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Analytic Properties of "Legendre Functions of Two Variables"

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(I) Since *Legendre functions* appear as zonal spherical functions on $SL(2,\mathbb{R})/SO(2)$, the class of zonal spherical functions on $SL(3,\mathbb{R})/SO(3)$ is regarded as a two variables analogue of Legendre functions. Moreover, there is a deep connection between zonal spherical functions on $SL(3,\mathbb{R})/SO(3)$ and Appell's hypergeometric function $F_1(\alpha,\beta,\beta',\gamma;x,y)$ [AK]. In my talk, I present some problems which I encounter when I study the system $\mathcal{M}_\lambda$ of differential equations (defined later) which governs a zonal spherical function on $SL(3,\mathbb{R})/SO(3)$. I don't explain the details of the background. The readers who are interested in this topic, refer to [S2].

(II) Consider the partial differential operators Δ_2, Δ_3 defined by

$$\Delta_2 = -4(\vartheta_1^2-\vartheta_1\vartheta_2+\vartheta_2^2) + \left(2\frac{1+x_1}{1-x_1} - \frac{1+x_2}{1-x_2} + \frac{1+x_1x_2}{1-x_1x_2}\right)\vartheta_1 + \left(2\frac{1+x_2}{1-x_2} - \frac{1+x_1}{1-x_1} + \frac{1+x_1x_2}{1-x_1x_2}\right)\vartheta_2 - 1,$$

$$\Delta_3 = -8\vartheta_1\vartheta_2(\vartheta_1-\vartheta_2) + 2\left(\frac{1+x_2}{1-x_2} + \frac{1+x_1x_2}{1-x_1x_2}\right)\vartheta_1^2 + 4\left(\frac{1+x_1}{1-x_1} - \frac{1+x_2}{1-x_2}\right)\vartheta_1\vartheta_2 - 2\left(\frac{1+x_1}{1-x_1} + \frac{1+x_1x_2}{1-x_1x_2}\right)\vartheta_2^2$$

$$-\frac{2(1+x_1)(1-x_1x_2^2)}{(1-x_1)(1-x_2)(1-x_1x_2)}\vartheta_1+\frac{2(1+x_2)(1-x_1^2x_2)}{(1-x_1)(1-x_2)(1-x_1x_2)}\vartheta_2,$$

where $\vartheta_j = x_j\partial_{x_j}$ $(j = 1, 2)$. Using these operators, define the system $\mathcal{M}_\lambda$ of differential equations

$\mathcal{M}_\lambda$: $(\Delta_j - L_j)u = 0$ $(j = 2, 3)$,

where $\lambda = (\lambda_1,\lambda_2,\lambda_3) \in \mathbb{C}^3$, $\lambda_1+\lambda_2+\lambda_3= 0$, $L_2 = \lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2$, $L_3 = \lambda_1\lambda_2\lambda_3$. Note that $[\Delta_2, \Delta_3] = 0$. *Is it easy to check this commutation relation by use of computer?* In the sequel, we always assume the condition (C) : $\frac{1}{2}(\lambda_i-\lambda_j) \notin \mathbb{Z}$ $(i \neq j)$.

Theorem 1. There is a unique power series $f(x;\lambda)$ of the form $f(x;\lambda) = \sum_{m,n=0}^{\infty} a_{m,n}(\lambda)x_1^m x_2^n$ with the following conditions:

(i) $f(x;\lambda)$ is convergent in a neighbourhood of $x = (0,0)$ and $f(0,0;\lambda) = 1$.

(ii) The function $\tilde{f}(x_1,x_2;\lambda) = x_1^{-\frac{1}{2}(\lambda_1-1)} x_2^{\frac{1}{2}(\lambda_3+1)} f(x_1,x_2;\lambda)$ is a solution to the system $\mathcal{M}_\lambda$.

(iii) $f(x,0;\lambda) = F(-\frac{1}{2}(\lambda_1-\lambda_2-1),\frac{1}{2},-\frac{1}{2}(\lambda_1-\lambda_2-2);x)$,
$f(0,x;\lambda) = F(-\frac{1}{2}(\lambda_2-\lambda_3-1),\frac{1}{2},-\frac{1}{2}(\lambda_2-\lambda_3-2);x)$.□

The statements (i),(ii) are due to Harish-Chandra and (iii) is shown in [HO]. The symmetric group $\mathfrak{S}_3$ of order 3 acts on $\mathcal{M}_\lambda$ as permutation of $\lambda=(\lambda_1,\lambda_2,\lambda_3)$. Under the condition (C), the set $\{\tilde{f}(x_1,x_2;s\lambda);\ s \in \mathfrak{S}_3\}$ forms a fundamental system of solutions to the system $\mathcal{M}_\lambda$ near $x = (0,0)$.

Proposition 2. There is a unique function $g_1(x_1,x_2;\lambda)$ defined by the power series $g_1(x_1,x_2;\lambda) = \sum_{m,n=0}^{\infty} c_{mn}(\lambda)(1-x_1)^m x_2^n$ with the following properties.

(i) The power series in question is convergent in a neighbourhood of $(1,0)$ and $c_{00}(\lambda) = 1$.

