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Bounding the Number of Columns (1, k-2, 1) in the Intersection Array of a Distance-Regular Graph

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1. Introduction.

Let $\Gamma = (X,R)$ be an undirected connected finite graph without loops and multiple edges, where X and R are the vertex and edge sets. For a vertex x, $\Gamma_i(x)$ denotes the set of vertices having distance i from x. Γ is said to be *distance-regular* if the numbers (which are called *intersection numbers*)

$$c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$$
$$a_i = |\Gamma_i(x) \cap \Gamma_1(y)|$$
$$b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$$

are independent of the choices of $x \in X$ and $y \in \Gamma_i(x)$. In what follows, we always assume that Γ is a distance-regular graph. The valency and diameter are denoted by k and d:

$$k = |\Gamma_1(x)|,$$
$$d = \mathrm{Max}\{ i \mid \Gamma_i(x) \neq \phi \}.$$

The intersection array is denoted by

$$\begin{bmatrix} * & c_1 & c_2 & \cdots & c_i & \cdots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \cdots & a_i & \cdots & a_{d-1} & a_d \\ k & b_1 & b_2 & \cdots & b_i & \cdots & b_{d-1} & * \end{bmatrix}.$$

Let

$$\ell = \ell (c, a, b) = \# \{ i \mid (c_i , a_i , b_i) = (c, a, b) \}.$$

Then by the relation [1, page 195]

$$1 = c_1 \leq c_2 \leq \cdots \leq c_d , \quad k = b_0 \geq b_1 \geq \cdots \geq b_{d-1} ,$$

these ℓ columns appear in the intersection array consecutively.

In [2] , it is conjectured that $\ell(c, a, b)$ is bounded by a function of the valency k and it is shown that if $c = b$,

$$\ell (c, a, b) \leq 10k2^k .$$

We shall improve the above bound when $c = b = 1$.

Theorem.

If $k \geq 5$, $\ell (1, k - 2, 1) < 46\sqrt{k - 3}$.

Notice that distance-regular graphs of valency 3 or 4 are classified by [3], [4]. Our result is useful for the classification of distance-regular graphs with small valencies, for example $k = 5, 6$.

2. Preliminaries.

In what follows, let the intersection array be

$$\left[\begin{matrix} * & 1 & \cdots & 1 & c_{\alpha+1} & \cdots & c_s & c & \cdots & c & c_{s+\ell+1} & \cdots & c_{d-1} & c_d \\ 0 & a_1 & \cdots & a_1 & a_{\alpha+1} & \cdots & a_s & a & \cdots & a & a_{s+\ell+1} & \cdots & a_{d-1} & a_d \\ k & \underbrace{b_1}_{} & \underbrace{\cdots}_{\alpha} & \underbrace{b_1}_{} & b_{\alpha+1} & \cdots & b_s & \underbrace{b}_{} & \underbrace{\cdots}_{\ell} & \underbrace{b}_{} & b_{s+\ell+1} & \cdots & b_{d-1} & * \end{matrix} \right].$$

The i-th adjacency matrix of Γ is denoted by A_i ($0 \leq i \leq d$) and we set $A = A_1$.

Proposition 1. ([2, Proposition 1])

Let $\ell = \ell(c, a, b)$. *Then there exists an eigenvalue* θ *of* A *such that*

$$k - b - c + 2\sqrt{bc}\cos\frac{\nu + 2}{\ell}\pi < \theta < k - b - c + 2\sqrt{bc}\cos\frac{\nu}{\ell}\pi$$

for each $\nu = 1, 2, \cdots, \ell-3$.

There exists a polynomial $v_i(x)$ of degree i such that $v_i(A) = A_i$, and we have $v_i(k) = k_i$. See [1].

Proposition 2. ([2, Proposition 2])

$v_s(x)$ *has roots all less than* $k - b_s - c_s + 2\sqrt{b_s c_s}$.

