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A multivariable quantum determinant over a commutative ring

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Recently, the quantum determinant (which was found for example in [NYM]) appeared in many interesting ways in the representations of the quantum groups, this notion was defined for the matrices whose components satisfy the quantum commutation relations. In this article, we consider the quantum determinant over a commutative ring, formally using the expression in the definition. Also we define a multivariable quantum determinant which contains several parameters $q_1, q_2, \cdots, q_n$ and coincides with the original quantum determinant if we specify $q = q_1 = q_2 = \cdots = q_n$. We find expansion formulas in terms of a refinement of inversion numbers.

§1 Definitions and some properties

First, we introduce some notations and define a multivariable quantum determinant.

DEFINITION 1.1.

Let $\mathfrak{S}_n$ be the symmetric group of degree n and let $[n]$ denote the set of positive integers up to n. For $w \in \mathfrak{S}_n$ and $i \in [n]$, we define the inversion set $L_i(w)$ at i and the inversion number $\ell_i(w)$ at i by

$$L_i(w) := \{(i,j); i < j, w(i) > w(j)\} \quad \text{and}$$

$$\ell_i(w) := \sharp L_i(w).$$

Then, the (total) inversion number $\ell(w)$ is defined by

$$\ell(w) := \ell_1(w) + \ell_2(w) + \cdots + \ell_n(w).$$

(Of course, $\ell_n(w)$=0 for all $w \in \mathfrak{S}_n$, but we use this notation in order to avoid the confusion in case n=1.)

Let K be a commutative ring and let q and q_i be variables (for all i ∈[n]) and $\mathbf{q}$ denotes the n-tuple of variables $(q_1, q_2, \cdots, q_n)$. For $A = (a_{ij}) \in M(n, K)$, the quantum determinant of A is, by definition, given by

$$\det_q A := \sum_{w \in \mathfrak{S}_n} (-q)^{\ell(w)} a_{1w(1)} a_{2w(2)} \cdots a_{nw(n)}.$$

Similarly, we introduce a multivariable quantum determinant defined by

$$\det_{\mathbf{q}} A := \sum_{w \in \mathfrak{S}_n} (-q_1)^{\ell_1(w)} (-q_2)^{\ell_2(w)} \cdots (-q_n)^{\ell_n(w)} \cdot a_{1w(1)} a_{2w(2)} \cdots a_{nw(n)}.$$

We call $\det_{\mathbf{q}} A$ the $\mathbf{q}$-determinant of A.

EXAMPLE 1.2.

$$\det_{\mathbf{q}} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei - q_2 afh - q_1 bdi + q_1 q_2 bfg + {q_1}^2 cdh - {q_1}^2 q_2 ceg.$$

From the definition, we have the next properties.

(i) For all $A \in M(n,K)$, we have

$$\det_q A = \det_{(q,q,\cdots,q)} A \text{ and } \det_1 A = \det A.$$

(ii) Both for the quantum determinant and the **q**-determinant, the multilinearities with respect to the rows and the columns are valid as in the case of the ordinary determinant.

(iii) In general, $\det_{\mathbf{q}} {}^tA \neq \det_{\mathbf{q}} A$, but $\det_q {}^tA = \det_q A$ because$\ell(w) = \ell(w^{-1})$ for all $w \in \mathfrak{S}_n$.

§2 Expansion formulas

First, we define a multivariable **q**-analogue of the complementary matrix of A in order to show expansion formulas of the **q**-determinant.

DEFINITION 2.1.

For $n \geq 2$, $A = (a_{ij}) \in M(n,K)$ and $1 \leq i,j \leq n$, we define the (i,j)-**q**-complementary matrix $A_{ij}(\mathbf{q})$ by

