$P_\theta(n^{1/2}(T - \theta) \leqq t) - P_\theta(n^{1/2}(\tilde{\theta}(t) - \theta) \leqq t)$$

$$= n^{-1}\phi(I_{1.1}^{1/2}(\theta)(t-C_0(\theta)))t$$
$$\times\ I_{1.1}^{5/2}(\theta)\{-A_{31}(\theta)+(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}t^2$$
$$+I_{1.1}^{5/2}(\theta)\ [4(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))$$
$$+\ \{4A_{31}(\theta)-3(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}C_0(\theta)\ ]\ t$$
$$+I_{1.1}^{3/2}(\theta)\ [12(A_{20}(C_0'(\theta),C_1'(\theta),\theta)-K_{22}(\theta))-3(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)$$
$$-8(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))I_{1.1}(\theta)C_0(\theta)$$
$$+\ \{-4A_{31}(\theta)+3(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}I_{1.1}(\theta)C_0^2(\theta)\ ]+o(n^{-1})$$

ただし

$$A_{20}(C_0'(\theta),C_1'(\theta),\theta)=-I_{1.1}^{-3}(\theta)(K_{3.1}(\theta)+4K_{2.1.1}(\theta)+K_{1.1.1.1}(\theta)+I_{1.1}^{2}(\theta))$$
$$+\frac{1}{2}I_{1.1}^{-4}(\theta)(7J_{2.1}^{2}(\theta)+14J_{2.1}(\theta)J_{1.1.1}(\theta)+5J_{1.1.1}^{2}(\theta))$$
$$+(C_0'(\theta))^2I_{1.1}^{-1}(\theta)+2C_1'(\theta)I_{1.1}^{-1}(\theta),$$

$$A_{21}(\theta)=I_{1.1}^{-3}(\theta)\ \{I_{1.1}(\theta)(K_{2.2}(\theta)-I_{1.1}^{2}(\theta))-J_{2.1}^{2}(\theta)\},$$

$$A_{30}(C_0'(\theta),C_0''(\theta),\theta)=-3C_0'(\theta)I_{1.1}^{-3}(\theta)(3J_{2.1}(\theta)+2J_{1.1.1}(\theta))$$
$$+3C_0''(\theta)I_{1.1}^{-2}(\theta),$$

$$A_{31}(\theta)=3I_{1.1}^{-4}(\theta)\ \{I_{1.1}(\theta)(K_{2.2}(\theta)-I_{1.1}^{2}(\theta))-J_{2.1}^{2}(\theta)\},$$

$$C_0(\theta,t)=C_0(\theta)\qquad(\text{say}),$$

$$C_1(\theta,t)=C_1(\theta)\qquad(\text{say}).$$

従って任意の $t\neq 0$ に対して

$$I_{1.1}^{5/2}(\theta)\{-A_{31}(\theta)+(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}t^2$$
$$+I_{1.1}^{5/2}(\theta)\ [4(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))$$
$$+\ \{4A_{31}(\theta)-3(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}C_0(\theta)\ ]\ t$$
$$+I_{1.1}^{3/2}(\theta)\ [12(A_{20}(C_0'(\theta),C_1'(\theta),\theta)-K_{22}(\theta))-3(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)$$
$$-8(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))I_{1.1}(\theta)C_0(\theta)$$
$$+\ \{-4A_{31}(\theta)+3(K^{*}_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}I_{1.1}(\theta)C_0^2(\theta)\ ]\ \leq 0$$

上の不等式を観察することによって,

$$K^*_{42}(\theta)-K_{42}(\theta)\le 0. \tag{2}$$

$$\begin{aligned}
&I_{1.1}(\theta)\{A_{31}(\theta)-(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}t^2(\theta)\\
&+12(A_{20}(C_0'(\theta),C_1'(\theta),\theta)-K_{22}(\theta))-3(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\\
&\quad -8(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))I_{1.1}(\theta)C_0(\theta)\\
&\quad +\{-4A_{31}(\theta)+3(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}I_{1.1}(\theta)C_0^2(\theta)\le 0,
\end{aligned}\tag{3}$$

を得る. ただし

$$\begin{aligned}
&t(\theta)\\
&=\frac{4(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))+(4A_{31}(\theta)-3(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta))C_0(\theta)}{2\{A_{31}(\theta)-(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}}
\end{aligned}$$