(ii) The function $\tilde{g}_1(x_1,x_2;\lambda) = g_1(x_1,x_2;\lambda)x_2^{\frac{1}{2}(\lambda_3+1)}$ is a solution to the system $\mathcal{M}_\lambda$ near $(x_1,x_2) = (1,0)$. Moreover,

$$\tilde{g}_1(x_1,x_2;\lambda)$$
$$= -\frac{1}{\pi}\{B(\tfrac{1}{2}(\lambda_1-\lambda_2),\tfrac{1}{2})\tilde{f}(x_1,x_2;\lambda) + B(\tfrac{1}{2}(\lambda_2-\lambda_1),\tfrac{1}{2})\tilde{f}(x_1,x_2;s_2\lambda)\},$$

where $s_2 \in \mathfrak{S}_3$ is a permutation of λ_1, λ_2.

(iii) $g_1(x,0;\lambda) = x^{-\frac{1}{2}(\lambda_1-1)}F(-\frac{1}{2}(\lambda_1-\lambda_2-1),\frac{1}{2},1;1-x)$.□

(The right-hand of (iii) is nothing but a Legendre function of the first kind.)

Proposition 3. Define the function $\tilde{g}_2(x_1,x_2;\lambda)$ on $\{(x_1,x_2);\ 0 < x_1 < 1,\ 0 < x_2 < 1\}$ by

$$\tilde{g}_2(x_1,x_2;\lambda) = \tan\frac{\pi}{2}(\lambda_1-\lambda_2)B(\tfrac{1}{2}(\lambda_1-\lambda_2),\tfrac{1}{2})\tilde{f}(x_1,x_2;\lambda)$$
$$- \kappa(\tfrac{1}{2}(\lambda_1-\lambda_2-1))\tilde{g}_1(x_1,x_2;\lambda),$$

where $\kappa(s) = \psi(-s)-\psi(1)-\log 4$, $\psi(s) = \frac{d}{ds}\log\Gamma(s)$. If $0 < x_1 < 1$, the function $\tilde{g}_2(x_1,x_2;\lambda)x_2^{-\frac{1}{2}(\lambda_3+1)}$ is real analytic near $x_2 = 0$ and

$$\lim_{x_2\to 0} \tilde{g}_2(x_1,x_2;\lambda)x_2^{-\frac{1}{2}(\lambda_3+1)}$$
$$= F(-\tfrac{1}{2}(\lambda_1-\lambda_2-1),\tfrac{1}{2},1;1-x_1)\log(1-x_1)+F^*(-\tfrac{1}{2}(\lambda_1-\lambda_2-1),\tfrac{1}{2},1;1-x_1),$$

where $F^*(a,b,c;x) = (\partial_a+\partial_b+2\partial_c)F(a,b,c;x)$.□

Due to the propositions above, we construct local solutions $\tilde{g}_j(x;\lambda)$ $(j = 1, 2)$ at $(x_1,x_2) = (1,0)$. Needless to say, permuting λ_1, λ_2, λ_3 of $\tilde{g}_j(x;\lambda)$, we can obtain another fundamental system of solutions to $\mathcal{M}_\lambda$ consisting of $\tilde{g}_j(x;\lambda)$'s.

(III) The system $\mathcal{M}_\lambda$ is a non-trivial and interesting example of systems of differential equations of two variables.

(a) The first and *trivial* problem is to rewrite the system $\mathcal{M}_\lambda$ in the form of two variables analogue of "systems of ordinary differential equations of *Okubo tyhe*" in the sense of T. Yokoyama [Y]. It seems a clever way to accomplish this program with the help of computer.

(b) It is possible to show that there is a unique solution $\varphi_\lambda(x)$ to $\mathcal{M}_\lambda$ which is real analytic at $(x_1,x_2) = (1,1)$ up to a constant factor. Usually, $\varphi_\lambda(x)$ is called *the zonal spherical function* on SL(3,$\mathbb{R}$)/SO(3). Write $\varphi_\lambda(x)$ as a linear combination of $\{\tilde{f}(x;s\lambda)\}$, that is, $\varphi_\lambda(x) = \sum_{s\in\mathfrak{S}_3} c_s(\lambda)\tilde{f}(x;s\lambda)$. Then the coefficient $c_s(\lambda)$ is the product of beta functions of λ. Then what about other local solutions near $(x_1,x_2) = (1,1)$? What I want to know is the behaviour of the solutions to $\mathcal{M}_\lambda$ near the point $x = (1,1)$. It is provable that there are analytic functions $p_j(x)$ $(j = 1, 2, 3, 4)$ near $x = (1,1)$ such that

(i) $\tilde{f}(x;\lambda)$

$$= p_1(x)+p_2(x)\log(1-x_1)+p_3(x)\log(1-x_2)+p_4(x)\log(1-x_1x_2).$$

(ii) $p_j(x)$ $(j = 2, 3, 4)$ are solutions to $\mathcal{M}_\lambda$.

Then *want to characterize the functions* $p_j(x)$. The following is true.

