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# *A Calculus on Some Self-Similar Sets*

by

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**Abstract**

The main object is the Laplace operator $\Delta$ on the Sierpinski gasket which is a well-known example of fractal. It will be defined as a limit of natural difference operators on the Sierpinski pre-gaskets. Also, harmonic functions, Green function and Dirichlet form are defined constructively and some ordinary relations expected from these concepts are obtained including the "Gauss-Green's formula", "maximum principle" for harmonic functions and so on. Especially, the Dirichlet problem of Poisson's equation is shown to be equivalent to an infinite system of finite difference equations. And so in the simple cases, for example $\Delta f = 0$ or $\Delta f = 1$, the solutions are explicitly calculated.

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## Introduction

The concept of fractals has been born as a new geometry of nature. And so, discussing physical phenomena in nature, diffusion, waves and so on, we should study not only the geometry of fractals but also various kinds of analysis on fractals. Especially, one of the most important objects must be a natural definition of "Laplace operator".

In this direction, Kusuoka[12] and Barlow-Perkins[4] have constructed and investigated Brownian motion on the Sierpinski gasket as a limit of simple random works on the pre-gaskets. In their viewpoint, the Laplace operator is formulated as the infinitesimal generator of Brownian motion.

On the other hand, in [10], we have found the direct and natural definition of "Laplace operator" on the Sierpinski gasket as a limit of difference operators on the pre-gaskets. The present paper reviews [10] with an explicit formulation of Green function and Dirichlet form which are not in [10].

In §1, we define the N-Sierpinski space and introduce a sequence of difference operators. We then define harmonic functions as the kernel of the difference operators and discuss the Dirichlet problem of harmonic functions and also show the "maximum principle" for harmonic functions.

In §2, we construct the theory of Laplace operator, Green function and Dirichlet form on the pre-Sierpinski spaces. And in §3, we consider these concepts on the Sierpinski space as the limits of those defined on the pre-Sierpinski spaces. We then discuss the Dirichlet problem of Poisson's equation and see that it is equivalent to an infinite system of finite difference equations. This fact has been pointed out by Hata-Yamaguti[8] and Yamaguti-Kigami[18] in the simplest case $N = 2$. We remark that if $N = 2$, then our theory becomes a reconstruction of the ordinary calculus on the interval $[0,1]$.

Now we mention some related works. In [11], we have established the theory as in the present paper on a class of self-similar sets called p.c.f. self-similar sets, which is almost the same concept as finitely ramified fractals and includes nested fractals studied by Lindstrøm[14]. An example of p.c.f. self-similar sets is seen in Figure 1.

Shima[16] and Fukushima-Shima[6] have studied the eigenvalue problem of the Laplace operator given in this paper. They apply "decimation method" and determine the eigenvalues and eigen vectors explicitly. By their results, let denote by $\rho(x)$ the number of eigen values of -(Laplace operator), taking the multiplicities into account, not exceed $x$, then if $N > 2$,

$$0 < \underline{\lim}_{x\to\infty} \rho(x)x^{-d_s/2} < \overline{\lim}_{x\to\infty} \rho(x)x^{-d_s/2} < \infty,$$

where $d_s = 2(\log N)/(\log N+2)$ is called the spectral dimension.

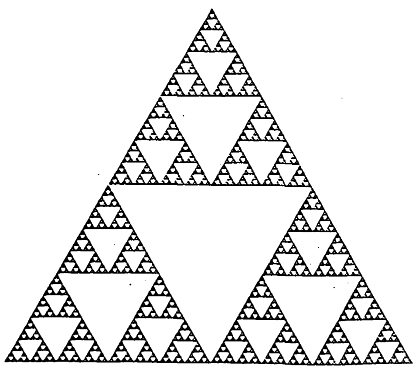
Barlow et al[1],[2],[3] have constructed and investigated Brownian motion on the Sierpinski carpet which is not finitely ramified fractal.