$A_{ij}(\mathbf{q}) :=$

$$\left(\begin{array}{ccc|ccc}
a_{1,1} & \cdots & a_{1,j-1} & -q_1 a_{1,j+1} & \cdots & -q_1 a_{1,n} \\
a_{2,1} & \cdots & a_{2,j-1} & -q_2 a_{2,j+1} & \cdots & -q_2 a_{2,n} \\
\vdots & & \vdots & \vdots & & \vdots \\
a_{i-1,1} & \cdots & a_{i-1,j-1} & -q_{i-1} a_{i-1,j+1} & \cdots & -q_{i-1} a_{i-1,n} \\
\hline
-q_i a_{i+1,1} & \cdots & -q_i a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\
-q_i a_{i+2,1} & \cdots & -q_i a_{i+2,j-1} & a_{i+2,j+1} & \cdots & a_{i+2,n} \\
\vdots & & \vdots & \vdots & & \vdots \\
-q_i a_{n,1} & \cdots & -q_i a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n}
\end{array}\right)$$

$\in M(n-1, K[\mathbf{q}])$,

where $a_{k,\ell} := a_{k\ell}$ $(1 \leq k, \ell \leq n)$.

Then, for $A = (a_{ij}) \in M(n, K)$ and $i \in [n]$, we have the following.

PROPOSITION 2.2.

$\det_{\mathbf{q}} A = \sum_{j=1}^{n} a_{ij} \det_{(q_1, q_2, \cdots, \widehat{q_i}, \cdots, q_n)} A_{ij}(\mathbf{q})$,

where $(q_1, q_2, \cdots, \widehat{q_i}, \cdots, q_n) := (q_1, q_2, \cdots, q_{i-1}, q_{i+1}, \cdots, q_n)$.

In particular,

$$\det_{\mathbf{q}} A =$$

$$\sum_{j=1}^{n} (-q_1)^{j-1} a_{1j}$$

$$\cdot \det_{(q_2, q_3, \cdots, q_n)} \left(\begin{array}{ccc|ccc} a_{2,1} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2,n} \\ a_{3,1} & \cdots & a_{3,j-1} & a_{3,j+1} & \cdots & a_{3,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n} \end{array} \right).$$

This formula is expansion formula with respect to the first row and it is much more similar to the expansion of the ordinary determinant.

PROOF:

This can be easily obtained from the multilinearity with respect to the rows and the following Lemma 2.3. ∎

LEMMA 2.3.

For $A = (a_{ij}) \in M(n, K)$ *and* $i \in [n]$, *we obtain*

$$
\det_{\mathbf{q}} \left(\begin{array}{ccc|c|ccc}
a_{1,1} & \cdots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \cdots & a_{1,n} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\
\hline
0 & \cdots & 0 & a_{i,j} & 0 & \cdots & 0 \\
\hline
a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
a_{n,1} & \cdots & a_{n,j-1} & a_{n,j} & a_{n,j+1} & \cdots & a_{n,n}
\end{array} \right)
$$

$$
= a_{ij} \det_{(q_1, q_2, \cdots, \widehat{q_i}, \cdots, q_n)} A_{ij}(\mathbf{q}).
$$

PROOF:

From the shape of the matrix of the left hand side, we may think only the case of $w(i) = j$ in the explicit expansion of the $\mathbf{q}$-determinant. If $k < i$ and $w(k) > j$ ($i < k$ and $j > w(k)$), then the pair $(k, i) \in L_k(w)$ ($(i, k) \in L_i(w)$). So, we have this lemma. ∎

Note that similar expansion formulas with respect to the columns also hold.

Next we show an analogue of the Laplace expansion formula of the $\mathbf{q}$-determinant. We introduce some more notations.

DEFINITION 2.4.

For $A = (a_{ij}) \in M(n, K)$, $1 \leq m \leq n$, $1 \leq r_1 < r_2 < \cdots < r_m \leq n$ and $1 \leq s_1 < s_2 < \cdots < s_m \leq n$,

we put

$$b_{ij} := a_{r_i s_j} \qquad (1 \leq i, j \leq m),$$

$$c_{ij} := (-q_{r_i})^{s_j - j} a_{r_i s_j} \qquad (1 \leq i, j \leq m),$$

$$D\begin{pmatrix} r_1, r_2, \cdots, r_m \\ s_1, s_2, \cdots, s_m \end{pmatrix} := (b_{ij}) \in M(m, K) \quad \text{and}$$

$$D_{\mathbf{q}}\begin{pmatrix} r_1, r_2, \cdots, r_m \\ s_1, s_2, \cdots, s_m \end{pmatrix} := (c_{ij}) \in M(m, K[\mathbf{q}]).$$

Then, for $k \in [n]$, we have the next formulas.