Claim. There are linearly independent local solutions $A(x)$, $B(x)$, $C(x)$, $D(x)$, $E(x)$, $F(x)$ to $\mathcal{M}_\lambda$ near $x = (1,1)$ with the following properties:

$$A(x) = A_1(x),$$
$$B(x) = B_1(x)+B_2(x)\log(1-x_1) \qquad\qquad +B_4(x)\log(1-x_1x_2),$$
$$C(x) = C_1(x)+C_2(x)\log(1-x_1) \qquad\qquad +C_4(x)\log(1-x_1x_2),$$
$$D(x) = D_1(x)+D_2(x)\log(1-x_1)+D_3(x)\log(1-x_2),$$
$$E(x) = E_1(x)+E_2(x)\log(1-x_1)+E_3(x)\log(1-x_2),$$
$$F(x) = F_1(x)+F_2(x)\log(1-x_1)+F_3(x)\log(1-x_2)+F_4(x)\log(1-x_1x_2).$$

The functions $A_j(x)$, $B_j(x)$, $C_j(x)$, etc. are analytic near $x = (1,1)$. In particular, $A(x)$ is the zonal spherical function. The functions $B_j(x)$, $C_j(x)$, etc. $(j > 1!)$ are solutions to $\mathcal{M}_\lambda$. There are two linear relations among B_2, C_2, D_2, E_2, F_2.□

Remark on this claim. The functions $A(x)$, $B(x)$, $C(x)$ are linear combinations of $\tilde{g}_1(x;\lambda_1,\lambda_2,\lambda_3)$, $\tilde{g}_1(x;\lambda_1,\lambda_3,\lambda_2)$, $\tilde{g}_1(x;\lambda_2,\lambda_3,\lambda_1)$.

Since $A(x)$, $B(x)$, $C(x)$ have no singularities along

$x_2 = 1$, it is possible to restrict them on the hyperplane $x_2 = 1$. As a result, $A(x_1,1)$, $B(x_1,1)$, $C(x_1,1)$ satisfy the same differential equation induced from $\mathcal{M}_\lambda$. Let (*) $P(x_1,\partial_{x_1})u = 0$ be the ordinary differential equation thus obtained. Then *is it possible to obtain connection formulas among the local solutions to* (*) $Pu = 0$ *near* $x_1 = 0$ *and those near* $x_1 = 1$. On the other hand, it is clear from the definition that $D(x)$, $E(x)$, $F(x)$ actually admit singularities along $x_2 = 1$. But, for example, consider $D(x)$. By definition, there are analytic functions $q(x_1)$, $r(x_1)$ such that

$$D(x) = q(x_1)\log(1-x_2)+r(x_1)+x_1 s(x_1,x_2)$$

for some function $s(x)$ which is non-singular with respect to x_1. Then what kind of ordinary differential equations do $q(x_1)$ and $r(x_1)$ satisfy? How to characterize $q(x_1)$ and $r(x_1)$?

(IV) Here I note a recent results concerning the 60 representations of Appell's $F_1(\alpha,\beta,\beta',\gamma;x,y)$ due to Vavasseur (cf. [AK], pp.62-64). The argument in the sequel is independent of the main text.

Let us consider the symmetric group of order 5 which is generated by reflections $s_1 = (12)$, $s_2 = (23)$, $s_3 = (34)$, $s_4 = (45)$. So put $\mathfrak{S}_5 = \langle s_i;\ i = 1,2,3,4\rangle$. On the other hand, we put $\mathbb{P}^2(\mathbb{C}) = \{(a_1:a_2:a_3)\}$ and consider six lines on $\mathbb{P}^2(\mathbb{C})$ defined by

$$L_i : a_i = 0 \ (i = 1,\ 2,\ 3),$$

L'_{ij} : $a_i = a_j$ $(i \neq j)$.

Put $X = \mathbb{P}^2(\mathbb{C})$ - {the six lines}. Then it is possible to embed $\mathfrak{S}_3$ into the automorphism group Aut(X) of X in the following way:

Proposition. We define the automorphisms g_i (i = 1, 2, 3, 4) of X by

$$g_1(a_1:a_2:a_3) = (a_1^{-1}:a_2^{-1}:a_3^{-1}),$$

$$g_2(a_1:a_2:a_3) = {}^t\left(\begin{bmatrix}-1 & 0 & 0\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}\right) = (a_1:a_1-a_2:a_1-a_3)$$

$$g_3(a_1:a_2:a_3) = {}^t\left(\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}\right) = (a_2:a_1:a_3)$$

$$g_4(a_1:a_2:a_3) = {}^t\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\end{bmatrix}\right) = (a_1:a_3:a_2).$$

Then the correspondence $s_i \to g_i$ gives an embedding of $\mathfrak{S}_5$ into Aut(X).

To prove the proposition, it suffices to show that

$$g_i^2 = 1 \quad (i = 1, 2, 3, 4)$$
$$(g_i g_{i+1})^3 = 1 \quad (i = 1, 2, 3)$$
$$g_i g_j = g_j g_i \quad \text{if } |i-j| > 1.$$

It is easy to show that X is identified with the space

$$Y = \{(x,y) \in \mathbb{C}^2;\ xy(x-1)(y-1)(x-y) \neq 0\}$$

by the correspondence $(a_1:a_2:a_3) \to (a_2/a_1, a_3/a_1)$. Then the action of g_i on X induces that on Y. In this way, we can realize an action of $\mathfrak{S}_5$ on Y.