Proposition 3. ([5])

Let $\alpha = \ell(c_1, a_1, b_1)$ *and* $\theta \neq \pm k$ *an eigenvalue of* A. *If* $a_1 = 0$, *then the multiplicity* $m(\theta)$ *of* θ *in* A *satisfies*

$$m(\theta) \geq k(k-1)^{r-1}$$

with $r = [(\alpha + 1)/2]$, *the integer part of* $(\alpha + 1)/2$.

Proposition 4. ([6])

Let $\alpha = \ell(c_1, a_1, b_1)$ *and* $\alpha' \leq \ell(c_u, a_u, b_u)$. *Suppose* $c_{u+\alpha'} = 1$ *and* $a_1 \neq a_u$. *Then the following hold*

(1) $\alpha' \leq \alpha$

(2) $\alpha' \leq \alpha - 1$ *if* $\alpha \geq 3$

(3) $\alpha' \leq \alpha - 2$ *if* $\alpha \geq 5$.

The results in Proposition 4 are also obtained by A.V.Ivanov (personal communication).

Proposition 5. ([7])

Let $\alpha = \ell(1, 0, k-1)$ *and* $\alpha_1 \leq \ell(1, 1, k-2)$. *Suppose* $c_{\alpha+\alpha_1+1} = 1$, $k \geq 4$ *and* $\alpha \geq 1$. *Then* $\alpha_1 \leq 2$.

Lemma 6. *If* $\ell(1, k-2, 1) \geq 2$, *then* $a_1 = 0$.

Proof. Suppose $a_1 \neq 0$. Then for an edge $\{y, z\}$ with $y \in \Gamma_{s+1}(x)$, $z \in \Gamma_{s+2}(x)$, there exists a triangle yzw, and so we have $b_{s+1} > 1$ or $c_{s+2} > 1$, which is a contradiction. □

3. Proof of Theorem.

Let Γ be a distance regular graph with valency k and diameter d. Let $\ell = \ell(1, k-2, 1) > 0$, and

$$(c_{s+1}, a_{s+1}, b_{s+1}) = \cdots = (c_{s+\ell}, a_{s+\ell}, b_{s+\ell}) = (1, k-2, 1).$$

Let $\alpha = \ell(c_1, a_1, b_1)$ and $r = [(\alpha + 1)/2]$.

By the relation $c_i \leq c_{i+1}$, $b_i \geq b_{i+1}$, we have the following intersection array :

$$\begin{bmatrix} * & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & c_{s+\ell+1} & \cdots & c_{d-1} & c_d \\ 0 & a_1 & \cdots & a_1 & a_{\alpha+1} & \cdots & a_s & k-2 & \cdots & k-2 & a_{s+\ell+1} & \cdots & a_{d-1} & a_d \\ k & \underbrace{b_1 & \cdots & b_1}_{\alpha} & b_{\alpha+1} & \cdots & b_s & \underbrace{1 & \cdots & 1}_{\ell} & 1 & \cdots & 1 & * \end{bmatrix}.$$

Firstly we need two lemmas to estimate the number of vertices n and the number s above.

Lemma 7. *Let n be the number of vertices. Then*

$$n < k_s \cdot \Big((k-1)/(k-2) + (\ell+1)(k-1) \Big).$$

Proof. Since $n = k_0 + k_1 + \cdots + k_d$, we evaluate k_i's using the property,

$$b_i k_i = c_{i+1} k_{i+1}.$$

For $i \le s$, since $c_i = 1$ and $k = b_0 \ge b_1 \ge \cdots \ge b_s > 1$, we have

$$k_{i-1} = k_i / b_{i-1} \le k_i / b_s \le k_{i+1} / b_s^{\,2} \le k_s / b_s^{\,s-(i-1)}.$$