Lindstrøm[14] has constructed Brownian motion on nested fractals by a probabilistic method.

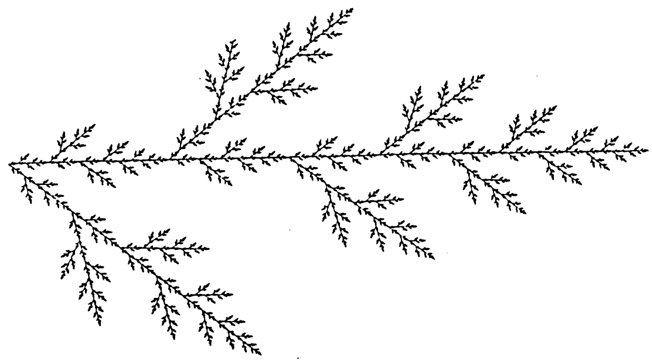
Kusuoka[13] has given an explicit expression of Dirichlet forms on a class of self-similar sets by using products of random matrices, which correspond to the matrices given in Theorem 1.7 in the present paper.

Finally we remark (1) The Laplace operator in this paper, let denote it by $\Delta$, is a little different from that in [10], let denote it by $\tilde{\Delta}$. The relation is given by $\Delta = \frac{N}{2}\tilde{\Delta}$. Of course, this difference is not essential.

(2) There are no proofs of theorems, propositions and lemmas in the present paper. Readers may refer to [10] or [11].



Sierpinski Gasket



Hata's tree-like Set(Hata[7])
an example of p.c.f. self-similar set

Figure 1

§ 1. Sierpinski Spaces and Harmonic Functions

In this section, we first define the N-Sierpinski Space $K^N$ as a self-similar set studied by Hutchinson[9] and Hata[7]. And next we introduce a sequence of difference operators $H_{m,p}$, whose limit will give the Laplace operator $\Delta$ on $K^N$.

**Definition 1.1** Let $p_1, p_2, \cdots, p_N \in \mathbb{R}^{N-1}$ satisfying $|p_i - p_j| = 1$ for each pair $(i,j)$ with $i \neq j$ and let

$$F_i(x) = \frac{1}{2}(x - p_i) - p_i$$

for $i = 1, 2, \cdots, N$. Then by the result in [9], there exists a unique compact set $K^N \subset \mathbb{R}^{N-1}$ with

$$K^N = F_1(K^N) \cup F_2(K^N) \cup\cdot\cdot\cup F_N(K^N).$$

$K^N$ is called the N-Sierpinski Space.

For example, $K^2 = [0,1]$ if $p_1 = 0$ and $p_2 = 1$, and $K^3$ is the Sierpinski Gasket found by Sierpinski[17]. See Figure 1. For ease of notation, we drop $N$ of $K^N$ hereafter.

We next give a sequence $\{V_m\}_{m\geq 0}$ of finite sets in $K$ which is a natural process of approximation of $K$. $V_m$ is called pre-Sierpinski space.

