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On the Chern character of SO(n)

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In [6] and [7] we described the Chern character homomorphism ch: $K^*(G) \longrightarrow H^{**}(G; \mathbf{Q})$ for $G = Spin(2n+1)$ and $SO(2n+1)$. The purpose of this paper is to study ch for $G = SO(2n)$ and $Spin(2n)$, where $n \geqq 1$.

§1. Representation rings.

In this section, for later use, we quote from [3] and [8] some results on the complex representation rings of classical Lie groups that concern us.

Let G be a compact, connected Lie group. Its representation ring $R(G)$ is the Grothendieck construction of the semiring of isomorphism classes $[V]$ of G-modules V over $\mathbf{C}$. It has an augumentation

$$\varepsilon: R(G) \longrightarrow \mathbf{Z}$$

which assigns to each class $[V]$ the dimension of V. Let T be a maximal torus of G. The Weyl group $W(G) = N(T)/T$ of G acts on T and therefore on $R(T)$. The inclusion $i: T \longrightarrow G$ induces a monomorphism $i^*: R(G) \longrightarrow R(T)$ and its image $i^*(R(G))$ coincides with the subring $R(T)^{W(G)}$ of elements left elementwise fixed by $W(G)$. Thus, through i^*, $R(G)$ can be regarded as a subring of $R(T)$. Besides, $R(G)$ is a λ-ring with operations

$$\lambda^k\colon R(G) \longrightarrow R(G) \quad \text{for} \quad k \geq 0$$

induced by the exterior powers of G-modules over **C**. Their properties needed in the sequel are: $\lambda^0(x) = 1$ for all $x \in R(G)$; if $\varepsilon(x) = n$, then $\varepsilon(\lambda^k(x)) = \binom{n}{k}$ and $\lambda^k(x) = 0$ for $k > n$.

Let T be the maximal torus of diagonal matrices in the unitary group U(n). If $\alpha_1, \cdots, \alpha_n$ denote the standard 1-dimensional representations of T, then

$$(1.1) \qquad R(T) = \mathbf{Z}[\alpha_1, \alpha_1^{-1}, \cdots, \alpha_n, \alpha_n^{-1}]/ (\alpha_1\alpha_1^{-1} - 1, \cdots, \alpha_n\alpha_n^{-1} - 1).$$

Put $\lambda_1 = [\mathbf{C}^n] \in R(U(n))$ and let $\lambda_k = \lambda^k(\lambda_1)$. Then

$$(1.2) \qquad R(U(n)) = \mathbf{Z}[\lambda_1, \cdots, \lambda_{n-1}, \lambda_n, \lambda_n^{-1}]/(\lambda_n\lambda_n^{-1} - 1)$$

and, as a subring of R(T), the relation

$$(1.3) \qquad \prod_{i=1}^{n} (1 + \alpha_i t) = \sum_{k=0}^{n} \lambda_k t^k$$

holds (in the polynomial ring R(T)[t]).

For the rotation groups SO(n), there is a fibre bundle

$$(1.4) \qquad SO(n) \xrightarrow{j_n} SO(n+1) \xrightarrow{q_n} SO(n+1)/SO(n) = S^n.$$

Let

$$(1.5) \qquad j_n'\colon U(n) \longrightarrow SO(2n)$$

be the real restriction mapping. Then $j_n'(T)$ becomes a maximal torus of SO(2n), which we denote by T again. Further, $j_{2n}(T)$ becomes a maximal torus of SO(2n+1), we denote by T also. Put $\mu_1' = [(\mathbf{R}^{2n+1})^{\mathbf{C}}] \in R(SO(2n+1))$ and let $\mu_k' = \lambda^k(\mu_1')$. Then

$$(1.6) \qquad R(SO(2n+1)) = \mathbf{Z}[\mu_1', \mu_2', \cdots, \mu_n']$$

and, as a subring of R(T) (which is just as in (1.1)), the relation

$$(1.7) \qquad (1 + t)\prod_{i=1}^{n} (1 + \alpha_i t)(1 + \alpha_i^{-1} t) = \sum_{k=0}^{2n+1} \mu_k' t^k$$

holds. Note that $\mu'_{2n+1-k} = \mu'_k$ for $k = 0, 1, \cdots, 2n+1$; in fact,

$$\sum_{k=0}^{2n+1} \mu'_{2n+1-k} t^k = (t+1)\prod_{i=1}^{n}(t+\alpha_i)(t+\alpha_i^{-1})$$
$$= (t+1)\prod_{i=1}^{n}(\alpha_i^{-1}t+1)\alpha_i\alpha_i^{-1}(\alpha_i t+1)$$
$$= (1+t)\prod_{i=1}^{n}(1+\alpha_i^{-1}t)(1+\alpha_i t) = \sum_{k=0}^{2n+1}\mu'_k t^k.$$

For $n \geqq 3$, the spinor group $Spin(n)$ is a universal covering group of $SO(n)$:

(1.8) $\quad S^0 = \{1, -1\} \longrightarrow Spin(n) \xrightarrow{p_n} SO(n).$

Then $p_n^{-1}(T)$ is a maximal torus of $Spin(n)$, which we denote by $\tilde{T}$. For this torus $\tilde{T}$ of $Spin(2n+1)$, there are 1-dimensional representations $\alpha_1, \cdots, \alpha_n$ of $\tilde{T}$ such that

(1.9) $\quad R(\tilde{T}) = Z[\alpha_1, \alpha_1^{-1}, \cdots, \alpha_n, \alpha_n^{-1}, (\alpha_1\cdots\alpha_n)^{1/2}]/$
$\quad (\alpha_1\alpha_1^{-1} - 1, \cdots, \alpha_n\alpha_n^{-1} - 1, ((\alpha_1\cdots\alpha_n)^{1/2})^2 - \alpha_1\cdots\alpha_n)$

and $p_{2n+1}: \tilde{T} \longrightarrow T$ induces the obvious inclusion $R(T) \longrightarrow R(\tilde{T})$ under the descriptions of (1.1) and (1.9). Set $\mu'_k = p_{2n+1}{}^*(\mu'_k) \in R(Spin(2n+1))$. Let Δ_{2n+1} be the spin representation of dimension 2^n. Then

(1.10) $\quad R(Spin(2n+1)) = Z[\mu'_1, \mu'_2, \cdots, \mu'_{n-1}, \Delta_{2n+1}]$

and in $R(\tilde{T})$,

(1.11) $\quad \Delta_{2n+1} = \prod_{i=1}^{n}(\alpha_i^{1/2} + \alpha_i^{-1/2}) = \sum_{\varepsilon_i=\pm 1} \alpha_1^{\varepsilon_1/2}\cdots\alpha_n^{\varepsilon_n/2}$