Hence

$$k_0 + k_1 + \cdots + k_s \le k_s \Big((1/b_s)^s + (1/b_s)^{s-1} + \cdots + 1 \Big)$$

$$= k_s \cdot \frac{b_s}{b_s - 1} \Big(1 - (1/b_s)^{s+1} \Big)$$

$$< \frac{b_s}{b_s - 1} \cdot k_s .$$

Obviously, it holds that

$$k_{s+1} = k_{s+2} = \cdots = k_{s+\ell} = b_s k_s .$$

For $i \ge s + \ell + 1$, since $b_i = 1$ and $1 < c_{s+\ell+1} \le \cdots \le c_d$, we have

$$k_{s+\ell+j} \le k_{s+\ell} / (c_{s+\ell+1})^{j}.$$

Hence

$$k_{s+\ell+1} + \cdots + k_d$$

$$\le k_{s+\ell} \left(1 / c_{s+\ell+1} + (1 / c_{s+\ell+1})^2 + \cdots + (1 / c_{s+\ell+1})^{d-(s+\ell)} \right)$$

$$= k_{s+\ell} \cdot \frac{1}{c_{s+\ell+1} - 1} \left(1 - (1 / c_{s+\ell+1})^{d-(s+\ell)} \right)$$

$$< \frac{1}{c_{s+\ell+1} - 1} \cdot k_{s+\ell}$$

$$\le k_{s+\ell} = b_s k_s .$$

Therefore we get

$$n < k_s \left(b_s / (b_s - 1) + (\ell + 1) b_s \right)$$

$$\le k_s \left(\frac{k - 1}{(k - 1) - 1} + (\ell + 1)(k - 1) \right).$$

Note that $(\ell + 1)x + \dfrac{x}{x - 1}$ is increasing if $x \ge 2$. □

Lemma 8. *Suppose* $\ell \ge 5$ *and* $k \ge 5$. *Then* $s \le \alpha(k - 3)$.

Proof. Let $\alpha' = \ell - 1$ in Proposition 4 and $u = s + 1$. Since $\alpha' \ge 4$, $\alpha \ge 5$. Let

$$\alpha_i = \ell (1, i, k - i - 1).$$

Then $\alpha_0 = \alpha$ and $\alpha_{k-2} = \ell$. Note that $a_1 = 0$ by Lemma 6.

By Proposition 5, $\alpha_1 \le 2$.

By Proposition 4.(3), $\alpha_i \le \alpha - 2 \quad i = 2, \cdots, k - 3$.

Therefore we get

$$
\begin{aligned}
s &\le \alpha + \alpha_1 + \alpha_2 + \cdots + \alpha_{k-3} \\
&\le \alpha + 2 + (k-4)(\alpha - 2) \\
&= \alpha(k-3) - 2(k-5) \\
&\le \alpha(k-3) \qquad \text{as} \quad k \ge 5. \quad \square
\end{aligned}
$$

Now we start the proof of Theorem.

By Proposition 1, there exists an eigenvalue θ of A such that

$$k - b - c + 2\sqrt{bc}\cos(3\pi/\ell) < \theta < k - b - c + 2\sqrt{bc}\cos(\pi/\ell).$$

Since $b = c = 1$, it holds that

$$
\begin{aligned}
k - 2 + 2\cos(3\pi/\ell) &\ge k - 2 + 2\left(1 - \frac{1}{2}(3\pi/\ell)^2\right) \\
&= k - (3\pi/\ell)^2.
\end{aligned}
$$

Thus we have an eigenvalue θ of A such that

$$k - \delta < \theta < k,$$

with $\delta = (3\pi/\ell)^2$.

By Proposition 2, $v_s(x)$ is positive for

$x \ge k - b_s - c_s + 2\sqrt{b_s c_s}$, while

$$
\begin{aligned}
k - b_s - c_s + 2\sqrt{b_s c_s} &= k - \left(\sqrt{b_s} - \sqrt{c_s}\right)^2 \\
&= k - \left(\sqrt{b_s} - 1\right)^2 \\
&\le k - \left(\sqrt{2} - 1\right)^2.
\end{aligned}
$$

Hence we get

$$v_s(x) > 0 \quad \text{for} \quad x \ge k - \left(\sqrt{2} - 1\right)^2. \qquad \cdots(1)$$

Now assume that $\ell > 46\sqrt{k-3}$. Since $\ell \geq 3\pi\left(\sqrt{2}+1\right)$, $\theta > k - \left(\sqrt{2}-1\right)^2$ and so $v_s(\theta) > 0$.