PROPOSITION 2.5.

(i) $$\det_{\mathbf{q}} A = \sum_{\{i_1,i_2,\cdots,i_k\}_< \cup \{i_{k+1},i_{k+2},\cdots,i_n\}_< = [n]} \det_{(q_1,q_2,\cdots,q_k)} D_{\mathbf{q}}\begin{pmatrix} 1, 2, \cdots, k \\ i_1, i_2, \cdots, i_k \end{pmatrix}$$

$$\cdot \det_{(q_{k+1},q_{k+2},\cdots,q_n)} D\begin{pmatrix} k+1, k+2, \cdots, n \\ i_{k+1}, i_{k+2}, \cdots, i_n \end{pmatrix}.$$

(ii) $$\det_{\mathbf{q}} A = \sum_{\{i_1,i_2,\cdots,i_k\}_< \cup \{i_{k+1},i_{k+2},\cdots,i_n\}_< = [n]} (-q_{i_{k+1}})^{k+1-i_{k+1}} (-q_{i_{k+2}})^{k+2-i_{k+2}} \cdots (-q_{i_n})^{n-i_n}$$

$$\cdot \det_{(q_{i_1},q_{i_2},\cdots,q_{i_k})} D\begin{pmatrix} i_1, i_2, \cdots, i_k \\ 1, 2, \cdots, k \end{pmatrix}$$

$$\cdot \det_{(q_{i_{k+1}},q_{i_{k+2}},\cdots,q_{i_n})} D\begin{pmatrix} i_{k+1}, i_{k+2}, \cdots, i_n \\ k+1, k+2, \cdots, n \end{pmatrix}.$$

PROOF:

We can easily obtain this proposition from next Lemma 2.6, Lemma 2.7 and Lemma 2.8. ∎

LEMMA 2.6.

We put

$$\Omega_1 := \{w \in \mathfrak{S}_n; w(i) < w(i+1) \text{ for all } i \in [n-1] \setminus \{k\}\},$$

$\Omega_2 := \{w \in \mathfrak{S}_n; w(i) = i \text{ for all } i \in [n] \setminus [k]\}\ (\cong \mathfrak{S}_k)$ *and*

$\Omega_3 := \{w \in \mathfrak{S}_n; w(i) = i \text{ for all } i \in [k]\}\ (\cong \mathfrak{S}_{n-k})$.

Then, we have

(i) $\mathfrak{S}_n = \Omega_1\Omega_2\Omega_3$ *and*

(ii) $\ell_i(w) = \ell_{\sigma_2(i)}(\sigma_1) + \ell_i(\sigma_2) + \ell_i(\sigma_3)$

$$= \begin{cases} \ell_{\sigma_2(i)}(\sigma_1) + \ell_i(\sigma_2) & \text{if } i \in [k] \\ \ell_i(\sigma_3) & \text{if } i \in [n] \setminus [k] \end{cases}$$

for all $i \in [n]$ *and all* $w = \sigma_1\sigma_2\sigma_3$ $(\sigma_j \in \Omega_j, j = 1, 2, 3)$.

PROOF:

(i) is a well known formula, so we will prove (ii).

First, for $w \in \mathfrak{S}_n$ and $i \in [n]$, we put

$L_i^{(1)}(w) := \{(i,j); 1 \le i \le k, k+1 \le j \le n, w(i) > w(j)\}$,

$L_i^{(2)}(w) := \{(i,j); 1 \le i < j \le k, w(i) > w(j)\}$ and

$L_i^{(3)}(w) := \{(i,j); k+1 \le i < j \le n, w(i) > w(j)\}$.

Then, we have the next formula.