Question. Do you know the structure of the group $\mathrm{Aut}(Y)$? Is it true that $\#\,\mathrm{Aut}(Y) < \infty$? Is it true that $\mathfrak{S}_5 = \mathrm{Aut}(Y)$?

Moreover, is the following true?

If φ is a birational map of $\mathbb{P}^2(\mathbb{C})$ such that $\varphi|X$ is an automorphism of X, then $\varphi \in \langle g_i ; i = 1, 2, 3, 4\rangle = \mathfrak{S}_5$.

Now we consider Appell's $F_1(\alpha,\beta,\beta',\gamma;x,y)$ and the system $\mathcal{M}_{\alpha,\beta,\beta',\gamma}$ of differential equations of F_1. We consider the system in question on the $(\alpha,\beta,\beta',\gamma,x,y)$-space. Then the action of $\mathfrak{S}_5$ on Y naturally induces an action of $\mathfrak{S}_5$ on the space of systems $\mathcal{M}_{\alpha,\beta,\beta',\gamma}$'s. As a result, we obtain 120 representations of Appell's F_1. Noting that the coordinate change $(x,y) \to (y,x)$ is realized by g_4, we restrict our attention to 60 representations of Appell's F_1. Then we restore Vavasseur's table (cf. [AK] p.62). We are going to write the variables. Let z_n $(1 \leq n \leq 60)$ be the functions in [AK],pp.62-64. Then we denote

$$z_n = x^{p_1} y^{p_2} (1-x)^{q_1} (1-y)^{q_2} (x-y)^r F_1(a,b,b',c;X_n,Y_n).$$

In the sequel, we give the concrete forms of (X_n,Y_n) if we put $x = a_2/a_1$, $y = a_3/a_1$.

z_n : (X_n,Y_n)	
z_1 : (x, y)	$\left(\frac{a_2}{a_1}, \frac{a_3}{a_1}\right)$
z_2 : $(1-x, 1-y)$	$\left(\frac{a_1-a_2}{a_1}, \frac{a_1-a_3}{a_1}\right)$
z_3 : $(\frac{1}{x}, \frac{1}{y})$	$\left(\frac{a_1}{a_2}, \frac{a_1}{a_3}\right)$
z_4 : $(\frac{x}{x-1}, \frac{x}{x-y})$	$\left(\frac{a_2}{a_2-a_1}, \frac{a_2}{a_2-a_3}\right)$
z_5 : $(\frac{y}{y-x}, \frac{y}{y-1})$	$\left(\frac{a_3}{a_3-a_2}, \frac{a_3}{a_3-a_1}\right)$
z_6 : $(\frac{x-1}{x}, \frac{x-1}{x-y})$	$\left(\frac{a_2-a_1}{a_2}, \frac{a_2-a_1}{a_2-a_3}\right)$
z_7 : $(\frac{y-1}{y-x}, \frac{y-1}{y})$	$\left(\frac{a_3-a_1}{a_3-a_2}, \frac{a_3-a_1}{a_3}\right)$
z_8 : $(\frac{1}{x}, \frac{y}{x})$	$\left(\frac{a_1}{a_2}, \frac{a_3}{a_2}\right)$
z_9 : $(\frac{x}{y}, \frac{1}{y})$	$\left(\frac{a_2}{a_3}, \frac{a_1}{a_3}\right)$
z_{10}: $(\frac{y-x}{y}, \frac{y-x}{y-1})$	$\left(\frac{a_3-a_2}{a_3}, \frac{a_3-a_2}{a_3-a_1}\right)$
z_{11}: $(\frac{x}{x-1}, \frac{y}{y-1})$	$\left(\frac{a_2}{a_2-a_1}, \frac{a_3}{a_3-a_1}\right)$
z_{12}: $(\frac{x-1}{x}, \frac{y-1}{y})$	$\left(\frac{a_2-a_1}{a_2}, \frac{a_3-a_1}{a_3}\right)$

z_{13}: $(\frac{1}{1-x}, \frac{1}{1-y})$ $\left(\frac{a_1}{a_1-a_2}, \frac{a_1}{a_1-a_3}\right)$

z_{14}: $(x, \frac{x}{y})$ $\left(\frac{a_2}{a_1}, \frac{a_2}{a_3}\right)$

z_{15}: $(\frac{y}{x}, y)$ $\left(\frac{a_3}{a_2}, \frac{a_3}{a_1}\right)$

z_{16}: $(1-x, \frac{1-x}{1-y})$ $\left(\frac{a_1-a_2}{a_1}, \frac{a_1-a_2}{a_1-a_3}\right)$

z_{17}: $(\frac{1-y}{1-x}, 1-y)$ $\left(\frac{a_1-a_3}{a_1-a_2}, \frac{a_1-a_3}{a_1}\right)$

z_{18}: $(\frac{1}{1-x}, \frac{y}{y-x})$ $\left(\frac{a_1}{a_1-a_2}, \frac{a_3}{a_3-a_2}\right)$

