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An inverse problem for 1-dimensional heat equations

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1 Introduction

In this note we study the uniqueness in an inverse problem for 1-dimensional heat equations.

For $p \in C^1[0,1]$ and $a \in L^2(0,1)$, both of which are real-valued, let $(E_{p,a})$ be the heat equation

$$(1.1) \qquad \frac{\partial u}{\partial t} + (p(x) - \frac{\partial^2}{\partial x^2})u = 0 \qquad (0 < x < 1, 0 < t < \infty),$$

with the Dirichlet boundary condition

$$(1.2) \qquad u|_{x=0} = u|_{x=1} = 0 \qquad (0 < t < \infty),$$

and the initial condition

$$(1.3) \qquad u|_{t=0} = a(x) \qquad (0 < x < 1).$$

Let $u = u(t,x)$ be a unique solution of $(E_{p,a})$. Fix $x_0 \in (0,1]$ and T_1, T_2 such that $0 \le T_1 < T_2 < \infty$. Our problem is to study to what extent the "observation" $\{(u_x(t,0), u_x(t,x_0)); T_1 \le t \le T_2\}$ determines the potential p and the initial data a. To formulate this problem, we define the map χ_{x_0} by

(1.4) $$\chi_{x_0} : (p, a) \longmapsto \{(u_x(t,0), u_x(t,x_0)); T_1 \le t \le T_2\},$$

and the set M_{p,a,x_0} by

(1.5) $$M_{p,a,x_0} = \{(q,b) \in C^1[0,1] \times L^2(0,1); \chi_{x_0}(q,b) = \chi_{x_0}(p,a)\}.$$

Then the observation determines uniquely (p, a) if and only if

(1.6) $$M_{p,a,x_0} = \{(p,a)\}.$$

Remark 1.1. We can replace the time interval $[T_1, T_2]$ by $(0, \infty)$ in (1.4) because of the analyticity of $u(t,x)$ with respect to $t \in (0, \infty)$.

Let A_p denote the self-adjoint realization in $L^2(0,1)$ of $p(x) - \partial^2/\partial x^2$ with the Dirichlet boundary condition. The eigenvalues and the eigenfunctions of A_p are denoted by $\{\lambda_n\}$ and $\{\varphi_n\}$, respectively, the latter being normalized as $\|\varphi_n\|_{L^2(0,1)} = 1$.

Definition 1.1. For $a \in L^2(0,1)$, the number

(1.7) $$N_{p,a} = \sharp\{n; (a, \varphi_n)_{L^2(0,1)} = 0\}$$

is called the degenerate number of a with respect to A_p.

The problem of uniqueness (1.6) is closely related to the degenerate number. In fact, Murayama [1] obtained the following result.

Theorem 0.1. (Murayama) If $x_0 = 1$, the observation determines (p, a)

uniquely if and only if $N_{p,a} = 0$.

One can also study the inverse problem for (1.1) with the Robin boundary condition:

$$\frac{\partial u}{\partial x} - hu|_{x=0} = \frac{\partial u}{\partial x} + Hu|_{x=1} = 0.$$

In this case, we aim at determining p, h, H and a through the observation $\{u(t,0), u(t,x_0); T_1 \leq t \leq T_2\}$. Then Suzuki [4] obtained the following result.

Theorem 0.2. (Suzuki) In the case of the Robin boundary condition, the observation determines p, h, H and a uniquely if and only if $x_0 = 1$ and the degenerate number is equal to 0.

The above two theorems suggest that the uniqueness depends on not only $N_{p,a}$ but also the position of x_0. The aim of this paper is to show that, in the case of the Dirichlet boundary condition, generically, the uniqueness does not hold if $0 < x_0 < 1$.

A reduction is necessary before going into the details. By the same argument as in Suzuki [4], one can show that , if $(q,b) \in M_{p,a,x_0}$, b is uniquely determined by q. So, if we let

$$\tilde{M}_{p,a,x_0} = \{q \in C^1[0,1];\ \textit{there exists some}\ b \in L^2(0,1)\ \textit{such that}\ (q,b) \in M_{p,a,x_0}\}, \tag{1.8}$$

(1.6) is equivalent to

$$\tilde{M}_{p,a,x_0} = \{p\}. \tag{1.9}$$

2 Main results

Our results are summarized in the following two theorems.

Theorem 1. For each $x_0 \in (0,1)$, there exists an open dense set $U_{x_0} \subseteq C^1[0,1]$ such that $p \in U_{x_0}$ implies $\tilde{M}_{p,a,x_0} \neq \{p\}$ for any $a \in L^2(0,1)$. In particular, when $x_0 \in (0,\frac{1}{2})$, we can choose $U_{x_0} = C^1[0,1]$.