**Definition 1.2.** For $w = i_1 i_2 \cdot\cdot i_m \in \{1,2,\cdot\cdot,N\}^m$, let

$$F_w = F_{i_1} \circ F_{i_2} \circ\cdot\cdot\circ F_{i_m}.$$

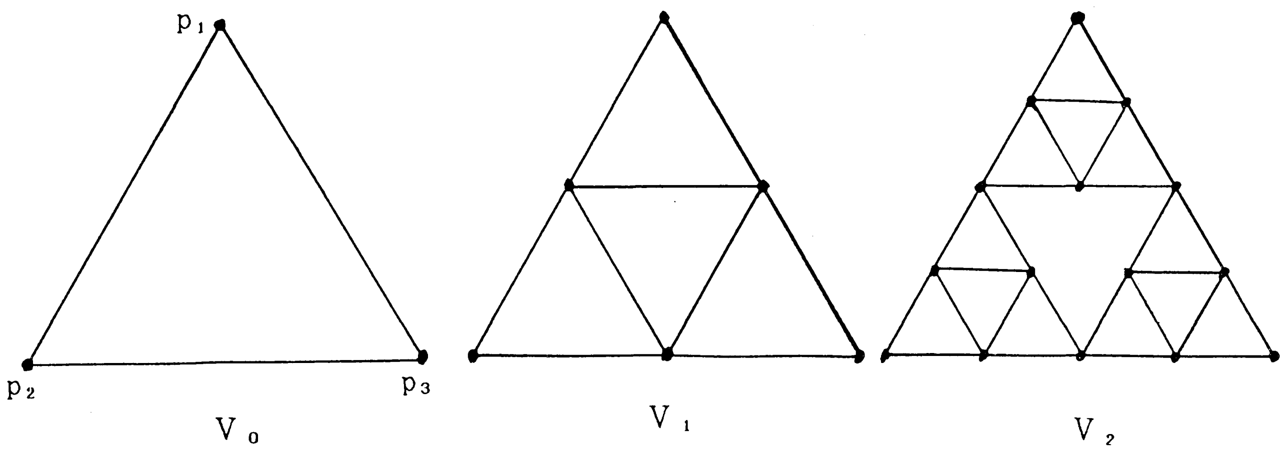
Then we define $V_m \subset K$ for $m \geq 0$ by

$$V_0 = \{p_1, p_2, \cdot\cdot, p_N\}$$

and, for $m \geq 1$, $\quad V_m = \bigcup_{w\in\{1,2,\cdot\cdot,N\}^m} F_w(V_0).$

Also we let $V_m^\circ = V_m - V_0$ and $V_* = \bigcup_{m\geq 0} V_m$. Further, for $p \in V_m$, the nearest neighbors of $p$ in $V_m$, $V_{m,p}$ is defined by

$$V_{m,p} = \{q \mid |p - q| = 2^{-m}\}.$$




the pre-Sierpinski Spaces for N=3

Figure 2

Remark. (1) $K$ is the closure of $V_*$.

(2) $$\#(V_{m,p}) = \begin{cases} 2(N-1) & \text{if } p \in V_m^\circ \\ N-1 & \text{if } p \in V_0, \end{cases}$$

where $\#(\cdot)$ denotes the number of the elements.

Notation. Let $V$ and $U$ be sets.

(1) $\ell(V) = \{f \mid f: V \to \mathbb{R}\}$. We use $(f)_p$ or $f_p$ to denote the value of $f \in \ell(V)$ at $p \in V$. For $p \in V$, $\chi_p \in \ell(V)$ is defined by

$$\chi_p(q) = \begin{cases} 1 & \text{if } q = p, \\ 0 & \text{otherwise.} \end{cases}$$

(2) Let $A: \ell(V) \to \ell(U)$ be linear, then we use $(A)_{pq}$ or $A_{pq}$ to denote $(A\chi_q)_p$ for $q \in V$ and $p \in U$. Note that

$$\sum_{q \in V} A_{pq} f_q = (Af)_p.$$

(3) $C(K) = \{f \mid f$ is a continuous function on $K.\}$

We now define the difference operators on $V_m$.

**Definition 1.3** (1) For $p \in V_m^\circ$ and $f \in \ell(V_m)$, we define

$$H_{m,p} f = \sum_{q \in V_{m,p}} f(q) - 2(N-1) f(p).$$

(2) For $p \in V_0$ and $f \in \ell(V_m)$, we define

$$D_{m,p} f = \sum_{q \in V_{m,p}} f(q) - (N-1) f(p).$$

By the definition of $V_{m,p}$, if $H_{m,p} f = 0$, then $f(p)$ is the arithmetic average of $f(q)$'s for $q \in V_{m,p}$. This fact teaches us how we can define harmonic functions on $K$.