Let $m(\theta)$ be the multiplicity of θ in A.

By Biggs' formula [1, page 72], it holds that

$$m(\theta) = n \Big/ \sum_{i=0}^{n} \{ v_i(\theta)^2 / k_i \}. \quad \cdots (2)$$

Since $\ell \geq 3\pi\left(\sqrt{2}+1\right) > 2$, $a_1 = 0$ by Lemma 6.

Hence we can apply the Terwilliger bound (Proposition 3)

$$m(\theta) \geq k(k-1)^{r-1}. \quad \cdots (3)$$

We shall find an upper bound of $m(\theta)$, and by comparing it with (3), we shall prove that ℓ can not exceed $46\sqrt{k-3}$.

Applying Lemma 7 to (2), we have

$$m(\theta) \leq n \cdot k_s / v_s(\theta)^2$$

$$\leq \left(k_s/v_s(\theta)\right)^2 \cdot \left((k-1)/(k-2) + (\ell+1)(k-1)\right).$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_s$ be the roots $v_s(x)$.

By (1), we may assume

$$\theta > k - \left(\sqrt{2}-1\right)^2 > \lambda_1 > \lambda_2 > \cdots > \lambda_s.$$

Since $(k-\lambda_i)/(k-\theta) = 1 + (k-\theta)/(\theta-\lambda_i)$ increases with λ_i, it follows that

$$\frac{k-\lambda_i}{\theta-\lambda_i} < \frac{k - \left\{k - \left(\sqrt{2}-1\right)^2\right\}}{\theta - \left\{k - \left(\sqrt{2}-1\right)^2\right\}}$$

$$= \frac{1}{1 + (\theta - k)\left(\sqrt{2}+1\right)^2}$$

$$< \frac{1}{1 - \delta\left(\sqrt{2} + 1\right)^2} \qquad \text{as} \quad k - \delta < \theta.$$

So we have

$$\frac{k_s}{\upsilon_s(\theta)} = \frac{\upsilon_s(k)}{\upsilon_s(\theta)} = \prod_{i=1}^{s} \frac{k - \lambda_i}{\theta - \lambda_i} < \frac{1}{\left\{ 1 - \delta\left(\sqrt{2} + 1\right)^2 \right\}^s} ,$$

$$m(\theta) < \frac{(k - 1)/(k - 2) + (\ell + 1)(k - 1)}{\left\{ 1 - \delta\left(\sqrt{2} + 1\right)^2 \right\}^{2s}} . \qquad \cdots (4)$$

Since $r = [(\alpha + 1)/2]$, $\alpha \le 2r$ and it follows from Lemma 8 that

$$s \le \alpha(k - 3) \le 2r(k - 3). \qquad \cdots (5)$$

By (3), (4) and (5), we have

$$k(k - 1)^{r-2} < \frac{1/(k - 2) + (\ell + 1)}{\left\{ 1 - \delta\left(\sqrt{2} + 1\right)^2 \right\}^{4r(k-3)}} , \qquad \cdots (6)$$

while $0 < 1 - \delta\left(\sqrt{2} + 1\right)^2 < 1$ by $\ell \ge 3\pi\left(\sqrt{2} + 1\right)^2$ and $\delta = (3\pi / \ell)^2$.

Since $\ell + 1 \le \alpha \le s$ by Proposition 4.

