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ON THE EXISTENCE OF HETEROCLINIC SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS ON A STRIP-LIKE DOMAIN

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1. Introduction. In this note, we consider the existence of heteroclinic solutions of semilinear elliptic equations on a strip-like domain. Let S be a strip-like domain, i.e., $S = R \times \Omega$ where $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$. We study the existence of solutions of the problem:

$$(P_0) \qquad \begin{cases} -\Delta u = g(u) & \text{in } S \\ u = 0 & \text{on } \partial S \end{cases}$$

where $g \in C(L^2(\Omega), L^2(\Omega))$ with $g'(\cdot) \in C(L^2(\Omega), L^2(\Omega))$.

The problem (P_0) appears in several problems in mechanics and physics. For example, the problem (P_0) describes waves in density-stratified channels[8]. It also considered as a model equation for viscous fluid flow between concentric cylinders(cf.[3],[7],[9]). The problem (P_0) can be rewritten as

$$(P) \qquad \begin{cases} -u_{tt} - \Delta_x u = g(u) & \text{in } R \times \Omega \\ u(t,x) = 0 & \text{for } x \in \partial\Omega \text{ and } t \in R \end{cases}$$

The t-stationally solutions of problem (P) is the solutions of the problem:

$$(P_s) \qquad \begin{cases} -\Delta_x u = g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The problem (P_s) is variational. That is the solutions of (P_s) are critical points of the functional

$$F(u) = \int_\Omega \left(\frac{1}{2} \mid \nabla_x u \mid^2 - G(u)\right) dx \quad \text{for } u \in H_0^1(\Omega) \qquad (1.1)$$

where $G(t) = \int_0^t g(s)ds$.

We are interested in the existence of heteroclinic solutions of (P). Existence of heteroclinic solutions is deeply related to the critical points(critical levels) of the functional F. Our purpose in this note is to show the existence of heteroclinic solutions of the problem (P). More precisely, we seak for a solution of (P) which converges to critical points of F as t tends to $\pm\infty$. The existence of non-periodic solutions for (P) has been proved by Amick[3], Bona *et al*[4], Kirchgassner[7] and Turner[9] for the case that g is odd. In [6], Esteban have shown the existence of non-periodic solutions without assuming oddness of g. In [5], Cannino have shown that if g is odd and (P_s) has a positive(negative) solution $v^+(v^-)$ which is a global minimum of F, there is a heteroclinic orbit of (P)(cf. also Kirchgassner[7]). We prove that if F has two global minumums, there exists a heteroclinic solution connecting the two points.

In the following, we denote by $\| \cdot \|$ and $| \cdot |_2$ the norms of the Sobolev space $H_0^1(\Omega)$ and $L^2(\Omega)$, respectively. For each x, $y \in L^2(\Omega)$, $< x,y >$ denotes the inner product of x, y in $L^2(\Omega)$. We denote by $\lambda_1 < \lambda_2 \leq \cdots$, the eigenvalues of the problem

$$-\Delta u = \lambda u, \qquad \text{in } \Omega, \qquad u = 0 \qquad \text{on } \partial\Omega. \tag{1.4}$$

Let $E(\lambda)$ denote the finite dimensional subspace of $H_0^1(\Omega)$ spanned by the eigenfunctions of the problem (1.4) corresponding to the eigenvalues smaller than or equal to λ. We denote by $M_{loc}(M_{glb})$ the set of local(global) minimums of F in $H_0^1(\Omega)$. We impose the following conditions on F:

(F1) *There exist* $a_1, a_2 > 0$ *such that*

$$< \nabla F(u), u > \geq a_1 \| u \|^2 \qquad \text{for all } u \text{ with } | u |_2 > a_2; \tag{1.2}$$

(F2) *there exists* $\lambda > 0$, $\alpha_1 > 0$ *and* $\alpha_2 > 0$ *satisfying that*

$$< \nabla F(u) - \nabla F(v), u - v > \geq -\alpha_1 \| u - v \|^2 \tag{1.3}$$

for each $u, v \in H_0^1(\Omega)$, *and*

$$< \nabla F(v + z_1) - \nabla F(v + z_2), z_1 - z_2 > \geq \alpha_2 \| z_1 - z_2 \|^2 \tag{1.4}$$

for each $v \in H_0^1(\Omega)$ *and* $z_1, z_2 \in E(\lambda)^\perp$;

(F3) M_{loc} *consists of finite number of points* $v_0, v_1, \cdots, v_n$*: Moreover,* F *is nondegenerate at each* v_i, $0 \le i \le n$.