Remark 2.1. Let $H = \{\frac{2k}{2k+1};\ k \in \mathbf{N}\}$. For $x_0 \in (0,1) \setminus H$, U_{x_0} contains all the constant functions. In other words, if $x_0 \in (0,1) \setminus H$ and p is a constant function, then $\tilde{M}_{p,a,x_0} \neq \{p\}$ for any $a \in L^2(0,1)$.

Theorem 2. Let p be constant and $N_{p,a} = 0$.
(i) In the case of $x_0 \in (\frac{1}{2}, 1)$, let

$$R_1 = \{q \in C^1[0,1]; q'(x_0) + q'(1) \leq 0\}. \tag{2.1}$$

Then $R_1 \cap \tilde{M}_{p,a,x_0} = \{p\}$.
(ii) In the case of $x_0 = \frac{1}{2}$, let

$$R_2 = \{q \in C^1[0,1]; q'(x_0) + q'(0) \geq 0\}. \tag{2.2}$$

Then $R_2 \cap \tilde{M}_{p,a,x_0} = \{p\}$.
(iii) In the case of $x_0 \in (0, \frac{1}{2})$, let

$$R_3 = R_2 \cap \{\ \textit{the real analytic functions on}\ (0,1)\}. \tag{2.3}$$

Then $R_3 \cap \tilde{M}_{p,a,x_0} = \{p\}$.

By Theorem 1, the uniqueness does not hold generically if $0 < x_0 < 1$. And, by the above theorems, it follows that there exists a potential which has the same observation in $C^1[0,1] \setminus R_1$ if p is constant, $N_{p,a} = 0$, and $x_0 \in (\frac{1}{2}, 1) \setminus H$. In the case of $x_0 = \frac{1}{2}$ or $x_0 \in (0, \frac{1}{2})$, the above statement holds for R_2 or R_3 instead of R_1, respectively.

3 A hyperbolic equation

The following propositions, which arise from Suzuki's deformation formula ([3] or [4]), are the key points of the proof of Theorems 1 and 2.

Let $D = \{(x,y) \in \mathbf{R}^2; 0 < y < x < 1\}$, and consider the following equations :

$$
(E)\begin{cases}
(3.1) & K_{xx} - K_{yy} + (p(y) - q(x))K = 0 \quad \text{on } D,\\
(3.2) & K(x,x) = \frac{1}{2}\int_0^x (q(s) - p(s))ds \quad (0 \le x \le 1),\\
(3.3) & K(x,0) = 0 \quad (0 \le x \le 1),\\
(3.4) & K(1,y) = 0 \quad (0 \le y \le 1),\\
(3.5) & K_x(x_0,y) = 0 \quad (0 \le y \le x_0),\\
(3.6) & K(x_0,x_0) = 0.
\end{cases}
$$

Proposition 1. If there exist $q \in C^1[0,1]$ and $K \in C^2(\bar{D})$ such that K does not vanish identically on $\bar{D}$ and satisfies the equation (E), then $\tilde{M}_{p,a,x_0} \neq \{p\}$ for any $a \in L^2(0,1)$.

Remark 3.1. For $q \in C^1[0,1]$ in Proposition 1, $q \in \tilde{M}_{p,a,x_0}$ and $q \neq p$ holds.

Proposition 2. When $N_{p,a} = 0$, $q \in \tilde{M}_{p,a,x_0}$ if and only if there exists $K \in C^2(\bar{D})$ satisfying (E).

We can show these propositions in the same way as in [4].

4 Proof of theorems

Sketch of proof of Theorem 2.

If $x_0 \in (\frac{1}{2}, 1)$, we see that $q \in \tilde{M}_{p,a,x_0}$ implies $q'(x_0)+q'(1) = \int_{x_0}^{1}(q-p)^2dx$ by Proposition 2 and a straightforward calculation. Therefore, $q \in R_1 \cap \tilde{M}_{p,a,x_0}$ implies $q \equiv p$ on $[x_0, 1]$, *i.e.* $K(x,x) = 0$ for $x \in [x_0, 1]$. By solving (E), we get $K \equiv 0$ on $\bar{D}$, so $K(x,x) = 0$ for $x \in [0,1]$. From (3.2), $q \equiv p$ on [0,1].

If $x_0 \in (0, \frac{1}{2}]$, by Proposition 2 we see that $q \in \tilde{M}_{p,a,x_0}$ implies $q'(x_0) + q'(0) = -\int_0^{x_0}(q-p)^2dx$. We then proceed in the same way as above.

