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On Some Quasilinear Elliptic Equations

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1 Introduction

Let us consider the following Sobolev-Poincaré-type inequality :

$$\text{(SP)} \qquad |u|_{L^q(\Omega)} \leq C\,|\nabla u|_{L^p(\Omega)} \qquad \forall u \in W_o^{1,p}(\Omega)$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. Suppose that $1 < q < p^*$ with $p^* = \infty$ for $p \geq N$ and $p^* = Np/(N-p)$ for $p < N$. Then Rellich's theorem assures that $W_o^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$, so it is easy to construct an element $u_o \in W_o^{1,p}(\Omega)\backslash\{0\}$ which attains the best possible constant for (SP). Furthermore it can be shown that u_o give a nontrivial solution of the equation :

$$-\Delta_p\, u \;=\; \lambda\, |u|^{q-2}u \qquad (\lambda > 0), \qquad \Delta_p \;=\; div\,(|\nabla u|^{p-2}\,\nabla u)$$

In this paper we consider the equations of more general form :

$$(E)_\lambda \qquad\qquad -\Delta_p\, u \;=\; \lambda\, g\,(x,u)$$

The case that $p = 2$, i.e., $\Delta_p = \Delta$, has been studied by many peoples. However it seems that the general case, $p \neq 2$, has not been investgated so vigorously. The purpose of this paper is to discuss the existence of nontrivial solutions of $(E)_\lambda$ and the number of solutions. Our argument will rely on a variant of the Ljusternik-Schnirelman theory due to Clark [1] and follow the idea of Rabinowitz [7]. In carrying out this, it should be noted that the Lagrangian derived from the Euler equation $(E)_\lambda$ is defined on $W_o^{1,p}(\Omega)$ which is not a Hilbert space, therefore we can not use the orthogonal decompositon or some nice properties of eigenfunctions ; and since the solution of $(E)_\lambda$ does not always belong to $C^2(\Omega)$ (see [5]), we must always work in the framework of weak solutions. To get over these difficulties, we need some delicate arguments based on the notion of Schauder basis, the duality map and the convex analysis.

2 Main Results and Basic Lemmas

2.1 Main Results

Our main results are stated in the following two theorems according to the behaviour of $g(\cdot,u)$ at $u=\infty$, roughly speaking, sub-principal case : $g(\cdot,z) = o(|z|^{p-1})$(Theorem 1) and super-principal case : $g(\cdot,z) = O(|z|^{p-1})$ (Theorem 2):

Theorem 1 *Assume the following* (g.1)-(g.3) :

(g.1) $g(x,z)$ *is continuous in* $(x,z)\in\Omega\times\mathbf{R}^1$ *and odd in* $z\in\mathbf{R}^1$.

(g.2) $^{\exists}\epsilon > 0$ *s.t.* $z\,g(x,z) > 0 \quad {}^{\forall}(x,z)\in\bar\Omega\times B(0,\epsilon)\backslash\{0\}$.

(g.3) *The following* (a) *or* (b) *holds*:

(a) $^{\exists}\bar z > 0$ *s.t.* $g(x,\bar z)\le 0 \quad {}^{\forall}x\in\bar\Omega$.

(b) $g(x,z)\,|z|^{-(p-1)}\to 0$ *as* $|z|\to\infty$ *uniformly in* $x\in\bar\Omega$.

Then for all $k\in\mathrm{N}$, *there exists* $\lambda_k > 0$ *such that for all* $\lambda\ge\lambda_k$, $(E)_\lambda$ *has* k *distinct nontrivial solutions.*

Theorem 2 *Assume* (g.1) *and the following* (g.4)-(g.7) :

(g.4) $|g(x,z)| \le C_1 + C_2|z|^{s-1} \quad p<s<p*$ *for* $p<N$,

$\le C_3\, e^{\psi(z)}$ *with* $\psi(z)\,|z|^{-N/(N-1)}\to 0 \quad (|z|\to\infty)$ *for* $p=N$.

(g.5) $g(x,z) = o(|z|^{p-1})$ *uniformly in* $x\in\bar\Omega$ *at* $z=0$.

(g.6) $g(x,z)\,|z|^{-(p-1)}\to\infty$ *as* $|z|\to\infty$ *uniformly in* $x\in\bar\Omega$.