**Definition 1.4** $f \in C(K)$ is said to be harmonic if

$$H_{m,p}f = 0$$

for all $m \geq 1$ and all $p \in V_m^\circ$.

For ease of later discussion, we rewrite the definition of $H_{m,p}$ using matrices. Let $X_m : \ell(V_m^\circ) \to \ell(V_m^\circ)$ be a linear operator defined by, for each $p$ and $q \in V_m^\circ$,

$$(X_m)_{pq} = \begin{cases} -2(N-1) & \text{if} \quad p = q, \\ 1 & \text{if} \quad q \in V_{m,p}, \\ 0 & \text{otherwise.} \end{cases}$$

And let $J_m : \ell(V_0) \to \ell(V_m^\circ)$ be a linear operator defined by

$$(J_m)_{pq} = \begin{cases} 1 & \text{if} \quad q \in V_{m,p}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $p \in V_m^\circ$ and $f \in \ell(V_m)$, we see that

$$H_{m,p}f = (J_m f|_{V_0} + X_m f|_{V_m^\circ})_p.$$

**Lemma 1.5** $X_m$ is invertible for each $m \geq 1$ and

$$(X_m^{-1})_{pp} < (X_m^{-1})_{pq} < 0$$

for all $p$ and $q \in V_m^\circ$ with $p \neq q$.

By Lemma 1.5, if $f$ is harmonic on $K$, then,

$$f|_{V_m^\circ} = -X_m^{-1} J_m f|_{V_0}.$$

Also, for each $\rho \in \ell(V_0)$ and each $m \geq 1$, we can show that

$$(-X_{m+1}^{-1} J_{m+1}\rho)|_{V_m} = -X_m^{-1} J_m \rho.$$

By using the above facts, we have

**Theorem 1.6** For given $\rho \in \ell(V_0)$, there exists a unique harmonic function $f$ with $f|_{V_0} = \rho$.

Further, for $w \in \{1,2,\cdots,N\}^m$, we can express $f|_{F_w(V_0)}$ by using a product of random matrices $A_i$'s as follows.

**Theorem 1.7** For each $i \in \{1,2,\cdots,N\}$, let $A_i$ be an $N \times N$ matrix defined by

$$(A_i)_{jk} = \begin{cases} 1 & \text{if } j = k = i, \\ 2/(N+2) & \text{if } j \neq k = i \text{ or } j = k \neq i, \\ 1/(N+2) & \text{if } k \neq i,\ j \neq i \text{ and } k \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Then , for each $w = i_1 i_2 \cdots i_m \in \{1,2,\cdots,N\}^m$,

$$\begin{pmatrix} f(F_w(p_1)) \\ \vdots \\ f(F_w(p_N)) \end{pmatrix} = A_{i_m} A_{i_{m-1}} \cdots A_{i_1} \begin{pmatrix} f(p_1) \\ \vdots \\ f(p_N) \end{pmatrix}.$$

Making use of Theorem 1.7, we can show the "maximum principle" for harmonic functions.

**Theorem 1.8** Let $f$ be harmonic on $K$. If there exists $p \in K-V_0$ and a neighborhood of $p$, $U$, such that $f(p) \geq f(q)$ for each $q \in U$, then $f$ is reduced to a constant on $K$.

In the rest of §1, we treat piecewise harmonic functions.

**Definition 1.9** $f \in C(K)$ is said to be m-harmonic if, and only if $f \circ F_w$ is harmonic for each $w \in \{1,2,\cdots,N\}^m$.

As an immediate consequence of Theorem 1.6, we have

**Theorem 1.10** For given $\rho \in \ell(V_m)$, there exists a unique m-harmonic function $f$ with $f|_{V_m} = \rho$. Further, for $p \in V_m$, let $\psi_p^m$ be the unique m-harmonic function satisfying $\psi_p^m|_{V_m} = \chi_p$. Then $f = \sum_{p \in V_m} \rho_p \psi_p^m$.