Proof of Theorem 1.

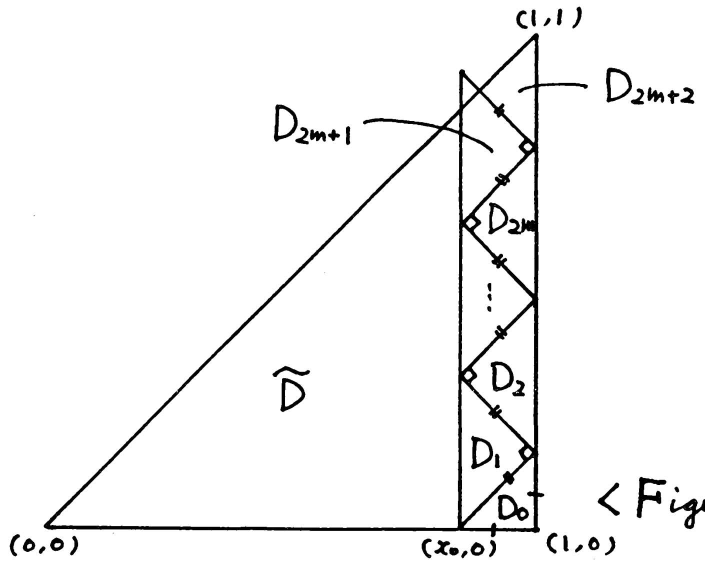
(I) *The case of* $x_0 \in [\frac{1}{2}, 1)$.

Let $G = \{g \in C^1[x_0, 1];\ g'(x_0) = g(1) = 0\}$.

< *Step* 1 > For $p, q \in C^1[0,1]$ and $g \in G$, we construct $K \in C^2(\bar{D})$ satisfying (3.1), (3.3), (3.4), (3.5) and

$$K_y(x,0) = g \quad (x_0 \le x \le 1). \tag{4.1}$$

This K is constructed as follows. We devide D into the pieces D_0, D_1, ..., D_{2m+2}, $\bar{D}$ (Figure 1) and solve the equation successively. Here, $g'(x_0) = g(1) = 0$ serves as a compatibility condition for the C^2-regularity of K. ([4])



< Figure 1 >

Notation. K in *Step* 1 is denoted by $K_g(x,y;q,p)$. In particular, when p is fixed, K is denoted by $K_g(x,y;q)$.

Remark 4.1.

(1) K_g is a $C^2(\bar{D})$-valued analytic function of q, g and p.

(2) K is linear with respect to g.

(3) There exists a monotone increasing continuous function $\tau : [0,\infty) \to (0,\infty)$ such that

$$\| K_g(\cdot,\cdot;p,q) \|_{C^2(\bar{D})} \leq \tau(\| p \|_{C^1[0,1]} + \| q \|_{C^1[0,1]}) \| g \|_{C^1[x_0,1]}$$

$$\| K_g(\cdot,\cdot;p_1,q_1) - K_g(\cdot,\cdot;p_2,q_2) \|_{C^2(\bar{D})}$$

$$\leq \tau(\| p \|_{C^1[0,1]} + \| q \|_{C^1[0,1]})(\| p_1 - p_2 \|_{C^1[0,1]} + \| q_1 - q_2 \|_{C^1[0,1]}) \| g \|_{C^1[x_0,1]}$$

for any $p, q \in C^1[0,1]$ and any $g \in G$. ([4])

< *Step* 2 > For fixed p, we consider the map

$$\begin{array}{rcl} T_g: \ C^1[0,1] & \longrightarrow & C^1[0,1] \\ q & \longmapsto & 2\frac{d}{dx}K_g(x,x;q)+p. \end{array}$$

By Remark 4.1 (3), there exists $\delta > 0$ such that , if $\|g\| < \delta$, T_g is a contraction map on some ball $U_B \subset C^1[0,1]$. So, T_g has a unique fixed point on U_B, denoted by $q(g)$. $K_g(x,y;q(g))$ satisfies (3.2).

Remark 4.2. $q(g)$ is analytic in g, so $K_g(x,y;q(g))$ is also analytic in g.

< *Step* 3 >

Proposition 3. If there exists $\tilde{g} \in G$ such that $K_{\tilde{g}}(x_0,x_0;p,p) \neq 0$, then $\tilde{M}_{p,a,x_0} \neq \{p\}$ for any $a \in L^2(0,1)$.