Moreover in $R(Spin(2n+1))$, the following relation holds:

(1.12) $$\Delta_{2n+1}^2 = \sum_{k=0}^{n}\mu'_k.$$

Therefore, $p_{2n+1}{}^*: R(SO(2n+1)) \longrightarrow R(Spin(2n+1))$ is given by

(1.13) $\quad p_{2n+1}{}^*(\mu'_k) = \mu'_k \qquad (k = 1, \cdots, n-1),$

$$p_{2n+1}{}^{*}(\mu_n') = \Delta_{2n+1}{}^{2} - \sum_{k=0}^{n-1} \mu_k'.$$

Put $\mu_1 = [(R^{2n})^{C}] \in R(SO(2n))$ and let $\mu_k = \lambda^k(\mu_1)$. In particular, μ_n can be halved, that is, there are two representations μ_n^+, μ_n^- of $SO(2n)$ such that

(1.14) $\mu_n = \mu_n^+ + \mu_n^-$ and $\varepsilon(\mu_n^+) = \varepsilon(\mu_n^-) = \frac{1}{2}\binom{2n}{n}$.

Then

(1.15) $R(SO(2n)) = Z[\mu_1, \mu_2, \cdots, \mu_{n-1}, \mu_n^+, \mu_n^-]/(\gamma_n)$

where

$$\gamma_n = (\mu_n^+ + \sum_{i\geq 1} \mu_{n-2i})(\mu_n^- + \sum_{i\geq 1} \mu_{n-2i}) - (\mu_{n-1} + \sum_{j\geq 1} \mu_{n-1-2j}).$$

Here the summations in the right side end at $\cdots + \mu_4 + \mu_2 + 1$ or $\cdots + \mu_3 + \mu_1$. As a subring of $R(T)$ (which is just as in (1.1)), the relations

(1.16) $\prod_{i=1}^{n} (1 + \alpha_i t)(1 + \alpha_i^{-1} t) = \sum_{k=0}^{2n} \mu_k t^k$

and

(1.17) $\mu_n^+ = \sum_{\Pi\varepsilon_i=1} \alpha_1^{\varepsilon_1}\cdots\alpha_n^{\varepsilon_n} + \Sigma'' \alpha_1^{\varepsilon_1}\cdots\hat{\alpha}_i^{\varepsilon_i}\cdots\hat{\alpha}_j^{\varepsilon_j}\cdots\alpha_n^{\varepsilon_n},$

$\mu_n^- = \sum_{\Pi\varepsilon_i=-1} \alpha_1^{\varepsilon_1}\cdots\alpha_n^{\varepsilon_n} + \Sigma'' \alpha_1^{\varepsilon_1}\cdots\hat{\alpha}_i^{\varepsilon_i}\cdots\hat{\alpha}_j^{\varepsilon_j}\cdots\alpha_n^{\varepsilon_n}$

hold, where the notation $\hat{\alpha}^{\varepsilon}$ means the replacement of α^{ε} by 1 and the number of $\hat{}$ in the summation Σ'' is even and positive.

Set $\mu_k = p_{2n}{}^{*}(\mu_k) \in R(Spin(2n))$. Let Δ_{2n}^+, Δ_{2n}^- be the half spin representations, each of dimension 2^{n-1}. Then

(1.18) $R(Spin(2n)) = Z[\mu_1, \mu_2, \cdots, \mu_{n-2}, \Delta_{2n}^+, \Delta_{2n}^-]$

and in $R(\tilde{T})$ (which is just as in (1.9)),

(1.19) $\Delta_{2n}^+ = \sum_{\Pi\varepsilon_i=1} \alpha_1^{\varepsilon_1/2}\cdots\alpha_n^{\varepsilon_n/2}$, $\Delta_{2n}^- = \sum_{\Pi\varepsilon_i=-1} \alpha_1^{\varepsilon_1/2}\cdots\alpha_n^{\varepsilon_n/2}$.

Moreover in $R(Spin(2n))$, the following relations hold:

(1.20)
$$\Delta_{2n}^{+}\Delta_{2n}^{-} = \mu_{n-1} + \sum_{\ell\geq 1}\mu_{n-1-2\ell},$$
$$\Delta_{2n}^{+\,2} = \mu_{n}^{+} + \sum_{k\geq 1}\mu_{n-2k},$$
$$\Delta_{2n}^{-\,2} = \mu_{n}^{-} + \sum_{k\geq 1}\mu_{n-2k}.$$

Therefore, $p_{2n}^{*}: R(SO(2n)) \longrightarrow R(Spin(2n))$ is given by

(1.21)
$$p_{2n}^{*}(\mu_k) = \mu_k \quad (k = 1, \cdots, n-2),$$
$$p_{2n}^{*}(\mu_{n-1}) = \Delta_{2n}^{+}\Delta_{2n}^{-} - \sum_{\ell\geq 1}\mu_{n-1-2\ell},$$
$$p_{2n}^{*}(\mu_{n}^{+}) = \Delta_{2n}^{+\,2} - \sum_{k\geq 1}\mu_{n-2k},$$
$$p_{2n}^{*}(\mu_{n}^{-}) = \Delta_{2n}^{-\,2} - \sum_{k\geq 1}\mu_{n-2k}.$$

The following propositions are immediate consequences of the above results.

Proposition 1.1. (1) $j_{2n}^{*}: R(SO(2n+1)) \longrightarrow R(SO(2n))$ satisfies

$$j_{2n}^{*}(\mu_1') = \mu_1 + 1.$$

(2) $j_{2n-1}^{*}: R(SO(2n)) \longrightarrow R(SO(2n-1))$ satisfies

$$j_{2n-1}^{*}(\mu_1) = \mu_1' + 1.$$

Proposition 1.2. (1) $\tilde{j}_{2n}^{*}: R(Spin(2n+1)) \longrightarrow R(Spin(2n))$ satisfies

$$\tilde{j}_{2n}^{*}(\mu_1') = \mu_1 + 1,$$
$$\tilde{j}_{2n}^{*}(\Delta_{2n+1}) = \Delta_{2n}^{+} + \Delta_{2n}^{-}.$$

(2) $\tilde{j}_{2n-1}^{*}: R(Spin(2n)) \longrightarrow R(Spin(2n-1))$ satisfies

$$\tilde{j}_{2n-1}^{*}(\mu_1) = \mu_1' + 1,$$
$$\tilde{j}_{2n-1}^{*}(\Delta_{2n}^{+}) = \tilde{j}_{2n-1}^{*}(\Delta_{2n}^{-}) = \Delta_{2n-1}.$$

Proposition 1.3. $j_n'^*: R(SO(2n)) \longrightarrow R(U(n))$ satisfies

$$j_n'^*(\mu_k) = \lambda_n^{-1} \sum_{i=0}^{k} \lambda_i \lambda_{n-k+i} \qquad (k = 1, \cdots, n).$$

Proposition 1.4. In $R(T)$,

$$\Delta_{2n}^{+\,2} - \Delta_{2n}^{-\,2} = \lambda_n^{-1} \Big(\sum_{i=0}^{n} \lambda_i\Big)\Big(\sum_{j=0}^{n} (-1)^{n-j} \lambda_j\Big).$$

§2. Cohomology rings

In this section we fix some notations concerning the integral cohomology of our groups G.