(g.7) $$\sup_{x\in\bar\Omega}\limsup_{|z|\to\infty}\frac{G(x,z)}{z\,g(x,z)} \le \theta < \frac{1}{p}, \qquad G(x,z) = \int_0^z g(x,t)dt.$$

Then for all $\lambda > 0$, $(E)_\lambda$ *with* $g(x,u)$ *replaced by* $a(x)\,|u|^{p-2}u+g(x,u)$ *has infinitely many solutions* $\{u_k\}_{k\in N}$ *with* $J(\lambda,u_k)\to+\infty$ *as* $k\to\infty$, *where* $a(\cdot)\in L^\infty(\Omega)$, $J(\lambda,u) = \int_\Omega\{\frac{1}{p}|\nabla u|^p - \frac{\lambda}{p}a(x)|u|^p - \lambda\, G(x,u(x))\}dx$.

Remark 1 (1) There is no growth condition in Theorem 1, if we assume (a) in (g.2).
(2) Typical examples for $g(x,z)$ are given by $g_1(x,z) = a(x)\,|z|^{q-2}z$ or $g_2(x,z) = a(x)|z|^{q-2}ze^{|z|^\alpha}$. $g_1(x,z)$ satisfies (b) of (g.3) if $1<q<p$ and (g.4) for $p<N$ if $p<q$, and $g_2(x,z)$ satisfies (g.4) for $p=N$ if $p<q$ and $\alpha<N/(N-1)$.
(3) Let $g(x,z) = g_1(x,z)$, $1<q<p$, and $a(\cdot)$ be continuous on $\bar\Omega$. Then Theorem 1 assures that for every $k\in\mathrm{N}$, $(E)_\lambda$ has k distinct solutions u_j, $(j=1,2,\cdots,k)$

for some $\lambda = \lambda_k$. Since $v_j = \lambda_k^{1/(p-q)} u_j$ gives a solution for $(E)_1 \quad -\Delta_p v = a(x)\,|v|^{q-2} v$, it is proved that $(E)_1$ has infinitely many solutions.

2.2 Genus and Basic Lemmas

In this paper we shall use a genus-version of the Lyusternik-Schnirelman theory. For topological spaces X and Y, we denote by $C(X, Y)$ and $C^1(X, Y)$ the space of continuous maps and continuously differentiable maps from X to Y respectively. Let V be a real Banach space and let $\Sigma'(V) \equiv \Sigma'$ denote the family of all closed symmetric subsets of $V \backslash \{0\}$. We define a mapping $\gamma : \Sigma' \to \mathbf{N}$ by $\gamma\,(A) = min\,\{n \in \mathbf{N} \mid {}^{\exists} f \in C(A, \mathbf{R}^N \backslash \{0\}),\ f(-z) = -f(z) \quad {}^{\forall} z \in A\}$, and we put $\gamma\,(\emptyset) = 0$, and $\gamma\,(A) = \infty$ if the minimum does not exist. Then we say that A has genus $\gamma\,(A)$. The properties of genus required later are listed below:

Lemma 1 Let $A, B \in \Sigma'$.

(1) *If there exists an odd* $f \in C(A, B)$, *then* $\gamma\,(A) \le \gamma\,(B)$.

(2) *If* $A \subset B$, *then* $\gamma\,(A) \le \gamma\,(B)$.

(3) *If* f *is an odd homeomorphism of* A *onto* B, *then* $\gamma\,(A) = \gamma\,(B)$.

(4) $\gamma\,(A \cup B) \le \gamma\,(A) + \gamma\,(B)$.

(5) *If* $\gamma\,(B) < \infty$, *then* $\gamma\,(\overline{A \backslash B}) \ge \gamma\,(A) - \gamma\,(B)$.

(6) *If* A *is compact, then* $\gamma\,(A) < \infty$ *and there exists a* $\delta > 0$ *such that* $\gamma\,(N_\delta(A)) = \gamma\,(A)$, *where* $N_\delta(A)$ *is the set of points in* V *whose distance from* A *is less or equal to* δ.

(7) *If* $\gamma\,(A) = k$, *then for all* $j < k$ *there exists* $A_j \subset A$ *such that* $\gamma\,(A_j) = j$.