Proof of Proposition 3. Let $\tilde{g}$ be as above. By Remark 4.1 (2), we can choose $\| \tilde{g} \|_{C^1[x_0,1]}$ sufficiently small. We set

$$f(t) = K_{t\tilde{g}}(x_0,x_0;q(t\tilde{g})) \ \ (= tK_{\tilde{g}}(x_0,x_0;q(t\tilde{g}))).$$

We remark that $f(t)$ is an entire function and $q(0) = p$. From the assumption, we have $f(0) = 0$ and $f'(0) = K_{\tilde{g}}(x_0,x_0;p,p) \neq 0$. So, there exist $t_1, t_2 \in \mathbf{R}$, whose absolute values are very small, such that $f(t_1) > 0$ and $f(t_2) < 0$ by the inverse function theorem. $S(g) = K_g(x_0,x_0;q(g))$ is continuous with respect to g. So, there exists $g_1 \in G$ such that $\| t_1\tilde{g} - g_1 \|_{C^1[x_0,1]}$ is very small and g_1 is linearly independent of $t_2\tilde{g}$ and that $S(g_1) > 0$. Since $S(g_1) > 0$ and $S(t_2\tilde{g}) < 0$, there exists $\hat{g} \in G$ such that $S(\hat{g}) = 0$, by the continuity of the function $S(\cdot)$. We remark that $\hat{g}$ does not vanish identically because g_1 is linearly independent of $t_2\tilde{g}$, and that $\| \hat{g} \|_{C^1[x_0,1]}$ is very small. Hence, satisfies (E), so $\tilde{M}_{p,a,x_0} \neq \{p\}$ for any $a \in L^2(0,1)$.

< *Step* 4 >

Lemma 1. If $x_0 \in [\frac{1}{2}, 1) \setminus H$ and p is a constant function, the assumption of Proposition 3 holds.

Lemma 2. If $x_0 \in H$, there exists $p_0 \in C^1[0,1]$ such that the assumption of Proposition 3 holds.

Admitting these lemmas for the moment, we continue the proof of Theorem 1.

If $x_0 \in [\frac{1}{2}, 1) \setminus H$, there exists $\hat{g} \in G$ such that $K_{\hat{g}}(x_0, x_0; 0, 0) \neq 0$ by Lemma 1. Let

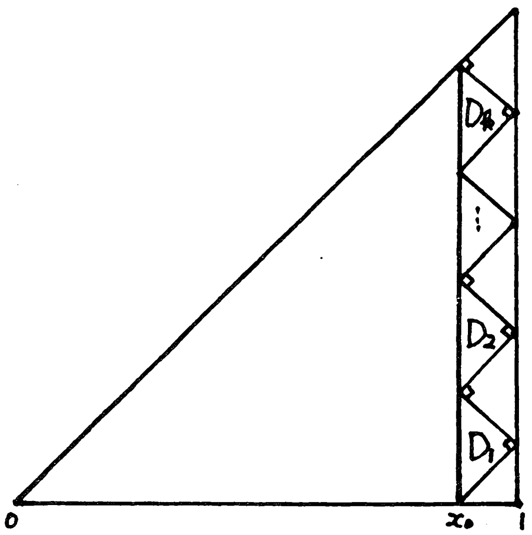
$$U_{x_0} = \{p \in C^1[0,1];\ K_{\hat{g}}(x_0, x_0; p, p) \neq 0\}.$$

Then U_{x_0} is an open set. $F(t) = K_{\hat{g}}(x_0, x_0; tp_0, tp_0)$ is an entire function with respect to t for any $p_0 \in C^1[0,1]$, so the zeros of F are discrete. Therefore U_{x_0} is dense in $C^1[0,1]$. And $p \in U_{x_0}$ implies that $\tilde{M}_{p,a,x_0} \neq \{p\}$ for any $a \in L^2(0,1)$ by Proposition 3 and Lemma 1.

If $x_0 \in H$, then we proceed in the same way as above. This completes the proof of Theorem 1 in the case of $x_0 \in [\frac{1}{2}, 1)$.

We next explain the proof of Lemma 1 and 2. Lemma 1 follows from a direct calculation, so we consider only Lemma 2.

Proof of Lemma 2. Let $x_0 = \frac{2k}{2k+1}$ and devide D as in Figure 2.