For $G = U(n)$, by Borel's transgression theorem, there exist elements $x_{2i-1} \in H^{2i-1}(U(n); \mathbf{Z})$, $i = 1, \cdots, n$, such that

$$H^*(U(n); \mathbf{Z}) = \Lambda_{\mathbf{Z}}(x_1, x_3, \cdots, x_{2n-1})$$

and

$$PH^*(U(n); \mathbf{Z}) = \mathbf{Z}\{x_1, x_3, \cdots, x_{2n-1}\}$$

where P denotes the primitive module functor. Thus

(2.1) $$H^*(U(n); \mathbf{Q}) = \Lambda_{\mathbf{Q}}(x_1, x_3, \cdots, x_{2n-1}).$$

$H^*(SO(n); \mathbf{Z})$ has 2-torsion only. For $G = SO(2n+1)$, by Poincaré duality and Borel's transgression theorem, there exist elements $x_{4i-1} \in H^{4i-1}(SO(2n+1); \mathbf{Z})$, $i = 1, \cdots, n$, such that

$$H^*(SO(2n+1); \mathbf{Z})/\mathrm{Tor} = \Lambda_{\mathbf{Z}}(x_3, x_7, \cdots, x_{4n-1})$$

and

$$PH^*(SO(2n+1); \mathbf{Q}) = \mathbf{Q}\{x_3, x_7, \cdots, x_{4n-1}\}.$$

Thus

(2.2) $$H^*(SO(2n+1); \mathbf{Q}) = \Lambda_{\mathbf{Q}}(x_3, x_7, \cdots, x_{4n-1}).$$

$H^*(Spin(n); Z)$ has 2-torsion if and only if $n \geqq 7$. For $G = Spin(2n+1)$, by similar reasons, there exist elements $\tilde{x}_{4i-1} \in H^{4i-1}(Spin(2n+1); Z)$, $i = 1, \cdots, n$, such that

$$H^*(Spin(2n+1); Z)/Tor = \Lambda_Z(\tilde{x}_3, \tilde{x}_7, \cdots, \tilde{x}_{4n-1})$$

and

$$PH^*(Spin(2n+1); \mathbf{Q}) = \mathbf{Q}\{\tilde{x}_3, \tilde{x}_7, \cdots, \tilde{x}_{4n-1}\}.$$

Thus

(2.3) $$H^*(Spin(2n+1); \mathbf{Q}) = \Lambda_{\mathbf{Q}}(\tilde{x}_3, \tilde{x}_7, \cdots, \tilde{x}_{4n-1}).$$

For $G = SO(2n)$, there exist elements $x_{4i-1} \in H^{4i-1}(SO(2n); Z)$, $i = 1, \cdots, n-1$, and $x'_{2n-1} \in H^{2n-1}(SO(2n); Z)$ such that

$$H^*(SO(2n); Z)/Tor = \Lambda_Z(x_3, x_7, \cdots, x_{4n-5}, x'_{2n-1})$$

and

(2.4) $$PH^*(SO(2n); \mathbf{Q}) = \mathbf{Q}\{x_3, x_7, \cdots, x_{4n-5}, x'_{2n-1}\}.$$

Thus

(2.5) $$H^*(SO(2n); \mathbf{Q}) = \Lambda_{\mathbf{Q}}(x_3, x_7, \cdots, x_{4n-5}, x'_{2n-1}).$$

For $G = Spin(2n)$, there exist elements $\tilde{x}_{4i-1} \in H^{4i-1}(Spin(2n); Z)$, $i = 1, \cdots, n-1$, and $\tilde{x}'_{2n-1} \in H^{2n-1}(Spin(2n); Z)$ such that

$$H^*(Spin(2n); Z)/Tor = \Lambda_Z(\tilde{x}_3, \tilde{x}_7, \cdots, \tilde{x}_{4n-5}, \tilde{x}'_{2n-1})$$

and

$$PH^*(Spin(2n); \mathbf{Q}) = \mathbf{Q}\{\tilde{x}_3, \tilde{x}_7, \cdots, \tilde{x}_{4n-5}, \tilde{x}'_{2n-1}\}.$$

Thus

(2.6) $$H^*(Spin(2n); \mathbf{Q}) = \Lambda_{\mathbf{Q}}(\tilde{x}_3, \tilde{x}_7, \cdots, \tilde{x}_{4n-5}, \tilde{x}'_{2n-1}).$$

The following two propositions are easy.

Proposition 2.1. $j_{2n-1}{}^*: H^*(SO(2n); Z) \longrightarrow H^*(SO(2n-1); Z)$ satisfies

$$j_{2n-1}{}^*(x_{4i-1}) = x_{4i-1} \quad (i = 1, \cdots, n-1),$$

$j_{2n-1}^{*}(x'_{2n-1}) = 0.$

Proposition 2.2. $\tilde{j}_{2n-1}^{*}: H^{*}(Spin(2n); Z) \longrightarrow H^{*}(Spin(2n-1); Z)$ satisfies

$$\tilde{j}_{2n-1}^{*}(\tilde{x}_{4i-1}) = \tilde{x}_{4i-1} \qquad (i = 1, \cdots, n-1),$$
$$\tilde{j}_{2n-1}^{*}(\tilde{x}'_{2n-1}) = 0.$$

Proposition 2.3. $j_n'^{*}: H^{*}(SO(2n); Z) \longrightarrow H^{*}(U(n); Z)$ satisfies

$$j_n'^{*}(x'_{2n-1}) = x_{2n-1}.$$

Furthermore, when $n = 2m$, not only $j_{2m}'^{*}(x'_{4m-1}) = x_{4m-1}$, but also $j_{2m}'^{*}(x_{4m-1}) = 0$.

Proof. Since

$$H^{*}(SO(2n)/U(n); Z) = Z[e_2, e_4, \cdots, e_{2n-2}]/ \left(e_{4k} + \sum_{i=1}^{2k-1} (-1)^i e_{2i}e_{4k-2i} : k = 1, \cdots, n-1\right) = \Delta_Z(e_2, e_4, \cdots, e_{2n-2})$$

where $\Delta_R(\ )$ denotes the algebra over a ring R having a simple system of generators, and $e_{2i} \in H^{2i}(SO(2n)/U(n); Z)$ (see [4, Chapter 3, Theorem 6.11]), the result follows from the spectral sequence argument for the integral cohomology of the fibre bundle

$$U(n) \xrightarrow{j_n'} SO(2n) \xrightarrow{q_n'} SO(2n)/U(n).$$

§3. K-rings

In this section we collect some results on the complex K-theory of our groups G.