## § 2. Calculus on the pre-Sierpinski Space $V_m$

In this section, we introduce Laplace operator $\Delta_m$, Green function $g_m$ and Dirichlet form $\mathcal{E}_m$ on $V_m$.

First we define a natural measure $\mu$ on K. By the results in Moran[15] and Hutchinson[9], the Hausdorff dimension of K is $\log N/\log 2$ and $0 < h(K) < \infty$, where h is the $(\log N/\log 2)$-dimensional Hausdorff measure. Now let $\mu = h/h(K)$, then $\mu$ is a probabilistic measure on K satisfying

$$\mu(F_w(K)) = N^{-m}$$

for each $w \in \{1,2,\cdots,N\}^m$. Further, a natural probabilistic measure $\mu_m$ on $V_m$ is defined by

$$\mu_m = \sum_{p\in V_m}\left(\int_K \psi_p^m d\mu\right)\delta_p,$$

where

$$\int_K \psi_p^m d\mu = \begin{cases} 2N^{-(m+1)} & \text{if } p \in V_m^\circ, \\ N^{-(m+1)} & \text{if } p \in V_0. \end{cases}$$

**Definition 2.1** (1) A linear operator $\Delta_m : \ell(V_m) \to \ell(V_m^\circ)$ is defined by, for each $p \in V_m^\circ$,

$$(\Delta_m f)_p = \frac{N}{2}(N+2)^m H_{m,p} f.$$

(2) $g_m : V_m \times V_m \to \mathbb{R}$ is defined by

$$g_m(p,q) = \begin{cases} -\left(\frac{N}{N+2}\right)^m (X_m^{-1})_{pq} & \text{if } p \text{ and } q \in V_m^\circ, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $g_m(p,q) = g_m(q,p)$ as $X_m$ is symmetric.

(3) A linear operator $G_m : \ell(V_m^\circ) \to \ell(V_m)$ is defined by ,for $p \in V_m$,

$$(G_m f)_p = \int_K g_m(p,q) f(q) \mu_m(dq).$$

(4) A bilinear form $\mathcal{E}$ on $\ell(V_m)$ is defined by

$$\mathcal{E}_m(u,v) = \sum_{p\in V_0} (u(p)\left(-\left(\frac{N+2}{N}\right)^m D_{m,p}f\right)) - \int_{V_m^\circ} u(\Delta_m v)d\mu_m.$$

Immediately, we have

**Proposition 2.2** (1) $\Delta_m \circ G_m$ = -identity of $\ell(V_m^\circ)$.

(2) $\Delta_m f = 0$ if, and only if there exists a harmonic function $\tilde{f}$ on K with $\tilde{f}|_{V_m} = f$.

(3) $\mathcal{E}_m$ is non-negative definite and symmetric. And let $g_m^p \in \ell(V_m)$ be defined by $g_m^p(q) = g_m(p,q)$, then, for each $f \in \ell(V_m)$ with $f|_{V_0} = 0$, we have

$$\mathcal{E}_m(g_m^p, f) = f(p).$$

(4) $\mathcal{E}_m(u,u) = 0$ if, and only if $u$ is constant on $V_m$.

By these results, we can see that $\Delta_m$, $g_m$ and $\mathcal{E}_m$ are well-formulated. The following facts will be helpful in the next section, where we will consider the limits of $\Delta_m$, $g_m$ and $\mathcal{E}_m$ as $m \to \infty$.

**Lemma 2.3** (1) $g_{m+1}|_{V_m\times V_m} = g_m$,

(2) For each $u \in \ell(V_{m+1})$,

$$\mathcal{E}_m(u|_{V_m}, u|_{V_m}) \le \mathcal{E}_{m+1}(u,u).$$

And the equality holds if, and only if there exists an m-harmonic function $\tilde{u}$ on K with $\tilde{u}|_{V_{m+1}} = u$.