(8) *If* A *is homeomorphic by an odd map to the boundary of a symmetric bounded open neibourhood of* 0 *in* $\mathbf{R}^m$, *then* $\gamma\,(A) = m$.

For the proofs of these properties, see [1] and [6].

The fundamental tool for our argument is provied by the following result of Clark [1].

Lemma 2 *Let* $J \in C^1(V, \mathbf{R}^1)$ *with* J *even and* $J\,(0) = 0$. *Suppose that* J *satisfies the property*

$(PS)_-$ *For every sequence* $\{x_n\}$ *in* V *such that* $J\,(x_n) < 0$, $J\,(x_n)$ *is bounded below and* $J'\,(x_n) \to 0$ *in* V^*, *then* $\{x_n\}$ *possesses a convergent subsequence in* V.

Let

$$(\sharp) \qquad d_j = \inf_{A \in \Sigma';\ \gamma(A) \ge j}\ \sup_{x \in A} J\,(x)$$

and let $K_d = \{x \in V \mid J\,(x) = d,\ J'\,(x) = 0\}$. *If* $-\infty < d_j < 0$, *then* K_{d_j} *is compact and nonempty. Moreover if* $-\infty < d_j = \cdots = d_{j+r} \equiv d < 0$, *then* $\gamma\,(K_d) \ge r + 1$.

Remark 2 (1) From the definition of d_j, d_j is a monotone increasing function of j. (2) Lemma 2 remains valid with Σ' replaced by Σ = the family of all compact subsets of Σ'. (3) By virtue of (7) of Lemma 1, d_j can be characterized by ($\sharp$) with $\gamma(A) \geq j$ replaced by $\gamma(A) = j$.

We can not apply the above result directly to the super-principal case. However, through some finite-dimensional approximation, we can treat our problem within the same framework. For this purpose, we need the following variant of Clark's lemma :

Lemma 3 *Let* $I \in C^1(\mathbf{R}^m, \mathbf{R}^1)$ *be even with* $I(0) = 0$. *Assume*

(2.1) $\quad {}^\exists R > 0$ s.t. $I(x) < 0$ for $|x| > R$.

Furthermore assume

(2.2)
$$C_k = \sup_{A \in \Sigma(\mathbf{R}^m), \gamma(A) \geq m-k+1} \min_{x \in A} I(x) > 0$$

Then, for any $j = k, \cdots, m$, C_j *is a critical value of* I ,i.e., $K_{C_j} = \{x \in \mathbf{R}^m \mid I(x) = C_j, I'(x) = 0\} \neq \emptyset$. *Moreover, if* $C_k = C_{k+1} = \cdots = C_{k+r} = C$, *then* $\gamma(K_C) \geq r+1$.

Proof. Take $E = \mathbf{R}^m, J = -I$. Then (2.1) implies $(PS)_-$. Since $C_j \geq C_k$ for $j \geq k$, (2.2) ensures that C_j is also a critical point of $I(\cdot)$. Lemma 3 now follows from Lemma 2 and Remark 2. [QED]

3 Proofs of Theorems

We begin with the proof of Theorem 1.

Proof of Theorem 1 Put $V = W_o^{1,p}(\Omega)$ and $J(u) = \mathcal{A}(u) - \lambda \mathcal{B}(u)$ with $\mathcal{A}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx$, $\mathcal{B}(u) = \int_\Omega G(x, u(x))\, dx$. Then it is easy to see that $J \in C^1(V, \mathbf{R}^1)$, J is even and $J(0) = 0$, and $J'(u) = 0$ is equivalent to $(E)_\lambda$. In order to apply Lemma 2, we are going to verify $(PS)_-$ and give an estimate for d_j.