The condition (F1) is imposed for the functional J *to satisfy Palais-Smale condition(hereafter refered by P-S). That is we may replace (F1) by*

(F1') *For each* $T > 0$, J *satisfies P-S in* H_T.

It is easy to see that if g satisfies that

$$\limsup_{|t|\to\infty} g(t)/t < \lambda_1, \tag{1.5}$$

then (F1) holds. The condition (F2) is fulfilled if g satisfies that for some $\lambda > 0$.

$$\sup_{t\in R} g'(t) < \lambda \tag{1.6}$$

We give an existence result for the case that the set M_{glb} consists of exactly two points. In this case, we can find a heteroclinical solution(orbit) without assuming any condition on critical points whose critical level are greater than the global minimal value.

Theorem 1. *Suppose that (F1) holds and that* F *has exact two global minimal points* $v_\pm$ *which are nondegenerate. Then there exists a solution* u *of (P) such that*

$$\lim_{t\to\pm\infty} u(t) = v_\pm \qquad \textit{in } L^2(\Omega).$$

Moreover, there exist sequences $\{t_n^\pm\}$ *with* $\lim_{n\to\infty} t_n^\pm = \pm\infty$ *such that*

$$\lim_{n\to\infty} u(t_n^\pm) = v_\pm \qquad \textit{in } H_0^1(\Omega).$$

By imposing (F2) and (F3), we can remove that condition that F has exactly two local minimals. That is we have

Theorem 2. *Suppose that (F1)-(F3) hold. Moreover suppose that There exists at least two global minimals of* F*. Then there exists a solution* u *of (1.1) such that*

$$\lim_{t\to\pm\infty} u(t) = v_\pm \quad \textit{in } L^2(\Omega),$$

where $v_\pm \in H_0^1(\Omega)$ *such that* $F(v_\pm) = \min\{F(v) : v \in H_0^1(\Omega)\}$.

Moreover, there exist sequences $\{t_n^\pm\}$ *with* $\lim_{n\to\infty} t_n^\pm = \pm\infty$ *such that*

$$\lim_{n\to\infty} u(t_n^\pm) = v_\pm \qquad \text{in } H_0^1(\Omega).$$

Remark 1. We may replace (F3) with the following condition which is slightly weaker than (F3):
(F3') (a): M_{loc} *consists of finite points ;*

(b): each point in $M_{loc} \backslash M_{glb}$ *is nondegenerate, and for each* $v \in M_{glb}$*, there exists a neighborhood* U *of* v *such that*

$$< \nabla F(v), u - v > \geq 0 \quad \text{for all } u \in U, \tag{1.7}$$

2. Proof of Theorems . *In the following, we assume for simplicity that* $\min\{F(v) : v \in H_0^1(\Omega)\} = 0$. *Let*

$$\Gamma = \{u \in L^2(R; H_0^1(\Omega)) \cap H^1(R; L^2(\Omega)) : \lim_{t\to\pm\infty} u(t) = v_\pm \text{ in } L^2(\Omega)\}$$

and

$$\Gamma_0 = \{u \in \Gamma : | u_t(t) |_2^2 = F(u(t)), \text{ for a.e. } t \in R\}.$$

We also set

$$J_\infty = \int_{-\infty}^{\infty} \int_\Omega | u_t |^2 \, dx dt + \int_{-\infty}^{\infty} F(u) dt.$$