$L_i(w) = L_i^{(1)}(w) \amalg L_i^{(2)}(w) \amalg L_i^{(3)}(w)$ (disjoint union)

(i.e. $\ell_i(w) = \sharp L_i^{(1)}(w) + \sharp L_i^{(2)}(w) + \sharp L_i^{(3)}(w)$)

So, we will show (a) $\sharp L_i^{(1)}(w) = \ell_{\sigma_2(i)}(\sigma_1)$, (b) $\sharp L_i^{(2)}(w) = \ell_i(\sigma_2)$ and (c) $\sharp L_i^{(3)}(w) = \ell_i(\sigma_3)$, where $w = \sigma_1\sigma_2\sigma_3$ $(\sigma_j \in \Omega_j, j$=1,2,3).

We define the mappings φ_1 from $L_i^{(1)}(w)$ to $L_{\sigma_2(i)}(\sigma_1)$, φ_2 from $L_i^{(2)}(w)$ to $L_i(\sigma_2)$ and φ_3 from $L_i^{(3)}(w)$ to $L_i(\sigma_3)$ as follows :

$\varphi_1((i,j)) := (\sigma_2(i), \sigma_3(j))$,

$\varphi_2((i,j)) := (i,j)$ and

$\varphi_3((i,j)) := (i,j)$.

Then, $\varphi_1, \varphi_2, \varphi_3$ are bijections. Hence (a),(b),(c) are valid. So, we obtain the first equation of (ii).

The second equation of (ii) follows immediately from the definitions.

∎

LEMMA 2.7.

Under the same assumptions as Lemma 2.6, we have

(i) $\mathfrak{S}_n = \Omega_2\Omega_3\Omega_1^{-1}$.

Moreover, for $j \in [n]$ and $w \in \mathfrak{S}_n$, we put

$\widetilde{L_j}(w) := \{(i,j); i < j, w(i) > w(j)\}$ *and*

$\widetilde{\ell}_j(w) := \sharp\widetilde{L_j}(w)$.

Then, we have the next formulas.

(ii) $\ell_i(w) = \widetilde{\ell}_{w(i)}(w^{-1})$ *for all $i \in [n]$ and all $w \in \mathfrak{S}_n$.*

(iii) $\ell_i(w) = \widetilde{\ell}_{\sigma_1^{-1}(i)}(\sigma_1) + \ell_{\sigma_1^{-1}(i)}(\sigma_2) + \ell_{\sigma_1^{-1}(i)}(\sigma_3)$

for all $i \in [n]$ and all $w = \sigma_2\sigma_3\sigma_1^{-1}$ ($\sigma_j \in \Omega_j, j = 1,2,3$).

PROOF:

(i) follows from $\Omega_2 = {\Omega_2}^{-1}$,$\Omega_3 = {\Omega_3}^{-1}$, $\Omega_2\Omega_3 = \Omega_3\Omega_2$ and Lemma 2.6-(i) easily. So we will show (ii) and (iii).

We define the mapping φ from $L_i(w)$ to $\widetilde{L_{w(i)}}(w^{-1})$ as follows :

$\varphi((i,j)) := (w(j), w(i))$.

Then φ is bijection, so we obtain (ii).

Next, for $w \in \mathfrak{S}_n$ and $i \in [n]$, we put

$\widetilde{L_i^{(1)}}(w) := \{(i,j) \in L_i(w); 1 \leq {\sigma_1}^{-1}(j) \leq k, k+1 \leq {\sigma_1}^{-1}(i) \leq n\}$

$\widetilde{L_i^{(2)}}(w) := \{(i,j) \in L_i(w); 1 \leq {\sigma_1}^{-1}(i), {\sigma_1}^{-1}(j) \leq k\}$ and

$\widetilde{L_i^{(3)}}(w) := \{(i,j) \in L_i(w); k+1 \leq {\sigma_1}^{-1}(i), {\sigma_1}^{-1}(j) \leq n\}$,

where $w=\sigma_2\sigma_3{\sigma_1}^{-1}$ ($\sigma_j \in \Omega_j$ j=1,2,3).

Then, since $\{(i,j) \in L_i(w); 1 \leq {\sigma_1}^{-1}(i) \leq k, k+1 \leq {\sigma_1}^{-1}(j) \leq n\}=\emptyset$, we have the following formula.