Verification of $(PS)_-$: First assume (b) of (g.3), then for any $\epsilon > 0$, there exists M_ϵ such that $|g(x, z)| \leq \epsilon\, |z|^{p-1}$ for $|z| \geq M_\epsilon$. Hence, by Poincaré's inequality, there exists a constant C such that

(3.1)
$$J(u) \geq \frac{1}{2p} |\nabla u|_{L^p}^p - C \qquad {}^\forall u \in V$$

Therefore $J(u_n) < 0$ implies that u_n is bounded in V and we can extract a subsequence u_{n_k} such that

(3.2) $\quad u_{n_k} \rightharpoonup u$ weakly in $W_o^{1,p}(\Omega)$,

(3.3) $\quad g(x, u_{n_k}) \rightarrow g(x, u)$ strongly in $L^{p/(p-1)}(\Omega)$ and V^*,

where we used (g.1), (b) of (g.3) and Egorov's theorem. On the other hand, making use of the definition of subdifferential, we get

$$\mathcal{A}(u) - \mathcal{A}(u_n) \geq < \mathcal{A}'(u_n), u - u_n > = < J'(u_n), u - u_n > + < g(x, u_n), u - u_n >$$

whence follows $\limsup_{n\to\infty} |\nabla u_n|_{L^p} \geq |\nabla u|_{L^p}$. Since V is uniformly convex, this relation and (3.2) assures that u_{n_k} converges to u strongly in V. Thus $(PS)_-$ is verified.

Estimate for d_j. Take linearly independent functions $e_1, e_2, \cdots, e_k \in V \bigcap L^\infty$ and put $A = \{u = \sum_{i=1}^k \alpha_i\, e_i(x) \mid \|\alpha\|_{\mathbf{R}^k} = \epsilon\}$. By virtue of (8) of Lemma 1, $\gamma(A) = k$; and by (g.2), $\mathcal{B}(u) > 0$ for any $u \in A$ for a sufficiently small ϵ. Since A is compact, $a_1(k) = \min_{u\in A} \mathcal{B}(u) > 0$. Similarly, $a_o(k) = \max_{u\in A} \mathcal{A}(u) < +\infty$. Then we derive

$$\max_{u\in A} J(u) \leq a_o(k) - \lambda\, a_1(k) < 0 \qquad \forall \lambda > \lambda_k = \frac{a_o(k)}{a_1(k)}$$

Furthermore, (3.1) assures that $d_j > -\infty$. Thus we can apply Lemma 2.
Assume now (a) instead of (b) in (g.3). We set $\bar{g}(x,z) = g(x,z)$ $|z| \geq \bar{z}$; $g(x,\bar{z})$ $z \geq \bar{z}$; $-g(x,\bar{z})$ $z \leq -\bar{z}$. Since $\bar{g}$ satisfies conditions (g.1),(g.2) and (b) of (g.3), the assertion of Theorem 1 holds true with $(E)_\lambda$ replaced by $(\bar{E})_\lambda$ $-\Delta_p\, u = \bar{g}(x,u)$. Let u be a solution of $(\bar{E})_\lambda$ and put $u_o \equiv \bar{z}$, then

$$-\Delta_p\, u - -\Delta_p\, u_o \leq \lambda\, \bar{g}(x,u) - \lambda \bar{g}(x,u_o)$$

Multiplying this by $[u - u_o]^+(x) = max(u(x) - u_o(x), 0) \in V$ (see [2]), we obtain

$$\int_{u>u_o} |\nabla(u-u_o)|^p dx = \int_{u>u_o} |\nabla u|^p dx = \int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u_o|^{p-2}\nabla u_o)\ \nabla[u-u_o]^+ dx \leq 0,$$

whence follows $[u - u_o]^+ \equiv 0$,i.e., $u(x) \leq \bar{z}$ $a.e. x \in \Omega$. Repeating the same argument as above for $-u$, we get $|u| \leq \bar{z}$, that is to say, u turns out to be a solution of $(E)_\lambda$. This completes the proof. [QED]

Proof of Theorem 2 In what follows we consider only the case where $\lambda = 1$. However exactly the same proof as below works for the general case. For the moment we also assume that $a \equiv 0$. Let $\{e_j\}_{j=1}^\infty$ be a Schauder basis of $W_o^{1,p}(\Omega)$ and V_m be the linear subspace of $W_o^{1,p}(\Omega)$ generated by $\{e_1, e_2, \cdots, e_m\}$. Put $J(u) = \mathcal{A}(u) - \mathcal{B}(u)$, $J_m = J|_{V_m}$ with $\mathcal{A}(u) = \frac{1}{p}|\nabla u|_{L^p}^p$ and $\mathcal{B}(u) = \int_\Omega \{\frac{1}{p}a(x)|u|^p + G(x,u(x))\}dx$. Since $u \in V_m$ has the form $u = \sum_{i=1}^m \alpha_i\, e_i$, $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbf{R}^m$ we define $I_m \in C^1(\mathbf{R}^m, \mathbf{R}^1)$ by $I_m(\alpha) = J_m(\sum \alpha_i\, e_i)$. To prove the theorem we need several lemmas.