一方

$$\begin{aligned}
&P_\theta(n^{1/2}(T-\theta)\le x)-P_\theta(n^{1/2}(\tilde\theta(t(\theta))-\theta)\le x)\\
&=\frac{1}{24n}\phi(I_{1.1}^{1/2}(\theta)(x-C_0(\theta)))I_{1.1}^{5/2}(\theta)x\\
&\times[(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)(x-t(\theta))^2-(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)t^2(\theta)\\
&\quad +12(A_{20}(C_0'(\theta),C_1'(\theta),\theta)-K_{22}(\theta))I_{1.1}^{-1}(\theta)+12A_{21}(\theta)I_{1.1}^{-1}(\theta)(C_0(\theta)\\
&\qquad -\frac{1}{2}t(\theta))^2\\
&\quad -8\{A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta)+A_{31}(\theta)(C_0(\theta)-\frac{1}{2}t(\theta))\}C_0(\theta)\\
&\quad +3(K^*_{42}(\theta)-K_{42}(\theta))(I_{1.1}(\theta)C_0^2(\theta)-1)]+o(n^{-1})\\
&=\frac{1}{24n}\phi(I_{1.1}^{1/2}(\theta)(x-C_0(\theta)))I_{1.1}^{5/2}(\theta)x\\
&\quad\times[(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)(x-t(\theta))^2\\
&\qquad -\{-A_{31}(\theta)+(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}t^2(\theta)\\
&\quad +12(A_{20}(C_0'(\theta),C_1'(\theta),\theta)-K_{22}(\theta))I_{1.1}^{-1}(\theta)-3(K^*_{42}(\theta)-K_{42}(\theta))\\
&\quad -8(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))C_0(\theta)\\
&\qquad -\{4A_{31}(\theta)-3(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}C_0^2(\theta)]+o(n^{-1})
\end{aligned}$$

不等式(2),(3)を用いて,

$x \geq 0$ のとき $P_\theta(n^{1/2}(T-\theta) \leq x) \leq P_\theta(n^{1/2}\{\tilde{\theta}(t(\theta))-\theta\} \leq x)+o(n^{-1})$

$x \leq 0$, のとき $P_\theta(n^{1/2}(T-\theta) < x)+o(n^{-1}) \geq P_\theta(n^{1/2}\{\tilde{\theta}(t(\theta))-\theta\} < x)$

が成り立つ. すなわちすべての $x_1 \geq 0$, $x_2 \geq 0$, に対して,

$$P_\theta(-x_1 \leq n^{1/2}(T-\theta) \leq x_2) \leq P_\theta(-x_1 \leq n^{1/2}(\tilde{\theta}(t(\theta))-\theta) \leq x_2)+o(n^{-1})$$

従って推定量Tがthird order admissibleであるためには

$$K_{42}(\theta)=K^*_{42}(\theta),$$

$$I_{1.1}(\theta)\{A_{31}(\theta)-(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}t^2(\theta)$$
$$[12(A_{20}(C_0'(\theta),C_1'(\theta),\theta)-K_{22}(\theta))-3(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)$$
$$-8(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))I_{1.1}(\theta)C_0(\theta)$$
$$+\{-4A_{31}(\theta)+3(K^*_{42}(\theta)-K_{42}(\theta))I_{1.1}(\theta)\}I_{1.1}(\theta)C_0^2(\theta)] = 0$$

整理して,

$$K_{42}(\theta)=K^*_{42}(\theta), \qquad (5)$$

$$(A_{30}(C_0'(\theta),C_0''(\theta),\theta)-K_{32}(\theta))^2 I_{1.1}(\theta)$$
$$+3\cdot A_{31}(\theta)(A_{20}(C_0'(\theta),C_1'(\theta),\theta)-K_{22}(\theta)) = 0 \qquad (6)$$

を満たすことである.

定理1. 推定量 $T \in C_E(2)$ が second order admissible ならば不等式(2),(3)を満たす.

<u>定理2</u>. 推定量 $T \in C_E(2)$ が third order admissible であるための条件は, $K_{20}(\theta) = I_{1.1}^{-1}(\theta)$, $K_{21}(\theta) = 2 I_{1.1}^{-1}(\theta) K'_{10}(\theta)$ であって, かつ 等式 (5), (6) を満たすことである.