[the right hand side of (6)]

$$\le \frac{1/(k - 2) + s}{\left\{ 1 - \delta\left(\sqrt{2} + 1\right)^2 \right\}^{4r(k-3)}}$$

$$\le \frac{s + 1}{\left\{ 1 - \delta\left(\sqrt{2} + 1\right)^2 \right\}^{4r(k-3)}}$$

$$\le \frac{2r(k - 3) + 1}{\left\{ 1 - \delta\left(\sqrt{2} + 1\right)^2 \right\}^{4r(k-3)}}$$

$$< \frac{2rk}{\left\{ 1 - \delta\left(\sqrt{2} + 1\right)^2 \right\}^{4r(k-3)}} .$$

Hence

$$\log k + (r - 2)\log (k - 1)$$

$$< \log 2 + \log r + \log k - 4r(k-3)\log\left\{1 - \delta\left(\sqrt{2}+1\right)^2\right\}. \qquad \cdots (7)$$

We want to show that (7) does not hold for large ℓ.
Namely we shall show the opposite inequality for $\ell \geq 46\sqrt{k-3}$:

$$(r-2)\log(k-1)$$

$$\geq \log 2 + \log r - 4r(k-3)\log\left\{1 - \delta\left(\sqrt{2}+1\right)^2\right\}. \qquad \cdots (8)$$

By $r = [(\alpha+1)/2]$ and Proposition 4.(3),

$$r \geq (\ell+1)/2 \geq \frac{46\sqrt{k-3}+1}{2} \geq \frac{46\sqrt{2}+1}{2}.$$

[the left hand side of (8)] - [the right hand side of (8)]

$$= (r-2)\log(k-1) - \log 2 - \log r$$
$$+ 4r(k-3)\log\left\{1 - (3\pi/\ell)^2\cdot\left(\sqrt{2}+1\right)^2\right\}$$
$$\geq (r-2)\log(k-1) - \log 2 - \log r$$
$$+ 4r(k-3)\log\left[1 - \left(\frac{3\pi}{46\sqrt{k-3}}\right)^2\cdot\left(\sqrt{2}+1\right)^2\right]$$
$$> (r-2)\log(k-1) - \log 2 - \log r$$
$$+ 4r(k-3)\cdot\left[-\frac{\left(3\pi/46\sqrt{k-3}\right)^2\cdot\left(\sqrt{2}+1\right)^2}{1-\left(3\pi/46\sqrt{k-3}\right)^2\cdot\left(\sqrt{2}+1\right)^2}\right]$$

$$\left(\text{by } \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \quad (\text{for } 0 < x < 1)\right.$$
$$> -x - x^2 - x^3 - \cdots$$
$$\left. = -\frac{x}{1-x}\right)$$

$$= (r-2)\log(k-1) - \log 2 - \log r$$
$$+ 4r(k-3)\cdot\left[-\frac{(3\pi/46)^2\left(\sqrt{2}+1\right)^2/(k-3)}{1-(3\pi/46)^2\left(\sqrt{2}+1\right)^2/(k-3)}\right]$$
$$> (r-2)\log(k-1) - \log 2 - \log r$$

$$+ 4r(k-3)\cdot\left[- \frac{0/2447/(k-3)}{1 - 0.2447/(k-3)} \right]$$

$$\left(\text{by } (3\pi/46)^2\cdot(\sqrt{2}+1)^2 \approx 0.244668445 \right)$$

$$= (r-2)\log(k-1) - \log 2 - \log r$$

$$- 4r \times 0.2447 \times \frac{k-3}{(k-3) - 0.2447}$$

$$\geq (r-2)\log 4 - \log 2 - \log r - 4r \times 0.2447 \times \frac{2}{2 - 0.2447}$$

$$(\text{by } k \geq 5)$$

$$\geq \left(\frac{46\sqrt{2}+1}{2} - 2 \right) \log 4 - \log 2 - \log\left(\frac{46\sqrt{2}+1}{2} \right)$$

$$- 4 \times 0.2447 \times \frac{46\sqrt{2}+1}{2} \times \frac{2}{2 - 0.2447}$$

$$\left(\text{as it is increasing with } r, \text{ where } r \geq \frac{46\sqrt{2}+1}{2} \right)$$

$$\approx 43.01243307 - 0.69314718 - 3.497322742 - 36.83329505$$

$$> 43.01 - 0.70 - 3.50 - 36.84$$

$$= 1.97 > 0.$$

This proves the theorem. □

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