Lemma 3.1 J_m has m distinct (modulo $\pm$) critical points.

Proof. Note that α is a critical point of I_m if and only if $u = \sum \alpha_i e_i$ is a critical point of J_m. We apply Lemma 3 with $I = I_m$ to find out the critical points of I_m. First of all, let us notice

(3.4) $$|u|_V \sim |u|_{L^r} \sim |\alpha|_{\ell^r} \sim |\alpha| \equiv \|\alpha\|_{\mathbf{R}^m}, \quad \forall r \in [1,\infty],$$

since the norms of every two m-dimensional Banach spaces are equivalent to each other. Condition (g.6) says that for any large number K, there exists $M_K > 0$ such that $|g(x,z)| \geq K\, p\, |u|^{p-1}$ for all $|u| \geq M_K$. Therefore there exists a C_k such that $G(x,z) \geq K\, |z|^p - C_k$ for all $z \in \mathbf{R}^1$. Then, for all unit vector $\bar{\alpha} \in S^{m-1}$,

$$(3.5) \qquad I_m(R\,\bar{\alpha}) \leq R^p\, \frac{1}{p}\, |\nabla \bar{u}|^p_{L^p} - R^p\, K\, |\bar{u}|^p_{L^p} + C_k\, |\Omega|,$$

where $\bar{u} = \sum_{i=1}^m \bar{\alpha}_i\, e_i$. By virtue of (3.4), there exist constants a_1, a_2 such that $|\nabla \bar{u}|_{L^p} \leq a_1 |\bar{\alpha}|$ and $|\bar{u}|_{L^p} \geq a_2 |\bar{\alpha}|$. Thus we obtain

$$(3.6) \qquad I_m(R\,\bar{\alpha}) \leq (\frac{a_1^p}{p} - K\, a_2^p)\, R^p\, |\bar{\alpha}|^p + C_k\, |\Omega|.$$

Then taking $K = 2\, a_1^p / p\, a_2^p$ and R sufficiently large enough, we can assure (2.1). On the other hand, (g.5) implies that for all $\epsilon > 0$, there exists a δ such that

$$(3.7) \qquad |g(x,z)| \leq \epsilon\, |z|^{p-1} \qquad \text{for all} \quad |z| < \delta.$$

Furthermore, using $|u|_{L^\infty} \leq a_3 |\alpha|$, we get $G(x,u(x)) \leq \frac{\epsilon}{p} |u(x)|^p$ for all $|\alpha| < \frac{\delta}{a_3}$. Consequently, for a sufficiently small $\rho > 0$, we have

$$(3.8) \qquad I_m(\alpha) \geq \frac{1}{p} |\nabla u|^p_{L^p} - \frac{\epsilon}{p} |u|^p_{L^p} > 0 \quad \text{for all} \quad 0 < |\alpha| \leq \rho,$$

which assures that $C_1 > 0$, since $\gamma\,(\{\alpha \in \mathbf{R}^m \mid |\alpha| = \rho\}) = m$. Thus $C_k^m = \sup_{A \in \Sigma(\mathbf{R}^m), \gamma(A) \geq m-k+1} \min_{\alpha \in A} I_m(\alpha)$ are critical values of I_m for all $k = 1, 2, \cdots, m$, ,i.e., there exist $\alpha_k^m \in R^m$ such that $I'_m(\alpha_k^m) = 0$. Therefore $u_k^m = \sum_{j=1}^m (\alpha_k^m)_j\, e_j$ satisfies $J'(u_k^m) = 0$, i.e.,

$$(3.9) \qquad \int_\Omega |\nabla u_k^m|^{p-2} \nabla u_k^m \cdot \nabla v dx = \int_\Omega g(x, u_k^m(x))\, v(x) dx \qquad {}^\forall v \in V_m$$

[QED]