Then we have

Lemma 2.1. $m = \inf\{J_\infty(u) : u \in \Gamma_0\} = \inf\{J_\infty(u) : u \in \Gamma\} < \infty.$

where J_∞ *is the functional defined in (3.8).*

Proof. Since $v_\pm$ are nondegenerate, there exist neighborhoods $U_\pm$ of $v_\pm$ and homeomorphisms $\varphi_\pm : U_\pm \to H_0^1(\Omega)$ satisfying that

$$F(\varphi_\pm(v)) = \| v \|^2 \qquad \text{for all } v \in U_\pm.$$

For simplicity, we assume that $\varphi_\pm(v) = v - v_\pm$. That is

$$F(v) = \| v - v_\pm \|^2 \qquad \text{for all } v \in U_\pm. \tag{2.1}$$

We first see that $m = \inf\{J_\infty(u) : u \in \Gamma\} < \infty$. Let $u : (-\infty,\infty) \to H_0^1(\Omega)$ be a function defined by

$$u(t) = (\frac{1}{2} - \beta(t))v_- + (\frac{1}{2} + \beta(t))v_+$$

where $\beta(t) = -t/2(|\ t\ | + 1)$ for $t \in R$. Then from the definition of β, we have that

$$\int_{-\infty}^{\infty} (|\ u_t\ |^2 + \|\ u\ \|^2)dt < \infty.$$

Therefore $u \in \Gamma$. By (2.1), it is easy to check that $J_\infty(u) < \infty$. Then we have shown that $m < \infty$. Let P_0 be the set of normalized pathes connecting v_- and v_+. That is

$$P_0 = \{p(\cdot) \in C([-1,1]; H_0^1(\Omega)) :|\ p'(s)\ |= const. \text{ in } (-1,1),$$
$$p(\pm 1) = v_\pm \text{ and } p(t) \notin \{v_-, v_+\} \text{ for } t \in (-1,1)\} \tag{2.2}$$

Let V be a set of mappings $\tau(\cdot) : [-1,1] \to (-\infty,\infty)$ satisfying that

$$t(\cdot) \text{ is strictly monotone increasing, and } \lim_{s\to\pm 1} \tau(s) = \pm\infty.$$

We note that if $\tau(\cdot) \in V$, τ is differentiable a.e. in $(-1,1)$. Let $\tau(\cdot) \in V$ and $p \in P_0$. For simplicity, we assume that $|\ p'(s)\ |= 1$ for all $s \in (-1,1)$. Now we set $u(t) = p(\tau^{-1}(t))$ for $t \in R$. Then from the relation that $t = \tau(s)$ for $s \in [-1,1]$, we have

$$\frac{dt}{ds} =|\ \frac{dp}{dt}\ |^{-1}, \quad (\frac{dp}{dt} = \frac{dp}{ds}\cdot\frac{ds}{dt}).$$

Then it follows that

$$J_\infty(u(t)) = \int_{-\infty}^{\infty} (|\ u_t\ |^2 + F(u(t)))dt = \int_{-1}^{1} (\frac{ds}{dt} + F(u(s))\frac{dt}{ds})ds.$$

Here we set

$$J_p(\tau) = \int_{-1}^{1} (\frac{ds}{d\tau} + F(v(s))\frac{d\tau}{ds})ds \quad \text{for each } \tau \in V\ . \tag{2.3}$$

We claim that for each $p \in P_0$, there exists $\tau_0 \in V$ such that $J_p(\tau_0) = m_p = \inf\{J_p(\tau) : \tau \in V\}$. Let $p \in P_0$ such that $m_p < \infty$. Then there exists a sequence $\{\tau_n\} \subset V$ such that $\lim_{n\to\infty} J_p(\tau_n) = m_p$. We may assume without any loss of generality that $\tau_n(0) = 0$ for all $n \geq 1$. We first see that there exist $\tau_0 \in V$ and a subsequence of $\{\tau_n\}$(again denoted by $\{\tau_n\}$) such that $\tau_n(t) \to \tau_0(t)$ for all $t \in [-1,1]$. Let t be an arbitrary number in $(-1,1)$. We may suppose by extracting subsequences that $\tau_n(t) \to c$, as $n \to \infty$. Since $p(s) \notin \{v_-, v_+\}$ for $s \in (-1,1)$, we have

$$c_0 = \inf\{F(p(s)) : 0 \leq s \leq | t |\} > 0. \tag{2.4}$$

Then

$$J_p(\tau_n) \geq \left| \int_0^t F(v(s))\frac{d\tau_n}{ds}ds \right| \geq c_0 \mid \tau_n(t) \mid \qquad \text{for each } n \geq 1.$$