Let

$$\beta\colon R(G) \longrightarrow K^{-1}(G)$$

be the map of [2]; then β has the following properties:

(3.1)(1) For each $\rho_1, \rho_2 \in R(G)$,

$$\beta(\rho_1 + \rho_2) = \beta(\rho_1) + \beta(\rho_2);$$

(2) If $n \in R(G)$ is the class of a trivial G-module of dimension n, then $\beta(n) = 0$;

(3) For each $\rho_1, \rho_2 \in R(G)$,

$$\beta(\rho_1\rho_2) = \varepsilon(\rho_2)\beta(\rho_1) + \varepsilon(\rho_1)\beta(\rho_2).$$

As will be seen below, the $Z/(2)$-graded K-rings $K^*(G)$ can be described by using this map β. Indeed, Hodgkin's theorem [2, Theorem A] says that if G is a compact connected Lie group with $\pi_1(G)$ torsion-free, then $K^*(G)$ is torsion-free and has the structure of a Hopf algebra over Z; more precisely, if

$$R(G) = Z[\rho_1, \rho_2, \cdots, \rho_\ell],$$

then

$$K^*(G) = \Lambda_Z(\beta(\rho_1), \beta(\rho_2), \cdots, \beta(\rho_\ell))$$

where each $\beta(\rho_i)$ is primitive. Therefore, for $G = U(n)$, it follows from (1.2) and (3.1) that $\beta(\lambda_n^{-1}) = -\beta(\lambda_n)$ and

(3.2) $$K^*(U(n)) = \Lambda_Z(\beta(\lambda_1), \cdots, \beta(\lambda_{n-1}), \beta(\lambda_n)).$$

Similarly, for $G = Spin(2n+1)$, it follows from (1.10) that

(3.3) $$K^*(Spin(2n+1)) = \Lambda_Z(\beta(\mu_1'), \cdots, \beta(\mu_{n-1}'), \beta(\Delta_{2n+1})).$$

For $G = SO(2n+1)$, by (1.6), $\beta(\mu_1'), \cdots, \beta(\mu_n') \in K^{-1}(SO(2n+1))$. According to [1], there exist other two elements $\varepsilon_{2n+1} \in K^{-1}(SO(2n+1))$ and $\mathcal{E}_{2n+1} \in K^0(SO(2n+1))$ such that

(3.4) $$p_{2n+1}^*(\varepsilon_{2n+1}) = 2\beta(\Delta_{2n+1})$$

and

$$K^*(SO(2n+1)) = \Lambda_Z(\beta(\mu_1'), \cdots, \beta(\mu_{n-1}'), \varepsilon_{2n+1}) \otimes$$

$$(\mathbf{Z}\{1\} \oplus \mathbf{Z}/(2^n)\{\xi_{2n+1}\})/(\varepsilon_{2n+1} \otimes \xi_{2n+1}).$$

Thus

(3.5) $\quad K^*(SO(2n+1))/Tor = \Lambda_{\mathbf{Z}}(\beta(\mu_1'), \cdots, \beta(\mu_{n-1}'), \varepsilon_{2n+1}).$

Similarly, for $G = Spin(2n)$, it follows from (1.18) that

(3.6) $\quad K^*(Spin(2n)) = \Lambda_{\mathbf{Z}}(\beta(\mu_1), \cdots, \beta(\mu_{n-2}), \beta(\Delta_{2n}^+), \beta(\Delta_{2n}^-)).$

For $G = SO(2n)$, by (1.15), $\beta(\mu_1), \cdots, \beta(\mu_{n-1}), \beta(\mu_n^+), \beta(\mu_n^-) \in K^{-1}(SO(2n))$. According to [1], there exist other three elements $\delta_{2n}, \varepsilon_{2n} \in K^{-1}(SO(2n))$ and $\xi_{2n} \in K^0(SO(2n))$ such that

(3.7) $\quad p_{2n}{}^*(\delta_{2n}) = \beta(\Delta_{2n}^+) - \beta(\Delta_{2n}^-), \; p_{2n}{}^*(\varepsilon_{2n}) = 2\beta(\Delta_{2n}^+)$

and

$$K^*(SO(2n)) = \Lambda_{\mathbf{Z}}(\beta(\mu_1), \cdots, \beta(\mu_{n-2}), \delta_{2n}, \varepsilon_{2n}) \otimes (\mathbf{Z}\{1\} \oplus \mathbf{Z}/(2^{n-1})\{\xi_{2n}\})/(\varepsilon_{2n} \otimes \xi_{2n}).$$

Thus

(3.8) $\quad K^*(SO(2n))/Tor = \Lambda_{\mathbf{Z}}(\beta(\mu_1), \cdots, \beta(\mu_{n-2}), \delta_{2n}, \varepsilon_{2n}).$

Using these results, we can deduce the following from the results of §1.

Proposition 3.1. (1) In $K^*(Spin(2n+1))$,

$$\beta(\mu_n') = 2^{n+1}\beta(\Delta_{2n+1}) - \sum_{k=1}^{n-1} \beta(\mu_k').$$

(2) In $K^*(Spin(2n))$,

$$\beta(\mu_{n-1}) = 2^{n-1}\beta(\Delta_{2n}^+) + 2^{n-1}\beta(\Delta_{2n}^-) - \sum_{\ell=1}^{[(n-2)/2]} \beta(\mu_{n-1-2\ell}).$$

Proposition 3.2. (1) In $K^*(SO(2n+1))/Tor$,

$$\beta(\mu_n') = 2^n\varepsilon_{2n+1} - \sum_{k=1}^{n-1} \beta(\mu_k').$$

(2) In $K^*(SO(2n))/Tor$,

$$\beta(\mu_{n-1}) = -2^{n-1}\delta_{2n} + 2^{n-1}\varepsilon_{2n} - \sum_{\ell=1}^{[(n-2)/2]} \beta(\mu_{n-1-2\ell}),$$

$$\beta(\mu_n^+) = 2^{n-1}\varepsilon_{2n} - \sum_{k=1}^{[(n-1)/2]} \beta(\mu_{n-2k}),$$

$$\beta(\mu_n^-) = -2^n\delta_{2n} + 2^{n-1}\varepsilon_{2n} - \sum_{k=1}^{[(n-1)/2]} \beta(\mu_{n-2k}).$$