Lemma 3.2 $\quad C_j^{m+1} \leq C_j^m \qquad 1 \leq {}^\forall j \leq m.$

Proof. For all $A \in \Sigma(\mathbf{R}^{m+1})$ with $\gamma\,(A) \geq m-j+2$, we see $\gamma\,(A \cap V_m) \geq m-j+1$. Indeed, since $A \cap V_m$ is also a compact set in V_{m+1}, there exists a δ-neibourhood $N_\delta(A \cap V_m)$ of $A \cap V_m$ in V_{m+1} such that $\gamma\,(A \cap V_m) = \gamma\,(N_\delta(A \cap V_m))$ by (6) of Lemma 1. Here we define the projection P from $A \setminus N_\delta(A \cap V_m)$ into $\mathbf{R}^1 \setminus \{0\}$ by $x = (x_1, \cdots, x_m, x_{m+1}) \mapsto P(x) = x_{m+1}$. Obviously P is odd and continuous, so $\gamma\,(\overline{A \setminus N_\delta(A \cap V_m)}) \leq 1$. Then, by (5) of Lemma 1, we get

$1 \geq \gamma(\overline{A \setminus N_\delta(A \cap V_m)}) \geq \gamma(A) - \gamma(N_\delta(A \cap V_m)) \geq m - j + 2 - \gamma(A \cap V_m)$, which gives $\gamma(A \cap V_m) \geq m - j + 1$. Hence

$C_j^{m+1} \leq \sup_{A \in \Sigma(\mathbf{R}^{m+1}), \gamma(A) \geq m-j+2} \min_{\alpha \in A \cap V_m} I_m(\alpha) \leq C_j^m$. [QED]

Lemma 3.3 Let $S = \{v \in V \setminus \{0\} \mid |\nabla v|_{L^p}^p = \int_\Omega g(x,u)\, u\, dx\}$. Then there exists a constant ρ such that

$$(3.10) \qquad |\nabla v|_{L^p} \geq \rho > 0 \quad {}^\forall v \in S$$

Proof. (The case $p > N$) Assume that there exists a sequence $v_m \in S$ such that $|\nabla u|_{L^p} \to 0$ as $n \to \infty$. Since V is continuously embedded in $L^\infty(\Omega)$, we have $|v_n|_{L^\infty} \to 0$. Then, by (3.7) and Poincaré's inequality, $|\nabla v_n|_{L^p}^p \leq \epsilon \int_\Omega |v_n|^p dx \leq \epsilon\, K\, |\nabla v_n|_{L^p}^p$, which implies $v_n = 0$ for sufficiently large n. This is a contradiction.

(The case $p < N$) It follows from (g.4) and (3.7) that for any $\epsilon > 0$, there exists C_ϵ such that

$$(3.11) \qquad |g(x,z)| \leq \epsilon\, |z|^{p-1} + C_\epsilon\, |z|^{s-1} \qquad \text{for all } z \in \mathbf{R}^1$$

Hence, by Poincaré's inequality and Sobolev's embedding theorem, we obtain $|\nabla v|_{L^p}^p \leq \epsilon\, |v|_{L^p}^p + C_\epsilon |v|_{L^s}^s \leq \epsilon\, K\, |\nabla v|_{L^p}^p + C_\epsilon |\nabla v|_{L^p}^s$, whence follows (3.10).

(The case $p = N$) First of all, we recall Moser's inequality (see [8]): There exist constants α_N, C_N depending only on N such that

$$(3.12) \qquad \frac{1}{|\Omega|} \int_\Omega \exp\left(\alpha_N \left(\frac{|v|}{|\nabla v|_{L^p}}\right)^{\frac{N}{N-1}}\right) dx \leq C_N \qquad {}^\forall v \in W_o^{1,N}(\Omega)$$

On the other hand, (g.4) and (3.7) enssure that for any $\epsilon > 0$, there exists C_ϵ such that

$$(3.13) \qquad |g(x,z)| \leq \epsilon\, |z|^{p-1} + C_\epsilon |z|^{2p-1} \exp\left(\frac{\alpha_N}{2} |z|^{\frac{N}{N-1}}\right) \qquad {}^\forall z \in \mathbf{R}^1$$