z_{19}: $(\frac{x}{x-y}, \frac{1}{1-y})$ $\left(\frac{a_2}{a_2-a_3}, \frac{a_1}{a_1-a_3}\right)$

z_{20}: $(\frac{x-y}{x}, \frac{x-y}{x-1})$ $\left(\frac{a_2-a_3}{a_2}, \frac{a_2-a_3}{a_2-a_1}\right)$

z_{21}: $(\frac{x}{x-1}, \frac{x-y}{x-1})$ $\left(\frac{a_2}{a_2-a_1}, \frac{a_2-a_3}{a_2-a_1}\right)$

z_{22}: $(\frac{x-1}{x}, \frac{x-y}{x})$ $\left(\frac{a_2-a_1}{a_2}, \frac{a_2-a_3}{a_2}\right)$

z_{23}: $(\frac{1}{1-x}, \frac{y-x}{y(1-x)})$ $\left(\frac{a_1}{a_1-a_2}, \frac{a_1(a_3-a_2)}{a_3(a_1-a_2)}\right)$

z_{24}: $(x, \frac{x(1-y)}{x-y})$ $\left(\frac{a_2}{a_1}, \frac{a_2(a_1-a_3)}{a_1(a_2-a_3)}\right)$

z_{25}: $(\frac{y}{x}, \frac{y(1-x)}{x(1-y)})$ $\left(\frac{a_3}{a_2}, \frac{a_3(a_1-a_2)}{a_2(a_1-a_3)}\right)$

z_{26}: $(1-x, \frac{y(1-x)}{y-x})$ $\left(\frac{a_1-a_2}{a_1}, \frac{a_3(a_1-a_2)}{a_1(a_3-a_2)}\right)$

z_{27}: $(\frac{1-y}{1-x}, \frac{x(1-y)}{y(1-x)})$ $\left(\frac{a_1-a_3}{a_1-a_2}, \frac{a_2(a_1-a_3)}{a_3(a_1-a_2)}\right)$

z_{28}: $(\frac{1}{1-x}, \frac{1-y}{1-x})$ $\left(\frac{a_1}{a_1-a_2}, \frac{a_1-a_3}{a_1-a_2}\right)$

z_{29}: $(\frac{x}{x-y}, \frac{x-1}{x-y})$ $\left(\frac{a_2}{a_2-a_3}, \frac{a_2-a_1}{a_2-a_3}\right)$

$$z_{30}:\ \left(\frac{x-y}{x},\ \frac{x-y}{x(1-y)}\right) \qquad \left(\frac{a_2-a_3}{a_2},\ \frac{a_1(a_2-a_3)}{a_2(a_1-a_3)}\right)$$

$$z_{31}:\ \left(\frac{y-x}{y-1},\ \frac{y}{y-1}\right) \qquad \left(\frac{a_3-a_2}{a_3-a_1},\ \frac{a_3}{a_3-a_1}\right)$$

$$z_{32}:\ \left(\frac{y-x}{y},\ \frac{y-1}{y}\right) \qquad \left(\frac{a_3-a_2}{a_3},\ \frac{a_3-a_1}{a_3}\right)$$

$$z_{33}:\ \left(\frac{x-y}{x(1-y)},\ \frac{1}{1-y}\right) \qquad \left(\frac{a_1(a_2-a_3)}{a_2(a_1-a_3)},\ \frac{a_1}{a_1-a_3}\right)$$

$$z_{34}:\ \left(\frac{x(1-y)}{y(1-x)},\ \frac{x}{y}\right) \qquad \left(\frac{a_2(a_1-a_3)}{a_3(a_1-a_2)},\ \frac{a_2}{a_3}\right)$$

$$z_{35}:\ \left(\frac{y(1-x)}{y-x},\ y\right) \qquad \left(\frac{a_3(a_1-a_2)}{a_1(a_3-a_2)},\ \frac{a_3}{a_1}\right)$$

$$z_{36}:\ \left(\frac{y(1-x)}{x(1-y)},\ \frac{1-x}{1-y}\right) \qquad \left(\frac{a_3(a_1-a_2)}{a_2(a_1-a_3)},\ \frac{a_1-a_2}{a_1-a_3}\right)$$

$$z_{37}:\ \left(\frac{x(1-y)}{x-y},\ 1-y\right) \qquad \left(\frac{a_2(a_1-a_3)}{a_1(a_2-a_3)},\ \frac{a_1-a_3}{a_1}\right)$$

$$z_{38}:\ \left(\frac{y-1}{y-x},\ \frac{y}{y-x}\right) \qquad \left(\frac{a_3-a_1}{a_3-a_2},\ \frac{a_3}{a_3-a_2}\right)$$

$$z_{39}:\ \left(\frac{1-x}{1-y},\ \frac{1}{1-y}\right) \qquad \left(\frac{a_1-a_2}{a_1-a_3},\ \frac{a_1}{a_1-a_3}\right)$$

$$z_{40}:\ \left(\frac{y-x}{y(1-x)},\ \frac{x-y}{x-1}\right) \qquad \left(\frac{a_1(a_3-a_2)}{a_3(a_1-a_2)},\ \frac{a_2-a_3}{a_2-a_1}\right)$$