Proposition 3.3. (1) $\tilde{j}_{2n}{}^*: K^*(Spin(2n+1)) \longrightarrow K^*(Spin(2n))$ satisfies

$$\tilde{j}_{2n}{}^*(\beta(\mu_1')) = \beta(\mu_1),$$
$$\tilde{j}_{2n}{}^*(\beta(\Delta_{2n+1})) = \beta(\Delta_{2n}^+) + \beta(\Delta_{2n}^-).$$

(2) $\tilde{j}_{2n-1}{}^*: K^*(Spin(2n)) \longrightarrow K^*(Spin(2n-1))$ satisfies

$$\tilde{j}_{2n-1}{}^*(\beta(\mu_1)) = \beta(\mu_1'),$$
$$\tilde{j}_{2n-1}{}^*(\beta(\Delta_{2n}^+)) = \tilde{j}_{2n-1}{}^*(\beta(\Delta_{2n}^-)) = \beta(\Delta_{2n-1}).$$

Proposition 3.4. (1) $j_{2n}{}^*: K^*(SO(2n+1))/Tor \longrightarrow K^*(SO(2n))/Tor$ satisfies

$$j_{2n}{}^*(\beta(\mu_1')) = \beta(\mu_1),$$
$$j_{2n}{}^*(\varepsilon_{2n+1}) = -2\delta_{2n} + 2\varepsilon_{2n}.$$

(2) $j_{2n-1}{}^*: K^*(SO(2n))/Tor \longrightarrow K^*(SO(2n-1))/Tor$ satisfies

$$j_{2n-1}{}^*(\beta(\mu_1)) = \beta(\mu_1'),$$
$$j_{2n-1}{}^*(\delta_{2n}) = 0, \quad j_{2n-1}{}^*(\varepsilon_{2n}) = \varepsilon_{2n-1}.$$

Proposition 3.5. $j_n'{}^*: K^*(SO(2n))/Tor \longrightarrow K^*(U(n))$ satisfies

$$j_n'{}^*(\beta(\mu_1)) = \beta(\lambda_1) + \beta(\lambda_{n-1}) - n\beta(\lambda_n),$$

$$j_n'{}^*(\delta_{2n}) = \sum_{k=1}^{n} (-1)^{n-k}\beta(\lambda_k),$$

$$j_n'{}^*(\varepsilon_{2n}) = \sum_{k=1}^{n-1} (1+(-1)^{n-k})\beta(\lambda_k) - 2(2^{n-2}-1)\beta(\lambda_n).$$

§4. The Chern character homomorphisms

In this section we prove our main results.

Let $\Phi: \mathbf{N} \times \mathbf{N} \times \mathbf{N} \to \mathbf{Z}$ be a function defined by

$$(4.1)\qquad \Phi(n,k,q) = \sum_{j=1}^{k} (-1)^{j-1}\binom{n}{k-j} j^{q-1} \quad \text{for } n,\ k,\ q \in \mathbf{N}.$$

Then, following Method I of [5, pp. 464-466] and using Lemma 1 of [5], we have

Proposition 4.1. In the notations of (3.2) and (2.1), ch: $K^*(U(n)) \longrightarrow H^{**}(U(n); \mathbf{Q})$ is given by

$$ch(\beta(\lambda_k)) = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{(i-1)!}\Phi(n,k,i)x_{2i-1} \qquad (k \geq 1).$$

Let $\wp = \{2^i \mid i = 0, 1, 2, \cdots\}$. For each $n \in \mathbf{N}$ there is a unique integer $s(n)$ such that $2^{s(n)-1} < n \leq 2^{s(n)}$. Let

$$q(n,i) = \begin{cases} 2 & \text{if } n \notin \wp,\ i = 2^{s(n)-1} \\ & \quad\text{or } n \in \wp,\ i = 2^{s(n)} \\ 1 & \text{otherwise} \end{cases}$$

Then Theorem 1 of [6] is

Theorem 4.2. In the notations of (3.3) and (2.3), ch: $K^*($ $Spin(2n+1)) \longrightarrow H^{**}(Spin(2n+1); \mathbf{Q})$ is given by

$$ch(\beta(\tilde{\mu}'_k)) = \sum_{i=1}^{n} \frac{(-1)^{i-1}2^{q(n,i)}}{(2i-1)!}\Phi(2n+1,k,2i)\tilde{x}_{4i-1},$$

$$ch(\beta(\Delta_{2n+1})) = \sum_{i=1}^{n} \frac{(-1)^{i-1}2^{q(n,i)}}{(2i-1)!}\left(\frac{1}{2^{n+1}}\sum_{k=1}^{n} \Phi(2n+1,k,2i)\right)\tilde{x}_{4i-1}.$$

While, Theorem 5·3 of [7] is

Theorem 4.3. In the notations of (3.5) and (2.2), ch: $K^*(SO(2n+1))/Tor \longrightarrow H^{**}(SO(2n+1); \mathbf{Q})$ is given by

$$ch(\beta(\mu'_k)) = \sum_{i=1}^{n} \frac{(-1)^{i-1}2}{(2i-1)!}\Phi(2n+1,k,2i)x_{4i-1},$$

$$ch(\varepsilon_{2n+1}) = \sum_{i=1}^{n} \frac{(-1)^{i-1}2}{(2i-1)!}\left(\frac{1}{2^n}\sum_{k=1}^{n} \Phi(2n+1,k,2i)\right)x_{4i-1}.$$

Corollary 4.4. (1) $p_{2n+1}{}^*: H^*(SO(2n+1); \mathbf{Z}) \longrightarrow H^*(Spin(2n+1); \mathbf{Z})$ satisfies

$$p_{2n+1}{}^*(x_{4i-1}) = 2^{q(n,i)-1}\tilde{x}_{4i-1} \quad (i = 1, \cdots, n).$$

(2) $p_{2n}{}^*: H^*(SO(2n); \mathbf{Z}) \longrightarrow H^*(Spin(2n); \mathbf{Z})$ satisfies

$$p_{2n}{}^*(x_{4i-1}) = 2^{q(n-1,i)-1}\tilde{x}_{4i-1} \quad (i = 1, \cdots, n-1),$$

$$p_{2n}{}^*(x'_{2n-1}) = \tilde{x}'_{2n-1}.$$

Theorem 4.5. In the notations of (3.8) and (2.5), ch: $K^*(SO(2n))/Tor \longrightarrow H^{**}(SO(2n); \mathbf{Q})$ is given by

$$ch(\beta(\mu_k)) = \sum_{i=1}^{n-1} \frac{(-1)^{i-1}2}{(2i-1)!}\Phi(2n,k,2i)x_{4i-1} + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!}\Phi(2n,k,n)x'_{2n-1},$$

$$ch(\delta_{2n}) = x'_{2n-1},$$

$$ch(\varepsilon_{2n}) = \sum_{i=1}^{n-1} \frac{(-1)^{i-1}2}{(2i-1)!}\left(\frac{1}{2^{n-1}}\sum_{\ell=0}^{[(n-2)/2]} \Phi(2n,n-1-2\ell,2i)\right)x_{4i-1} + \left(1 + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!}\left(\frac{1}{2^{n-1}}\sum_{\ell=0}^{[(n-2)/2]} \Phi(2n,n-1-2\ell,n)\right)\right)x'_{2n-1}.$$