$L_i(w) = \widetilde{L_i^{(1)}}(w) \amalg \widetilde{L_i^{(2)}}(w) \amalg \widetilde{L_i^{(3)}}(w)$ (disjoint union).

So, we will show

(a) $\sharp\widetilde{L_i^{(1)}}(w) = \ell_i({\sigma_1}^{-1})$,

(b) $\sharp\widetilde{L_i^{(2)}}(w) = \ell_{{\sigma_1}^{-1}(i)}(\sigma_2)$ and

(c) $\sharp\widetilde{L_i^{(3)}}(w) = \ell_{{\sigma_1}^{-1}(i)}(\sigma_3)$.

We define the mappings ψ_1 from $\widetilde{L_i^{(1)}}(w)$ to $L_i({\sigma_1}^{-1})$, ψ_2 from $\widetilde{L_i^{(2)}}(w)$ to $L_{{\sigma_1}^{-1}(i)}(\sigma_2)$ and ψ_3 from $\widetilde{L_i^{(3)}}(w)$ to $L_{{\sigma_1}^{-1}(i)}(\sigma_3)$ as follows :

$\psi_1((i,j)) := (i,j)$,

$\psi_2((i,j)) := ({\sigma_1}^{-1}(i), {\sigma_1}^{-1}(j))$ and

$\psi_3((i,j)) := ({\sigma_1}^{-1}(i), {\sigma_1}^{-1}(j))$.

Then, ψ_1, ψ_2, ψ_3 are bijections. Hence (a),(b),(c) are valid. So, we have (iii) from (a),(b),(c) and (ii). ∎

LEMMA 2.8.

For $\sigma_1 \in \Omega_1, m \in [k]$ and $s \in [n] \setminus [k]$, we have

$\ell_m(\sigma_1) = \sigma_1(m) - m$ *and* $\widetilde{\ell_s}(\sigma_1) = s - \sigma_1(s)$.

PROOF:

This Lemma follows from the definition of Ω_1 easily. ∎

COROLLARY 2.9.

For $A = (a_{ij}) \in M(n, K)$, $k \in [n]$, we have the following formulas.

$$\text{(i) } \det_q A = \sum_{\{i_1,i_2,\cdots,i_k\}_< \cup \{i_{k+1},i_{k+2},\cdots,i_n\}_< = [n]} (-q)^{\sum_{m=1}^{k}(i_m - m)} \det_q D\begin{pmatrix} 1,2,\cdots,k \\ i_1,i_2,\cdots,i_k \end{pmatrix}$$

$$\cdot \det_q D\begin{pmatrix} k+1,k+2,\cdots,n \\ i_{k+1},i_{k+2},\cdots,i_n \end{pmatrix}.$$

$$\text{(ii) } \det_q A = \sum_{\{i_1,i_2,\cdots,i_k\}_< \cup \{i_{k+1},i_{k+2},\cdots,i_n\}_< = [n]} (-q)^{\sum_{m=1}^{k}(i_m - m)} \det_q D\begin{pmatrix} i_1,i_2,\cdots,i_k \\ 1,2,\cdots,k \end{pmatrix}$$

$$\cdot \det_q D\begin{pmatrix} i_{k+1},i_{k+2},\cdots,i_n \\ k+1,k+2,\cdots,n \end{pmatrix}.$$

§3 Some applications

We will give an extension of the length generating function for certain subsets of $\mathfrak{S}_n$.

DEFINITION 3.1.

For $k \in [n]$, we put

$$S_n^{(k)} := \{w \in \mathfrak{S}_n; w(i) \leq i + k - 1 \text{ for all i} \in [n]\}.$$

Then, we have

PROPOSITION 3.2.

$\sum_{w\in S_n^{(k)}} q_1^{\ell_1(w)} q_2^{\ell_2(w)} \cdots q_n^{\ell_n(w)} = \prod_{i=1}^{n-k}(k)_{q_i} \prod_{j=1}^{k}(j)_{q_{n-j+1}}$, where $(\ell)_q := 1+q+q^2+\cdots+q^{\ell-1}$.