$$z_{41}:\ \left(x,\ \frac{y-x}{y-1}\right) \qquad \left(\frac{a_2}{a_1},\ \frac{a_3-a_2}{a_3-a_1}\right)$$

$$z_{42}:\ \left(1-x,\ \frac{y-x}{y}\right) \qquad \left(\frac{a_1-a_2}{a_1},\ \frac{a_3-a_2}{a_3}\right)$$

$$z_{43}:\ \left(\frac{1}{x},\ \frac{x-y}{x(1-y)}\right) \qquad \left(\frac{a_1}{a_2},\ \frac{a_1(a_2-a_3)}{a_2(a_1-a_3)}\right)$$

$$z_{44}:\ \left(\frac{x}{x-1},\ \frac{x(1-y)}{y(1-x)}\right) \qquad \left(\frac{a_2}{a_2-a_1},\ \frac{a_2(a_1-a_3)}{a_3(a_1-a_2)}\right)$$

$$z_{45}:\ \left(\frac{y}{y-x},\ \frac{y(1-x)}{y-x}\right) \qquad \left(\frac{a_3}{a_3-a_2},\ \frac{a_3(a_1-a_2)}{a_1(a_3-a_2)}\right)$$

$$z_{46}:\ \left(\frac{x-1}{x},\ \frac{y(1-x)}{x(1-y)}\right) \qquad \left(\frac{a_2-a_1}{a_2},\ \frac{a_3(a_1-a_2)}{a_2(a_1-a_3)}\right)$$

$$z_{47}:\ \left(\frac{y-1}{y-x},\ \frac{x(1-y)}{x-y}\right) \qquad \left(\frac{a_3-a_1}{a_3-a_2},\ \frac{a_2(a_1-a_3)}{a_1(a_2-a_3)}\right)$$

$$z_{48}:\ \left(\frac{1}{x},\ \frac{y-1}{y-x}\right) \qquad \left(\frac{a_1}{a_2},\ \frac{a_3-a_1}{a_3-a_2}\right)$$

$$z_{49}:\ \left(\frac{x}{y},\ \frac{1-x}{1-y}\right) \qquad \left(\frac{a_2}{a_3},\ \frac{a_1-a_2}{a_1-a_3}\right)$$

$$z_{50}:\ \left(\frac{y-x}{y},\ \frac{y-x}{y(1-x)}\right) \qquad \left(\frac{a_3-a_2}{a_3},\ \frac{a_1(a_3-a_2)}{a_3(a_1-a_2)}\right)$$

$$z_{51}:\ \left(\frac{x-y}{x-1},\ y\right) \qquad \left(\frac{a_2-a_3}{a_2-a_1},\ \frac{a_3}{a_1}\right)$$

$$z_{52}:\ \left(\frac{x-y}{x},\ 1-y\right) \qquad \left(\frac{a_2-a_3}{a_2},\ \frac{a_1-a_3}{a_1}\right)$$

$$z_{53}:\ \left(\frac{y-x}{y(1-x)},\ \frac{1}{y}\right) \qquad \left(\frac{a_1(a_3-a_2)}{a_3(a_1-a_2)},\ \frac{a_1}{a_3}\right)$$

$$z_{54}:\ \left(\frac{x(1-y)}{x-y},\ \frac{x}{x-y}\right) \qquad \left(\frac{a_2(a_1-a_3)}{a_1(a_2-a_3)},\ \frac{a_2}{a_2-a_3}\right)$$

$$z_{55}:\ \left(\frac{y(1-x)}{x(1-y)},\ \frac{y}{y-1}\right) \qquad \left(\frac{a_3(a_1-a_2)}{a_2(a_1-a_3)},\ \frac{a_3}{a_3-a_1}\right)$$

$$z_{56}:\ \left(\frac{y(1-x)}{y-x},\ \frac{x-1}{x-y}\right) \qquad \left(\frac{a_3(a_1-a_2)}{a_1(a_3-a_2)},\ \frac{a_2-a_1}{a_2-a_3}\right)$$

$$z_{57}:\ \left(\frac{x(1-y)}{y(1-x)},\ \frac{y-1}{y}\right) \qquad \left(\frac{a_2(a_1-a_3)}{a_3(a_1-a_2)},\ \frac{a_3-a_1}{a_3}\right)$$

$$z_{58}:\ \left(\frac{1-y}{1-x},\ \frac{y}{x}\right) \qquad \left(\frac{a_1-a_3}{a_1-a_2},\ \frac{a_3}{a_2}\right)$$

$$z_{59}:\ \left(\frac{x-1}{x-y},\ \frac{1}{y}\right) \qquad \left(\frac{a_2-a_1}{a_2-a_3},\ \frac{a_1}{a_3}\right)$$

$$z_{60}:\ \left(\frac{x-y}{x(1-y)},\ \frac{y-x}{y-1}\right) \qquad \left(\frac{a_1(a_2-a_3)}{a_2(a_1-a_3)},\ \frac{a_3-a_2}{a_3-a_1}\right)$$

Now we are going to state a problem presented by Prof. K. Okubo. Before stating the problem, we recall Gaussian hypergeometric functions. "To obtain all the connection formulas for Gaussian hypergeometric functions, it is sufficient to prove Gauss's formula

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

and Kummer's 24 representations of solutions to Gaussian hypergeometric differential equation."