## § 3. Calculus on the Sierpinski Space K

In this section, we are concerned with the Laplace operator $\Delta$, Green function $g$ and Dirichlet form $\mathcal{E}$ on $K$ defined as a natural limits of those on $V_m$.

First we introduce the Green function $g$ on $K$.

**Lemma 3.1** Let $\tilde{g}_m : K \times K \to \mathbb{R}$ be defined by

$$\tilde{g}_m(x,y) = \sum_{p,q\in V_m} g_m(p,q)\psi_p^m(x)\psi_q^m(y).$$

Then $\{\tilde{g}_m\}_{m\geq 1}$ converges uniformly on $K$ as $m \to \infty$. We denote this limit by $g$.

By Lemma 3.1, $g$ is continuous on $K$. Also, obviously $\tilde{g}_m|_{V_m\times V_m} = g_m$, therefore by Lemma 2.3, $g(x,y) = g(y,x) \geq 0$ for all $x$ and $y \in K$.

Secondly, we introduce the Laplace operator $\Delta$ and construct the solution of the Dirichlet problem of Poisson's equation by using the Green function $g$.

**Definition 3.2** Let $f \in C(K)$. If there exists $\varphi \in C(K)$ satisfying that, as $m \to \infty$,

$$\max_{p\in \overset{\circ}{V}_m} |(\Delta_m(f|_{V_m}))_p - \varphi(p)| \to 0,$$

then we define $\Delta f = \varphi$. The domain of $\Delta$ is denoted by $\mathcal{D}$.

**Theorem 3.3** For given $\varphi \in C(K)$ and given $\rho \in \mathcal{L}(V_0)$, there exists a unique $f \in \mathcal{D}$ such that

$$\text{(PD)} \qquad \begin{cases} \Delta f = \varphi, \\ f|_{V_0} = \rho. \end{cases}$$

And this $f$ is given by, for each $x \in K$,

$$f(x) = \sum_{\rho \in V_0} \rho_p \psi_p^0(x) - \int_K g(x,y)\varphi(y)\mu(dy).$$

**Corollary 3.4** $f$ is harmonic if, and only if $\Delta f = 0$.

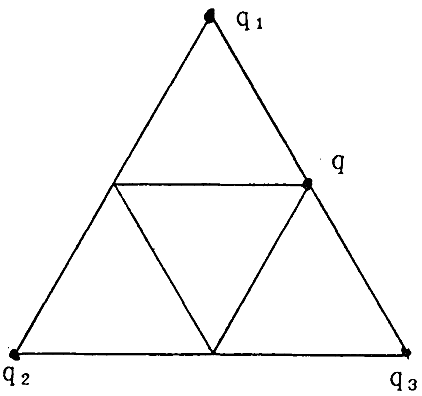
Remark. The Dirichlet problem of Poisson's equation (PD) is equivalent to the following infinite system of finite difference equations, that is,

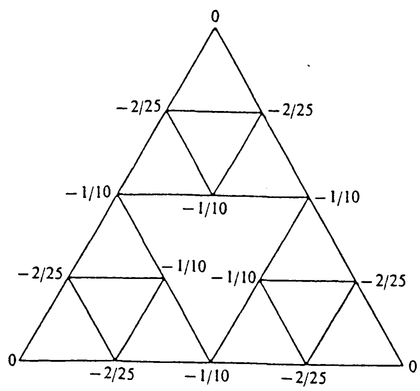
$$\begin{cases} f|_{V_0} = \rho \\ H_{m,p} f = \left(\frac{N}{N+2}\right)^m \int_K \psi_p^m \varphi d\mu \quad \text{for } m \geq 0 \text{ and } p \in \overset{\circ}{V}_m. \end{cases}$$