Proof. Since $\beta(\mu_1)$ is primitive in the Hopf algebra $K^*(SO(2n))/Tor$ (see [1]), by (2.4) we may set

(4.2) $$ch(\beta(\mu_1)) = \sum_{i=1}^{n-1} a_i x_{4i-1} + a'x'_{2n-1}$$

for some a_i, $a' \in \mathbf{Q}$. Let us compute these coefficients. Apply $j_{2n-1}{}^*$ to (4.2). Then the left hand side is

$$\begin{aligned} j_{2n-1}{}^*\mathrm{ch}(\beta(\mu_1)) &= \mathrm{ch}(j_{2n-1}{}^*(\beta(\mu_1))) \\ &= \mathrm{ch}(\beta(\mu_1')) && \text{by Proposition 3.4(2)} \\ &= \sum_{i=1}^{n-1} \frac{(-1)^{i-1}2}{(2i-1)!} x_{4j-1} && \text{by Theorem 4.3,} \end{aligned}$$

and the right hand side is

$$j_{2n-1}{}^*\left(\sum_{i=1}^{n-1} a_i x_{4i-1} + a' x_{2n-1}'\right) = \sum_{i=1}^{n-1} a_i x_{4i-1}$$

by Proposition 2.1. Hence $a_i = (-1)^{i-1}2/(2i-1)!$ for $i = 1, \cdots, n-1$. Apply $j_n'^*$ to (4.2). Then the left hand side is

$$\begin{aligned} j_n'^*\mathrm{ch}(\beta(\mu_1)) &= \mathrm{ch}(j_n'^*(\beta(\mu_1))) \\ &= \mathrm{ch}(\beta(\lambda_1) + \beta(\lambda_{n-1}) - n\beta(\lambda_n)) \quad \text{by Proposition 3.5} \\ &= \sum_{i=1}^{n} \frac{(-1)^{i-1}}{(i-1)!}(1 + \Phi(n,n-1,i) - n\Phi(n,n,i))x_{2i-1} \end{aligned}$$

by Proposition 4.1. Since $\Phi(n,k,1) = \binom{n-1}{k-1}$, we have

$$1 + \Phi(n,n-1,1) - n\Phi(n,n,1) = 1 + (n-1) - n\cdot 1 = 0$$

If $2 \leqq i \leqq n$, then $\Phi(n,n-k,i) = (-1)^i\Phi(n,k,i)$ and $\Phi(n,k,i) = 0$ for $k \geqq n$ by (4·14) and (4·15) of [7]. Therefore

$$\begin{aligned} 1 + \Phi(n,n-1,i) - n\Phi(n,n,i) &= 1 + (-1)^i\Phi(n,1,i) - 0 \\ &= 1 + (-1)^i. \end{aligned}$$

Consequently, for $i = 1, \cdots, n$,

$$1 + \Phi(n,n-1,i) - n\Phi(n,n,i) = 1 + (-1)^i.$$

Thus

$$j_n'^*\mathrm{ch}(\beta(\mu_1)) = \cdots + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!}x_{2n-1}.$$

On the other hand, the right hand side is

$$j_n'^*\left(\sum_{i=1}^{n-1} a_i x_{4i-1} + a' x_{2n-1}'\right) = \cdots + a' x_{2n-1}$$

by Proposition 2.3. Hence $a' = (-1)^{n-1}(1+(-1)^n)/(n-1)!$. This proves the first equality for $k = 1$, and that for $k > 1$ follows from it and Lemma 1 of [5].

Next we set

$$\text{(4.3)} \qquad ch(\delta_{2n}) = \sum_{i=1}^{n-1} a_i x_{4i-1} + a'x'_{2n-1}$$

for some a_i, $a' \in \mathbf{Q}$. Let us compute these coefficients. Apply $j_{2n-1}{}^*$ to (4.3). Then the left hand side is

$$j_{2n-1}{}^* ch(\delta_{2n}) = ch j_{2n-1}{}^*(\delta_{2n}) = ch(0) = 0$$

by Proposition 3.4(2), and the right hand side is

$$j_{2n-1}{}^*\left(\sum_{i=1}^{n-1} a_i x_{4i-1} + a'x'_{2n-1}\right) = \sum_{i=1}^{n-1} a_i x_{4i-1}$$

by Proposition 2.1. Hence $a_i = 0$ for $i = 1, \cdots, n-1$. Apply $j_n'^*$ to (4.3). Then the left hand side is

$$j_n'^* ch(\delta_{2n}) = ch(j_n'^*(\delta_{2n}))$$

$$= ch\left(\sum_{k=1}^{n} (-1)^{n-k}\beta(\lambda_k)\right) \qquad \text{by Proposition 3.5}$$

$$= \sum_{k=1}^{n} (-1)^{n-k} ch(\beta(\lambda_k))$$

$$= \sum_{k=1}^{n} (-1)^{n-k}\left(\sum_{i=1}^{n} \frac{(-1)^{i-1}}{(i-1)!}\Phi(n,k,i)x_{2i-1}\right) \qquad \text{by Proposition 4.1}$$

$$= \sum_{i=1}^{n} \frac{(-1)^{i-1}}{(i-1)!}\left(\sum_{k=1}^{n} (-1)^{n-k}\Phi(n,k,i)\right)x_{2i-1}.$$

But

$$\sum_{k=1}^{n} (-1)^{n-k}\Phi(n,k,i) = \sum_{k=1}^{n} (-1)^{n-k}\left(\sum_{j=1}^{k} (-1)^{j-1}\binom{n}{k-j} j^{i-1}\right) \qquad \text{by (4.1)}$$

$$= \sum_{j=1}^{n} \sum_{k=j}^{n} (-1)^{n-k+j-1}\binom{n}{k-j} j^{i-1}$$

$$= \sum_{j=1}^{n} \sum_{\ell=0}^{n-j} (-1)^{n-\ell-1}\binom{n}{\ell} j^{i-1}$$

$$= (-1)^{n-1}\sum_{j=1}^{n}\left(\sum_{\ell=0}^{n-j} (-1)^{\ell}\binom{n}{\ell}\right) j^{i-1}$$

$$= (-1)^{n-1} \sum_{j=1}^{n} (-1)^{n-j} \binom{n-1}{n-j} j^{i-1}$$

$$= \sum_{j=1}^{n} (-1)^{j-1} \binom{n-1}{n-j} j^{i-1}$$

$$= \Phi(n-1,n,i)$$

$$= \begin{cases} 0 & \text{if } i = 1, \cdots, n-1 \\ (-1)^{n-1}(n-1)! & \text{if } i = n \end{cases} \quad \text{by [7].}$$