Okubo's Problem. Find "basic connection formulas" for Appell's F_1 so that all the connection formulas for F_1 follow from these basic ones and Vavasseur's 60 representations.

(Since I don't know the history, I am not sure whether the naming of the problem is well-suited or not.)

We are now going to sate a conjecture to this problem. Let us consider Appell's $F_1(\alpha,\beta,\beta',\beta;x,y)$. Put $x_1 = x$, $x_2 = \frac{y}{x}$. Note that if $(x,y) = (a_2/a_1, a_3/a_1)$, then $(x_1,x_2) = (a_2/a_1, a_3/a_2)$.

Let $\mathcal{M} = \mathcal{M}(\alpha,\beta,\beta',\gamma)$ be the system of differential equations on (x_1,x_2)-space for the function $F_1(\alpha,\beta,\beta',\gamma;\underline{x_1,x_1x_2})$.

<u>For a moment, we assume that</u> x_1, x_2 <u>are real variables</u>.

(I) Let $F(x_1,x_2) = {}^t(f_1(x_1,x_2), f_2(x_1,x_2), f_3(x_1,x_2))$ be the three vector consisting of functions $f_j(x_1,x_2)$ such that

(1) $f_j(x_1,x_2)$ is a solution to $\mathcal{M}$ in a neighbourhood of

(0,0).

(2) $f_j(x_1,x_2) = x_1^p x_2^q \sum_{m,n=0}^{\infty} a_{mn} x_1^m x_2^n$ is convergent for some p, q and $a_{00}=1$.

Then we may take

$$f_1(x_1,x_2) = \sum_{m,n=0}^{\infty} a_{mn} x_1^m x_2^n \qquad (a_{00} = 1)$$

$$f_2(x_1,x_2) = x_1^{1-\gamma} \sum_{m,n=0}^{\infty} a'_{mn} x_1^m x_2^n \qquad (a'_{00} = 1)$$

$$f_3(x_1,x_2) = x_1^{1-\gamma} x_2^{\beta-\gamma+1} \sum_{m,n=0}^{\infty} a''_{mn} x_1^m x_2^n \qquad (a''_{00} = 1).$$

(II) Let $G(x_1,x_2) = {}^t(g_1(x_1,x_2),g_2(x_1,x_2),g_3(x_1,x_2))$ be the three vector consisting of functions $g_j(x_1,x_2)$ such that

(1) $g_j(x_1,x_2)$ is a solution to $\mathcal{M}$ in a neighbourhood of (1,0).

(2) $g_j(x_1,x_2) = (1-x_1)^p x_2^q \sum_{m,n=0}^{\infty} a_{mn}(1-x_1)^m x_2^n$ is convergent for some p, q and $a_{00}=1$.

Then we may take

$$g_1(x_1,x_2) = \sum_{m,n=0}^{\infty} b_{mn}(1-x_1)^m x_2^n \qquad (b_{00} = 1)$$

$$g_2(x_1,x_2) = x_2^{\beta-\gamma+1} \sum_{m,n=0}^{\infty} b'_{mn}(1-x_1)^m x_2^n \qquad (b'_{00} = 1)$$

$$g_3(x_1,x_2) = (1-x_1)^{\gamma-\alpha-\beta} \sum_{m,n=0}^{\infty} b''_{mn}(1-x_1)^m x_2^n \qquad (b''_{00} = 1).$$

(III) Let $H(x_1,x_2) = {}^t(h_1(x_1,x_2),h_2(x_1,x_2),h_3(x_1,x_2))$ be the three vector consisting of functions $h_j(x_1,x_2)$ such that

(1) $h_j(x_1,x_2)$ is a solution to $\mathcal{M}$ in a neighbourhood of

(0,1).

(2) $h_j(x_1,x_2) = x_1^p(1-x_2)^q \sum_{m,n=0}^{\infty} a_{mn}x_1^m(1-x_2)^n$ is convergent for some p, q and $a_{00}=1$.

Then we may take

$$h_1(x_1,x_2) = \sum_{m,n=0}^{\infty} c_{mn}x_1^m(1-x_2)^n \quad (c_{00} = 1)$$

$$h_2(x_1,x_2) = x_1^{1-\gamma} \sum_{m,n=0}^{\infty} c'_{mn}x_1^m(1-x_2)^n \quad (c'_{00} = 1)$$

$$h_3(x_1,x_2) = x_1^{1-\gamma}(1-x_2)^{1-\beta-\beta'} \sum_{m,n=0}^{\infty} c''_{mn}x_1^m(1-x_2)^n \quad (c''_{00} = 1).$$

In the above discussion, we assumed that $0 < x_1 < 1$, $0 < x_2 < 1$. Then we are going to consider their analytic continuation to the complex domain. Put $U_{\sigma,\sigma',\sigma''} = \{(x_1,x_2)\in\mathbb{C}^2;\ \sigma \mathrm{Im}\ x_1 > 0,\ \sigma' \mathrm{Im}\ x_2 > 0,\ \sigma'' \mathrm{Im}\ x_1x_2 > 0\}$. It is easy to show that $U_{\sigma,\sigma',\sigma''}$ is simply connected for any signatures $\sigma, \sigma', \sigma'' = \pm$. Noting this, we consider the complex analytic extensions of $F(x_1,x_2)$, $G(x_1,x_2)$, $H(x_1,x_2)$ to $U_{\sigma,\sigma',\sigma''}$ and denote the extensions by the same notation.