In particular,

$$\sum_{w\in S_n^{(k)}} q^{\ell(w)}$$
$$= (1+q+q^2+\cdots+q^{k-1})^{n-k}\prod_{i=1}^{k}(1+q+q^2+\cdots+q^{i-1}).$$

PROOF:

Let us consider the following matrix.

$$M_n^{(k)} := \left(\begin{array}{ccccc|cccc}
\multicolumn{5}{c|}{\overbrace{\hphantom{1\ 1\ 1\ \cdots\ 1}}^{k\cdot\text{times}}} & & & & \\
1 & 1 & 1 & \cdots & 1 & 0 & \cdot & \cdots & 0 \\
1 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 0 \\
\hline
1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1
\end{array}\right) \in M(n, \mathbf{R}).$$

From the shape of the matrix, non zero terms occurring in the explicit expansion of the $\mathbf{q}$-determinant correspond to the elements of $S_n^{(k)}$.

Thus $\quad \det_{\mathbf{q}} M_n^{(k)} = \sum_{w\in S_n^{(k)}}(-q_1)^{\ell_1(w)}(-q_2)^{\ell_2(w)}\cdots(-q_n)^{\ell_n(w)}.$

On the other hand, we obtain the next formula by the expansion formula with respect to the 1-st row and induction.

$$\det_{\mathbf{q}} M_n^{(k)} = \prod_{i=1}^{n-k}(k)_{-q_i}\prod_{j=1}^{k}(j)_{-q_{n-j+1}}.$$

Hence

$\sum_{w\in S_n^{(k)}} q_1^{\ell_1(w)} q_2^{\ell_2(w)} \cdots q_n^{\ell_n(w)} = \det_{-\mathbf{q}} M_n^{(k)}$

$= \prod_{i=1}^{n-k} (k)_{q_i} \prod_{j=1}^{k} (j)_{q_{n-j+1}}$. ∎

We obtain the following formulas from Proposition 3.2

COROLLARY 3.3.

(i) $\sum_{w\in\mathfrak{S}_n} q_1^{\ell_1(w)} q_2^{\ell_2(w)} \cdots q_n^{\ell_n(w)}$

$= \prod_{j=1}^{n}(1 + q_{n-j+1} + q_{n-j+1}^2 + \cdots + q_{n-j+1}^{j-1})$.

(ii) $\sum_{w\in\mathfrak{S}_n} q^{\ell(w)} = \prod_{j=1}^{n}(1 + q + q^2 + \cdots + q^{j-1})$.

Note that (ii) is a well known formula (for example, see [S]).

By a similar argument, we can also show that the Fibonacci number f_n is given in the following manner.

$f_n = \sharp\{w \in \mathfrak{S}_n; i-1 \leq w(i) \leq i+1 \text{ for all } i \in [n]\}$.

We would like to conclude this article with the following remark on multiplicativity for **q**-determinant.

Let K be a commutative ring with unit. Suppose we could define a **q**-product "$*_{\mathbf{q}}$" over $M(n, K(\mathbf{q}))$ having the following properties :

(1) The **q**-product "$*_{\mathbf{q}}$" coincides with the ordinary matrix product if we specify **q**=(1,1,$\cdots$,1).

(2) For all $A, B \in M(n, K(\mathbf{q}))$, $\det_{\mathbf{q}}(A *_{\mathbf{q}} B) = (\det_{\mathbf{q}} A)(\det_{\mathbf{q}} B)$.

Then we would be able to find more interesting properties about the q-determinant. But so far, we can only define such a product for n≤2.

REMARK 3.4.

For n ≥ 3, the product of the next form seems to be natural and satisfies the condition (1) above, but unfortunately this does not satisfy the condition (2) above.

For $A = (a_{ij}), B = (b_{ij}) \in M(n, K(\mathbf{q}))$,

$$A *_{\mathbf{q}} B = \sum_{1 \le i,j,k \le n} a_{ik} b_{kj} q_1^{r_{ij}^k(1)} q_2^{r_{ij}^k(2)} \cdots q_n^{r_{ij}^k(n)} E_{ij},$$

where E_{ij} $(1 \le i,j \le n)$ are the matrix units and $r_{ij}^k(m)$ $(1 \le i,j,k,m \le n) \in \mathbf{R}$.

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