Thus

$$j_n'^{*}ch(\delta_{2n}) = \frac{(-1)^{n-1}}{(n-1)!}(-1)^{n-1}(n-1)!x_{2n-1} = x_{2n-1}.$$

On the other hand, the right hand side is

$$j_n'^{*}\left(\sum_{i=1}^{n-1} a_i x_{4i-1} + a'x_{2n-1}'\right) = a'x_{2n-1}$$

by Proposition 2.3. Hence $a' = 1$. This proves the second equality.

The third equality is obtained from the first and second equalities by using the relation

$$2^{n-1}\varepsilon_{2n} = 2^{n-1}\delta_{2n} + \sum_{\ell=0}^{[(n-2)/2]} \beta(\mu_{n-1-2\ell})$$

of Proposition 3.2(2).

Corollary 4.6. $j_n'^{*}: H^{*}(SO(2n); Z) \longrightarrow H^{*}(U(n); Z)$ satisfies

$$j_n'^{*}(x_{4i-1}) = \begin{cases} (-1)^i x_{4i-1} & \text{if } i = 1, \cdots, [(n-1)/2] \\ 0 & \text{if } i = [(n-1)/2]+1, \cdots, n-1. \end{cases}$$

Corollary 4.7. $j_{2n}^{*}: H^{*}(SO(2n+1); Z) \longrightarrow H^{*}(SO(2n); Z)$ satisfies:

(i) if $n = 2m + 1$,

$$\begin{cases} x_{4i-1} \quad (i = 1, \cdots, 2m) \end{cases}$$

$$j_{4m+2}^{*}(x_{4i-1}) = \begin{cases} \\ 0 & (i = 2m+1) \end{cases} ;$$

(ii) if $n = 2m$,

$$j_{4m}^{*}(x_{4i-1}) = \begin{cases} x_{4i-1} & (i = 1, \cdots, 2m-1 \text{ and } i \neq m) \\ x_{4m-1} + (-1)^m x'_{4m-1} & (i = m) \\ 0 & (i = 2m) \end{cases}$$

Corollary 4.8. In the notations of (3.6) and (2.6), ch: $K^*($ Spin(2n)) $\longrightarrow H^{**}$(Spin(2n); **Q**) is given by

$$ch(\beta(\mu_k)) = \sum_{i=1}^{n-1} \frac{(-1)^{i-1}2^{q(n-1,i)}}{(2i-1)!}\Phi(2n,k,2i)\tilde{x}_{4i-1} + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!}\Phi(2n,k,n)\tilde{x}'_{2n-1},$$

$$ch(\beta(\Delta^+_{2n})) =$$
$$\sum_{i=1}^{n-1} \frac{(-1)^{i-1}2^{q(n-1,i)}}{(2i-1)!}\left(\frac{1}{2^n}\sum_{\ell=0}^{[(n-2)/2]} \Phi(2n,n-1-2\ell,2i)\right)\tilde{x}_{4i-1} +$$
$$\left(\frac{1}{2} + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!}\left(\frac{1}{2^n}\sum_{\ell=0}^{[(n-2)/2]} \Phi(2n,n-1-2\ell,n)\right)\right)\tilde{x}'_{2n-1},$$

$$ch(\beta(\Delta^-_{2n})) =$$
$$\sum_{i=1}^{n-1} \frac{(-1)^{i-1}2^{q(n-1,i)}}{(2i-1)!}\left(\frac{1}{2^n}\sum_{\ell=0}^{[(n-2)/2]} \Phi(2n,n-1-2\ell,2i)\right)\tilde{x}_{4i-1} +$$
$$\left(\frac{-1}{2} + \frac{(-1)^{n-1}(1+(-1)^n)}{(n-1)!}\left(\frac{1}{2^n}\sum_{\ell=0}^{[(n-2)/2]} \Phi(2n,n-1-2\ell,n)\right)\right)\tilde{x}'_{2n-1}.$$

Corollary 4.9. $\tilde{j}_{2n}^{*}$: H^*(Spin(2n+1); **Z**) $\longrightarrow$ H^*(Spin(2n); **Z**) satisfies:

(i) if $n = 2m + 1$,

$$\tilde{j}_{4m+2}^{*}(\tilde{x}_{4i-1}) = \begin{cases} \tilde{x}_{4i-1} & (i = 1, \cdots, 2m) \\ 0 & (i = 2m+1) \end{cases} ;$$

(ii) if $n = 2m$,

$$\tilde{j}_{4m}{}^{*}(\tilde{x}_{4i-1}) = \begin{cases} \tilde{x}_{4i-1} & (i = 1, \cdots, 2m-1 \text{ and } i \neq m) \\ \tilde{x}_{4m-1} + (-1)^m \tilde{x}'_{4m-1} & (i = m \text{ and } m \notin \wp) \\ 2\tilde{x}_{4m-1} + (-1)^m \tilde{x}'_{4m-1} & (i = m \text{ and } m \in \wp) \\ 0 & (i = 2m) \end{cases}$$

Remark. Let $\Phi_t\colon \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{Z}$ be a function defined by

$$\Phi_t(n,q) = \sum_{k=1}^{n} \Phi(n,k,q) \quad \text{for } n,\ q \in \mathbf{N}.$$

Then $\Phi_t(n,1) = 2^{n-1}$, $\Phi_t(n,2i) = 2\Phi_t(n-1,2i)$ and $\Phi_t(n,2i+1) = 0$ for $i = 1, \cdots, [(n-1)/2]$. With this notation we find that

$$2\sum_{k=1}^{n} \Phi(2n+1,k,2i) = \Phi_t(2n+1,2i) \quad \text{for } i = 1, \cdots, n$$

and

$$4\sum_{\ell=0}^{[(n-2)/2]} \Phi(2n,n-1-2\ell,2i) = \Phi_t(2n,2i) =$$

$$2\Phi(2n,n,2i) + 4\sum_{k=1}^{[(n-1)/2]} \Phi(2n,n-2k,2i) \quad \text{for } i = 1, \cdots, n-1$$

(cf. Theorems 4.3 and 4.5).