Thus we get

$$|\nabla v|_{L^p}^p \leq \epsilon\, |v|_{L^p}^p + C_\epsilon \int_\Omega |v|^{2p} \exp\left(\frac{\alpha_N}{2} |v|^{\frac{N}{N-1}}\right) dx$$

$$\leq \epsilon\, K\, |\nabla v|_{L^p}^p + C_\epsilon |\nabla v|_{L^p}^{2p}\, C_N\, |\Omega| \quad {}^\forall v \in S \text{ with } |\nabla v|_{L^p} \leq 1,$$

which assures (3.10). [QED]

Lemma 3.4 The set $S_d = \{v \in S \mid J(v) \leq d\}$ is bounded in V.

Proof. By condition (g.7), there exist numbers M and $\bar\theta > \frac{1}{p}$ such that $G(x,z) \leq \bar\theta\, g(x,z)$ for all $|z| \geq M$. Then

$$d \geq J(v) \geq \frac{1}{p}|\nabla v|_{L^p}^p - \bar\theta \int_{|v|\geq M} g(x,v)v\,dx - C_M \geq (\frac{1}{p} - \bar\theta)\,|\nabla v|_{L^p}^p - C_M$$

Therefore $|\nabla v|_{L^p}^p \leq (C_M + d)/(\frac{1}{p} - \bar\theta)$. [QED]

Lemma 3.5 $\gamma(S_d) < \infty$ for all $d > 0$

Proof. Suppose that $\gamma(S_d) = \infty$. Then there exist $w_n \in S_d$ (n=1,2,$\cdots$) such that

(3.14) $$w_n \in N(w_1^*)\bigcap N(w_2^*)\bigcap\cdots\bigcap N(w_{n-1}^*)\bigcap S_d \qquad k = 2,3,\cdots$$

where $w_n^* = F(w_n)$, F is the duality map from $L^q(\Omega)$ onto $L^{q'}$ with $\frac{1}{q}+\frac{1}{q'} = 1$, $p < q < p^*$ defined by $F(w) = |w|^{q-2}\, w/|w|_{L^q}^{q-2}$, and $N(w_j^*) = \{w \in V \mid < w_j^*, w >= 0\}$. If we can not take w_n satisfying (3.14), we deduce $S_d \subset \bigoplus_{j=1}^{n-1} \{w_j\} = L_{n-1} \subset V$, since $L^q(\Omega)$ is spanned by $w_1, w_2, \cdots, w_{n-1}$ and $N(w_{n-1}^*)$. Hence $\gamma(S_d) \leq n-1$, which is a contradiction. Noting that S_d is bounded in V and V is compactly embedded in $L^q(\Omega)$, we can extract a subsequence w_{n_k} which converges to w strongly in $L^q(\Omega)$. Then, by (3.14), $< w_n^*, w >= \lim_{k\to\infty} < w_n^*, w_{n_k} >= 0$ $\forall n \in \mathrm{N}$. Furthermore, recalling that F is a continuous map from $L^q(\Omega)$ onto $L^{q'}(\Omega)$, we obtain $|w|_{L^q}^2 =$ $< F(w), w > = \lim_{k\to\infty} < w_{n_k}^*, w > = 0$,i.e., $w = 0$. Now , using Egorov's theorem, we can show that $\int_\Omega g(x, w_{n_k})\, w_{n_k} dx \to 0$, whence follows $|\nabla w_{n_k}|_{L^p} \to 0$, which contradicts Lemma 3.3. [QED]

Proof of Theorem 2 (contined) Relation (3.9) with $v = u_k^m$ implies $u_k^m \in S$ and moreover, by Lemma 3.2, $J(v_k^m) = C_k^m \leq C_k^k$ for all $m \geq k$. Then Lemma 3.4 assures that $\{u_k^m\}$ is bounded in V. Therefore, by the same verification as for (3.2) and (3.3), we see $u_k^{m_j} \rightharpoonup u_k$ weakly in V ; $g(x, u_k^{m_j}) \to g(x, u_k)$ strongly in $L^{p/(p-1)}(\Omega)$ and in V^*. Hence

$$\mathcal{A}(v) - \mathcal{A}(u_k) \geq \mathcal{A}(v) - \liminf_{j\to\infty} \mathcal{A}(u_k^{m_j}) \geq \lim_{j\to\infty} < g(x, w_k^{m_j}), v - u_k^{m_j} >$$