Then since $\{J_p(\tau_n)\}$ is bounded, we find that $\{\tau_n(t)\}$ is also bounded. Thus we obtain that $| c |< \infty$. Since τ_n is monotone increasing, it follows that $\tau_n(s)$ is convergent for all $s \in [0,t]$. Since t is arbitrary, we have by repeating the argument above that there exists a subsequence of $\{\tau_n\}$ (denoted by $\{\tau_n\}$) such that $\lim \tau_n(s)$ exists for all $s \in [-1,1]$. Here we put $\tau_0(s) = \lim \tau_n(s)$ for all $s \in [-1,1]$. Then τ_0 is monotone increasing. We next see that τ_0 is strictly monotone. Suppose that $\tau_0(a) = \tau_0(b)$ for some $a < b$. By (2.3), we have

$$J_p(\tau_n) \geq \int_{-1}^1 (\frac{d\tau_n}{ds})^{-1}ds \geq \int_a^b (\frac{d\tau_n}{ds})^{-1}ds.$$

Then since $\lim_{n\to\infty} \tau_n(a) = \lim_{n\to\infty} \tau_n(b)$, the right hand side of the inequality above tends to infinity as $n \to \infty$. Since $\{J_p(\tau_n)\}$ is bounded, this is a contradiction. Therefore we have shown that $\tau_0 \in V$. Since τ_n and τ are monotone increasing, we also obtain that $d\tau_n/ds \to d\tau/ds$ a.e. on R. Then recalling that $d\tau_n/ds > 0$ for all $n \geq 1$, we obtain from Fatou's lemma that

$$J_p(\tau_0) = \int_{-1}^1 (\frac{ds}{d\tau_0} + F(p(s))\frac{d\tau_0}{ds})ds \leq \liminf \int_{-1}^1 (\frac{ds}{d\tau_n} + F(p(s))\frac{d\tau_n}{ds})ds$$

Thus we have shown that $J_p(\tau_0) = \lim_{n\to\infty} J_p(\tau_n) = m_0$. This implies that for each $\sigma \in V$,

$$\int_{-1}^{1} (J_p(\tau_0))'(\sigma)(s)ds = \int_{-1}^{1} (-(\frac{d\tau_0}{ds})^2 + F(p(s)))(\frac{d\sigma}{ds})ds \geq 0.$$

Since $\sigma \in V$ is arbitrary, it is easy to see that

$$(\frac{d\tau_0}{ds})^2 = F(p(s)) \quad \text{a.e. on } (-1,1). \tag{2.5}$$

Now we put $u(t) = p(\tau^{-1}(t))$ for $t \in R$. Then,

$$| u_t |^2 = | \frac{dp}{ds} \cdot \frac{d\tau}{dt} |^2 = F(u(t)) \quad \text{a.e. on } R.$$

Therefore, $u(\cdot) \in \Gamma_0$. Thus we have shown that the assertion holds. ∎

Lemma 2.2. *If $u \in \Gamma$ satisfies $J_\infty(u) = m$, then u is a solution of (P).*

Proof. Let $f(t) \in C_0^1([0,1])$, $\varphi \in H_1^0(\Omega)$ and $a, b \in R$ with $a < b$. We define a function $v \in \Gamma$ by

$$v(t) = \begin{cases} 0 & \text{on } (-\infty, a) \cup (b, \infty), \\ f((t-a)/(b-a))\varphi & \text{on } [a,b] \end{cases}$$

Then since $u + sv \in \Gamma$ for $s \in [0,1]$, we find that

$$< J'_\infty(u), v > = < u_{tt} + \Delta u + g(u), v > \geq 0.$$

From the definition of v, we have that

$$< \varphi, \int_a^b (u_{tt} + \Delta u + g(u)) f((t-a)/(b-a))dt > \geq 0.$$

Since $\varphi \in H_0^1(\Omega)$, $f \in C_0^1([0,1])$ and the intereval $[a,b]$ are arbitrary , we find that u is a solution of (P). ∎