<Figure 2>

We then have

(4.2) $$K_g(x_0, x_0; p, p) = 2\sum_{j=1}^{k}(-1)^{k+j-1}\iint_{D_j} R(p)K_g(p)dxdy,$$

where $R(p)(x, y) = p(x) - p(y)$, $K_g(p) = K_g(x, y; p, p)$. Let $g = x^2 - 2x_0x + 2x_0 - 1 \in G$, and assume that $K_g(x_0, x_0; p, p) = 0$ for any $p \in C^1[0, 1]$. We differentiate (4.2) at $p = 0$, then we have

(4.3) $$\sum_{j=1}^{k}(-1)^j \iint_{D_j} R(p)K_g(0)dxdy = 0$$

for any $p \in C^1[0, 1]$. We now put $p(x) = x$ in the left-hand side of (4.3), then we have "the left-hand side of (4.3)"$= \frac{(x_0-1)^5(89+61x_0)}{180} \neq 0$. This is a contradiction, so there exists p_0 such that $K_g(x_0, x_0; p_0, p_0) \neq 0$.

(II) *The case of* $x_0 \in (0, \frac{1}{2})$.

Let $f \in C^1[0,1]$, $f(1) = 0$, $f = 0$ on $[0, 2x_0]$ and f does not vanish identically on [0,1]. For $p, q \in C^1[0,1]$ and f, there exists $K \in C^2(\bar{D})$ satisfying (3.1), (3.3), (3.4) and $K_y(x,0) = f$ $(0 \leq x \leq 1)$. K is uniquely determined. We remark that K satisfies (3.5) and (3.6) by the assumptions on f. We now consider the map

$$T_f : q \longmapsto 2\frac{d}{dx}K(x,x) + p.$$

If $\| f \|_{C^1[0,1]}$ is sufficiently small, then T_f is a contraction map on some ball in $C^1[0,1]$. We can then argue as before.

5 Other observations and stability

We briefly explain what occurs when we take different observations. We first consider:

(1) $$\{u_x(t,0), u(t,x_0);\ T_1 \leq t \leq T_2\} \quad (x_0 \in (0,1]).$$

For this observation, we define M'_{p,a,x_0}, $\tilde{M'}_{p,a,x_0}$ in the same way as M_{p,a,x_0}, $\tilde{M}_{p,a,x_0}$, respectively. In this case, we have

Theorem 3. For each $x_0 \in (0,1]$,

$$\{p \in C^1[0,1];\ \tilde{M'}_{p,a,x_0} \neq \{p\}\ for\ any\ a \in L^2(0,1)\} = C^1[0,1].$$

We next consider:

(2) $$\{u_x(t,0), u_x(t,x_0), u(t,x_0);\ T_1 \leq t \leq T_2\} \quad (x_0 \in (0,1]).$$

We define M^*_{p,a,x_0}, $\tilde{M}^*{}_{p,a,x_0}$ in the same way as above. Then we have

Theorem 4.

(i) If $x_0 = 1$, $\tilde{M}^*{}_{p,a,x_0} = \{p\}$ holds if and only if $N_{p,a} = 0$.

(ii) If $x_0 \in (\frac{1}{2}, 1)$ and $N_{p,a} < +\infty$, then $\tilde{M}^*{}_{p,a,x_0} = \{p\}$.

(iii) If $x_0 = \frac{1}{2}$, $\tilde{M}^*{}_{p,a,x_0} = \{p\}$ holds if and only if $N_{p,a} \leq 1$.

(iv) If $x_0 \in (0, \frac{1}{2})$, for any $p \in C^1[0,1]$ and any $a \in L^2(0,1)$, we have $\tilde{M}^*{}_{p,a,x_0} \neq \{p\}$.

For $q \in C^1[0,1]$, we consider a bounded operator

$$\begin{array}{rccc} \Lambda_q : & L^2(0,1) & \longrightarrow & C^0(I) \times C^0(I) \\ & a & \longmapsto & (u_x(t,0), u_x(t,1)), \end{array}$$

where $u = u(t,x)$ is the solution of $(E_{q,a})$ and $I = [T_1, T_2]$, $T_1 > 0$. By Theorem 0.1, it is easy to see that $\Lambda_{q_0} = \Lambda_{q_1}$ implies $q_0 = q_1$. So, the map $q \mapsto \Lambda_q$ is injective. To study the continuity of the inverse map is an interesting problem. Using the result of [2], we obtain :

Theorem 5. Let $\{q_j\}_{j=1}^{\infty} \subset C^1[0,1]$ and $\sup_j \|q_j\|_{L^2(0,1)} < +\infty$, then $\Lambda_{q_j} \to \Lambda_{q_0}$ in $B(L^2(0,1), C^0(I) \times C^0(I))$ if and only if $q_j \to q_0$ in $L^2(0,1)$ weakly.

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