$$= < g(x, u_k), v - u_k > \quad \forall v \in V_m,\ \forall m \in \mathrm{N}$$

Then the standard argument shows that u_k is a solution of $(E)_1$ and $C_k^m = J(u_k^m) \downarrow J(v_k) \equiv C_k$. From the definition of C_k^m, Lemma 3.2 and Lemma 3.3, we get $0 < \rho \leq C_1 \leq \cdots \leq C_{k-1} \leq C_k \leq \cdots$. We are now going to show that $C_k \uparrow \infty$ as $k \uparrow \infty$.

Suppose that $C_k \le \bar{C}$ for all k, and put $d = \bar{C} + 1$. In view of (g.5) and (g.6), we can show that every continuous path in V_m which connects 0 with ∞ must meet $S \bigcap V_m$, which means that $S \bigcap V_m$ separates 0 and ∞ in V_m. Hence, by (8) of Lemma 1, $\gamma\ (S \bigcap V_m) = m$. Since $S_{m,d} = S_d \bigcap V_m$ is compact by Lemma 3.4 and there exists an integer k independent of m such that $\gamma\ (S_{m,d}) \le k$ by Lemma 3.5 , we can take a δ-neibourhood $N_\delta(S_{m,d})$ of $S_{m,d}$ satisfying $\gamma\ (N_\delta(S_{m,d})) \le k$. Therefore $\gamma\ (\overline{S \bigcap V_m \setminus N_\delta(S_{m,d})}) \ge m - k$, by (5) of Lemma 1. Thus we derive $C^m_{k+1} \ge \min_{u \in \overline{S \bigcap V_m \setminus N_\delta(S_{m,d})}} J_m(u) \ge d$ for all $m > k + 1$.

Letting $m \to \infty$, we have $C_{k+1} \ge d \ge \bar{C} + 1 \ge C_{k+1} + 1$. This is a contradiction.

As for the case where $a(\cdot) \not\equiv 0$, we rely on the following lemma.

Lemma 3.6 Let

$$P_k \ : \quad v = \sum_{j=1}^{\infty} \alpha_j\, e_j \mapsto \sum_{j=k}^{\infty} \alpha_j\, e_j,$$

then

$$|P_k\, v|_{L^p} \le \epsilon_k\, |\nabla P_k\, v|_{L^p} \quad {}^\forall v \in V \quad \text{with} \quad \lim_{k \to \infty} \epsilon_k = 0$$

Proof. Suppose that the assertion does not hold. Then there exist $w_{n_k} = P_{n_k}\, v_{n_k}$ such that $|\nabla w_{n_k}|_{L^p} = 1$ and $|w_{n_k}|_{L^p} \ge \delta > 0$. Hence we can extract a subsequence of w_{n_k} denoted again by w_{n_k} such that $w_{n_k} \rightharpoonup w$ weakly in V and $w_{n_k} \to w$ strongly in $L^p(\Omega)$. Furthermore, by virtue of Mazur's theorem, we can choose convex combinations of w_{n_k} staisfying $u_m = \sum_{k=m}^{n_m} \beta_k\, w_{n_k} \to w$ strongly in V. Since $\{e_n\}$ is a Schauder basis, the mapping $e^*_n : u = \sum_{i=1}^{\infty} \alpha_i\, e_i \mapsto \alpha_n$ becomes a bounded linear functional. Therefore we find that $< e^*_n, w > = \lim_{m \to \infty} < e^*_n, u_m > = 0$ for all $n \in \mathbf{N}$,i.e., $w = 0$. This contradicts the fact that $|w_{n_k}|_{L^p} \to |w|_{L^p} \ge \delta > 0$. [QED]

For the general case, we work on $V_{m,k}$ = the linear subspace of V generated by $\{e_k, e_{k+1}, \cdots, e_m\}$ instead of V_m. If we take k sufficiently large enough, Lemma 3.6 assures that $a(\cdot)\ |u|^{p-2}\, u$ can be controled by $\epsilon\ |\nabla u|^p_{L^p}$ in $V_{m,k}$. Thus we can repeat the same argument as before. [QED]

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