Proof of Theorem 1. Let $\{u_n\} \subset \Gamma$ be a sequence such that

$\lim_{n\to\infty} J_\infty(u_n) = m$. By Lemma 4.1, we may assume that $\{u_n\} \subset \Gamma_0$. Since $\lim_{\|v\|\to\infty} F(v) = \infty$ by (F1), we may assume that for some $r > 0$,

$$\| u_n(t) \| \leq r \quad \text{for all } n \geq 1 \text{ and } t \in R \tag{2.6}$$

Then since $\{u_n\} \subset \Gamma_0$, we also have that there exists $r_0 > 0$ such that

$$| u_{nt}(t) | < r_0 \quad \text{for all } n \geq 1 \text{ and }, t \in R \tag{2.7}$$

Let $B(d)_\pm$ be open balls in $L^2(\Omega)$ centered $v_\pm$ with radius $d > 0$. Let $d_0 > 0$ such that $B(d_0)_- \cap B(d_0)_+ = \phi$. Since $u_n \in P_0$, we have $\{u_n(t) : t \in R\} \not\subset B(d_0)_- \cup B(d_0)_+$. Then we may assume without any loss of generality that

$$u_n(0) \notin B(d_0)_- \cap B(d_0)_+ \qquad \text{for all } n \geq 1.$$

On the other hand, we have from the assumption that for each $d > 0$, there exists $\epsilon(d) > 0$ such that

$$F(v) > \epsilon(d) \quad \text{for all } v \notin B(d)_- \cup B(d)_+$$

Recalling that $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, we obtain by (2.6) and (2.7) that u_n converges to $u \in C(-\infty, \infty : L^2(\Omega))$ uniformly on each bounded interval $I \subset R$. It also follows that u_{nt} converges to u_t weakly in $L^2(-\infty, \infty; L^2(\Omega))$ and that $u_n(t)$ converges to $u(t)$ weakly in $H_0^1(\Omega)$, for a.e. $t \in R$. Then we obtain from the upper semicontinuity of J_∞ that $J_\infty(u) \leq \lim_{n\to\infty} J_\infty(u_n)$. We next show that $u \in \Gamma$. That is we see that $\lim_{t\to\pm\infty} u(t) = v_\pm$ in $L^2(\Omega)$. Let $0 < d < d_0$. Then since F is greater than $\epsilon(d)$ ouside of $B(d)_- \cup B(d)_+$ and $\{J_\infty(u_n)\}$ is bounded, we have that there exists t_0 such that

$$u_n(t) \in B(d)_\pm \quad \text{for all } n \geq 1 \text{ and } \; | t | \geq t_0. \tag{2.8}$$

Then it follows from (2.8) that $\lim_{t\to\pm\infty} u(t) = v_\pm$ in $L^2(\Omega)$. Thus we have shown that $u \in \Gamma$. Therefore $J_\infty(u) = m$ and by Lemma 4.2, u is a solution of (P). This completes the proof. ∎

Sketch of the proof of Theorem 2. The proof of Theorem 2 is long and complicated. Here we give a sketch of the proof. For each $n \geq 1$, we out

$$J_n(u) = \left(\frac{1}{2}\int_{-nT}^{nT}\int_\Omega | u_t |^2 \, dxdt + \int_{-nT}^{T} F(u)dt\right), \tag{2.9}$$

for $u \in L^2(0, 2nT; H_0^1(\Omega))$. Then we can find a critical point u_n of J_n such that

$$d_n = (1/nT)J_n(u_n) \to m, \text{ as } n \to \infty$$

where m is the global minimal value of F. It then follows that

$$\lim_{n\to\infty} F(u_n(\pm nT)) = m.$$

It also follows that

$$\sup\{\| u_n(t) \|: n \geq 1, -nT \leq t \leq nT\} < \infty.$$

Then it is not so difficult to see that there exists a subsequence $\{u_{n_i}\}$of $\{u_n\}$ and $u \in L^2(R; H_0^1(\Omega))$ such that

$$u_{n_i}(t) \to u(t) \qquad \text{for all } t \in R.$$

Therefore u satisfies the assertion of Theorem 2.

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