Finally, we display a list of $\mathrm{ch}\colon K^*(G) \longrightarrow H^{**}(G;\mathbf{Q})$ for $G = SO(n)$ and $Spin(n)$ with $3 \leqq n \leqq 9$:

SO(3)

$\mathrm{ch}(\varepsilon_3) = x_3$

Spin(3)

$\mathrm{ch}(\beta(\Delta_3)) = \tilde{x}_3$

SO(4)

$\mathrm{ch}(\delta_4) = x'_3$

$\mathrm{ch}(\varepsilon_4) = x_3$

Spin(4)

$\mathrm{ch}(\beta(\Delta_4^+)) = \tilde{x}_3$

$\mathrm{ch}(\beta(\Delta_4^-)) = \tilde{x}_3 - \tilde{x}'_3$

SO(5)

$\mathrm{ch}(\beta(\mu'_1)) = 2x_3 - \frac{1}{3}x_7$

Spin(5)

$\mathrm{ch}(\beta(\mu'_1)) = 2\tilde{x}_3 - \frac{2}{3}\tilde{x}_7$

$$\mathrm{ch}(\varepsilon_5) = 2x_3 + \tfrac{1}{6}x_7$$

$$\mathrm{ch}(\beta(\Delta_5)) = \tilde{x}_3 + \tfrac{1}{6}\tilde{x}_7$$

SO(6)

$$\mathrm{ch}(\beta(\mu_1)) = 2x_3 - \tfrac{1}{3}x_7$$
$$\mathrm{ch}(\delta_6) = x_5'$$
$$\mathrm{ch}(\varepsilon_6) = 2x_3 + x_5' + \tfrac{1}{6}x_7$$

Spin(6)

$$\mathrm{ch}(\beta(\mu_1)) = 2\tilde{x}_3 + \tilde{x}_5' - \tfrac{2}{3}\tilde{x}_7$$
$$\mathrm{ch}(\beta(\Delta_6^+)) = \tilde{x}_3 + \tfrac{1}{2}\tilde{x}_5' + \tfrac{1}{6}\tilde{x}_7$$
$$\mathrm{ch}(\beta(\Delta_6^+)) = \tilde{x}_3 - \tfrac{1}{2}\tilde{x}_5' + \tfrac{1}{6}\tilde{x}_7$$

SO(7)

$$\mathrm{ch}(\beta(\mu_1')) = 2x_3 - \tfrac{1}{3}x_7 + \tfrac{1}{60}x_{11}$$
$$\mathrm{ch}(\beta(\mu_2')) = 10x_3 + \tfrac{1}{3}x_7 - \tfrac{5}{12}x_{11}$$
$$\mathrm{ch}(\varepsilon_7) = 4x_3 + \tfrac{1}{3}x_7 + \tfrac{1}{30}x_{11}$$

Spin(7)

$$\mathrm{ch}(\beta(\mu_1')) = 2\tilde{x}_3 - \tfrac{2}{3}\tilde{x}_7 + \tfrac{1}{60}\tilde{x}_{11}$$
$$\mathrm{ch}(\beta(\mu_2')) = 10\tilde{x}_3 + \tfrac{2}{3}\tilde{x}_7 - \tfrac{5}{12}\tilde{x}_{11}$$
$$\mathrm{ch}(\beta(\Delta_7)) = 2\tilde{x}_3 + \tfrac{1}{3}\tilde{x}_7 + \tfrac{1}{60}\tilde{x}_{11}$$

SO(8)

$$\mathrm{ch}(\beta(\mu_1)) = 2x_3 - \tfrac{1}{3}x_7 - \tfrac{1}{3}x_7' + \tfrac{1}{60}x_{11}$$
$$\mathrm{ch}(\beta(\mu_2)) = 12x_3 - \tfrac{2}{5}x_{11}$$
$$\mathrm{ch}(\delta_8) = x_7'$$
$$\mathrm{ch}(\varepsilon_8) = 4x_3 + \tfrac{1}{3}x_7 + \tfrac{4}{3}x_7' + \tfrac{1}{30}x_{11}$$

Spin(8)

$$\mathrm{ch}(\beta(\mu_1)) = 2\tilde{x}_3 - \tfrac{2}{3}\tilde{x}_7 - \tfrac{1}{3}\tilde{x}_7' + \tfrac{1}{60}\tilde{x}_{11}$$
$$\mathrm{ch}(\beta(\mu_2)) = 12\tilde{x}_3 - \tfrac{2}{5}\tilde{x}_{11}$$
$$\mathrm{ch}(\beta(\Delta_8^+)) = 2\tilde{x}_3 + \tfrac{1}{3}\tilde{x}_7 + \tfrac{2}{3}\tilde{x}_7' + \tfrac{1}{60}\tilde{x}_{11}$$
$$\mathrm{ch}(\beta(\Delta_8^-)) = 2\tilde{x}_3 + \tfrac{1}{3}\tilde{x}_7 - \tfrac{1}{3}\tilde{x}_7' + \tfrac{1}{60}\tilde{x}_{11}$$

SO(9)

$$\mathrm{ch}(\beta(\mu_1')) = 2x_3 - \tfrac{1}{3}x_7 + \tfrac{1}{60}x_{11} - \tfrac{1}{2520}x_{15}$$
$$\mathrm{ch}(\beta(\mu_2')) = 14x_3 - \tfrac{1}{3}x_7 - \tfrac{23}{60}x_{11} + \tfrac{119}{2520}x_{15}$$
$$\mathrm{ch}(\beta(\mu_3')) = 42x_3 + 3x_7 - \tfrac{3}{20}x_{11} - \tfrac{1071}{2520}x_{15}$$

$$\mathrm{ch}(\varepsilon_9) = 8x_3 + \frac{2}{3}x_7 + \frac{1}{15}x_{11} + \frac{17}{2520}x_{15}$$

Spin(9)

$$\mathrm{ch}(\beta(\mu_1')) = 2\tilde{x}_3 - \frac{1}{3}\tilde{x}_7 + \frac{1}{60}\tilde{x}_{11} - \frac{1}{1260}\tilde{x}_{15}$$

$$\mathrm{ch}(\beta(\mu_2')) = 14\tilde{x}_3 - \frac{1}{3}\tilde{x}_7 - \frac{23}{60}\tilde{x}_{11} + \frac{119}{1260}\tilde{x}_{15}$$

$$\mathrm{ch}(\beta(\mu_3')) = 42\tilde{x}_3 + 3\tilde{x}_7 - \frac{3}{20}\tilde{x}_{11} - \frac{1071}{1260}\tilde{x}_{15}$$

$$\mathrm{ch}(\Delta_9) = 4\tilde{x}_3 + \frac{1}{3}\tilde{x}_7 + \frac{1}{30}\tilde{x}_{11} + \frac{17}{2520}\tilde{x}_{15}.$$

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