We can solve this equations inductively from $f|_{V_0}$ to $f|_{V_1}, f|_{V_2}, \cdots$. For example, in the case that $N = 3$, $\rho \equiv 0$ and $\varphi \equiv 3/2$, let $q_1, q_2, q_3 \in V_m$ and $q \in V_{m+1}$ be located as in Figure 3. Then,

$$f(q) = \frac{1}{5}(2f(q_1) + 2f(q_3) + f(q_2)) - 5^{-m}.$$

Using this formula, the solution on $V_2$ is given in Figure 3.




the solution of

$$\begin{cases} \Delta f = 3/2 \\ f|_{V_0} = 0 \end{cases}$$

Figure 3

Finally, we define a form $\mathcal{E}$ on $K$ as the limit of $\mathcal{E}_m$ on $V_m$. By virtue of Lemma 2.3, for $f \in \ell(V_*)$, we can define

$$\mathcal{E}(f,f) = \lim_{m\to\infty} \mathcal{E}_m(f|_{V_m}, f|_{V_m})$$

if we allow $\infty$ as the value of limit. Now let

$$\mathcal{F} = \{f \mid f \in \ell(V_*) \text{ and } \mathcal{E}(f,f) < \infty\},$$

then we see that, for each $u$ and $v \in \ell(V_*)$,

$$\mathcal{E}(u,v) = \lim_{m\to\infty} \mathcal{E}_m(u|_{V_m}, v|_{V_m})$$

is finite and well-defined.

By Proposition 2.2, $\mathcal{E}$ is non-negative symmetric form on $\mathcal{F}$. Further, $\mathcal{E}(u,u) = 0$ if, and only if $u$ is constant on $K$. Note that $C(K)$ can be thought as a subset of $\ell(V_*)$ through the restriction map $f \to f|_{V_*}$ because $V_*$ is dense in $K$.

Now we show "Gauss-Green's formula". For $f \in \mathcal{D}$, we define the "normal derivative", $(df)_p$ at $p \in V_0$.

**Lemma 3.5** For each $f \in \mathcal{D}$ and each $p \in V_0$,

$$\lim_{m\to\infty} -\left(\frac{N+2}{N}\right)^m D_{m,p}f = -D_{0,p}f + \int_K \psi_p^0 \Delta f d\mu.$$

We denote the above limit by $(df)_p$.

Recalling the definition of $\mathcal{E}_m$, we can see

**Theorem 3.6** $\mathcal{D} \subset \mathcal{F}$ and for each $u$ and $v \in \mathcal{D}$,

$$\mathcal{E}(u,v) = \sum_{p\in V_0} u(p)(dv)_p - \int_K u\Delta v d\mu.$$

**Corollary 3.7**(Gauss-Green's formula) For each $u$ and $v \in \mathcal{D}$,

(1) $$\sum_{p\in V_0} (u(p)(dv)_p - v(p)(du)_p) = \int_K (u\Delta v - v\Delta u) d\mu.$$

(2) $$\sum_{p\in V_0}(dv)_p = \int_K \Delta v d\mu .$$

At last we collect the results on Dirichlet form. See Fukushima[5] for the definition and the applications of Dirichlet forms.

**Theorem 3.8** (1) $\mathcal{D} \subset \mathcal{F} \subset C(K)$.

(2) $(\mathcal{F},\mathcal{E})$ is a regular local Dirichlet space on $L^2(K,\mu)$.

(3) For each $x \in K$, let $g^x$ be defined by $g^x(y) = g(x,y)$. Then $g^x \in \mathcal{F}$ and for each $u \in \mathcal{F}$,

$$\mathcal{E}(g^x,u) = u(x) - \sum_{p\in V_0} u(p)\psi_p^0(x).$$

(4) $(\mathcal{F},\mathcal{E})$ is the minimal closed extension of $(\mathcal{D},\mathcal{E})$.

(5) $\mathcal{D}$ is dense in $C(K)$.

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