Let us consider the transformation $(x_1,x_2) \to (x'_1,x'_2)$, where $x'_1 = 1/x_1$, $x'_2 = x_1x_2$. This transfromation follows from the permutation $a_1 \to a_2$, $a_2 \to a_1$. Sinc both $G(x_1,x_2)$, $G(x'_1,x'_2)$ form fundamental systems of solutions to $\mathcal{M}$ near (1,0), there is a matrix L_1 such that

$$G(x_1',x_2') = L_1 G(x_1,x_2).$$

Question. To determine L_1, is it sufficient to use the 60 representations for F_1 ?

I think that $G(x_1',x_2') = L_1 G(x_1,x_2)$ is an anologue of Kummer's formula $F(a,b,c;x) = (1-x)^{-a} F(a,c-b,c;x/(x-1))$. Threfore I conjecture that $L_1 = I_3$.

Let us consider the transformation $(x_1,x_2) \to (x_1'',x_2'')$, where $x_1'' = x_1 x_2$, $x_2'' = 1/x_2$. This transfromation is nothing but the permutation $a_2 \to a_3$, $a_3 \to a_2$. Sinc both $H(x_1,x_2)$, $H(x_1',x_2')$ form fundamental systems of solutions to $\mathcal{K}$ near $(0,1)$, there is a matrix L_2 such that

$$H(x_1'',x_2'') = L_2 H(x_1,x_2).$$

From the definition, there is a matrix M_1 such that

$$G(x_1,x_2) = M_1 F(x_1,x_2).$$

Similarly, there is a matrix M_2 such that

$$H(x_1,x_2) = M_2 F(x_1,x_2).$$

Conjecture. All the connection formulas among the local fundamental systems for the 15 points in the space of the blowing up (which will be explained in Remark (ii) below) are obtained by using the matrices M_1, M_2, L_1, L_2 and Vavasseur's 60 representations. In particular, if $L_1 = L_2 = I_3$ is true, the <u>basic connection formulas</u> in Okubo's Problem are M_1 and M_2.

Explicit forms of the matrices M_1, M_2 follow from the next proposition.

Proposition.

$$f_1(x_1,x_2)$$
$$= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}g_1(x_1,x_2) + \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}g_2(x_1,x_2)$$

$$f_2(x_1,x_2)$$
$$= \frac{\Gamma(2-\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)}g_1(x_1,x_2) + \frac{\Gamma(2-\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha-\gamma+1)\Gamma(\beta-\gamma+1)}g_2(x_1,x_2)$$

$$f_3(x_1,x_2) = g_2(x_1,x_2)$$

$$f_1(x_1,x_2) = h_1(x_1,x_2)$$

$$f_2(x_1,x_2)$$
$$= \frac{\Gamma(\gamma)\Gamma(1-\beta-\beta')}{\Gamma(1-\beta)\Gamma(\gamma-\beta-\beta')}h_2(x_1,x_2) + \frac{\Gamma(\gamma-\beta)\Gamma(\beta+\beta'-1)}{\Gamma(\gamma-1)\Gamma(\beta')}h_3(x_1,x_2)$$

$$f_3(x_1,x_2)$$
$$= \frac{\Gamma(\beta-\gamma+2)\Gamma(1-\beta-\beta')}{\Gamma(2-\gamma)\Gamma(1-\beta')}h_2(x_1,x_2) + \frac{\Gamma(\beta-\gamma+2)\Gamma(\beta+\beta'-1)}{\Gamma(\beta)\Gamma(\beta+\beta'-\gamma+1)}h_3(x_1,x_2)$$

Remark (i). The discussion above is based on the idea in [S2].

(ii) The author understands that Prof. Takayama (Kóbe University) has obtained connection formulas for Appell's $F_1(\alpha,\beta,\beta',\gamma;x,y)$ in the following sense. Let X be the projective manifold which is obtained by the blowing up of $\mathbb{P}^2(\mathbb{C}) = \{(a_1:a_2:a_3)\}$ so that the divisor $a_1a_2a_3(a_2-a_3)(a_3-a_1)(a_1-a_2) = 0$ becomes the normal crossing type. Then there are 10 lines in X projecting to the divisor in $\mathbb{P}^2(\mathbb{C})$ defined above. Each line in X thus obtained intersects 3 other lines and there are 15 points in X as the intersections of these lines. To each intersecting point, there associates a fundamental system of solutions defined at the point. Then we have 15 fundamental systems of solutions to $\mathcal{M}$ which is the systems differential equations for $F_1(\alpha,\beta,\beta',\gamma;x,y)$ in X. Prof. N. Takayama informed the author at the meeting that he has computed the connection formulas among the 15 fundamental systems.

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