数理解析研究所講究録
第 785 巻 1992 年 134-138

京都大学数理解析研究所
Kyōto University, Research Institute for Mathematical Sciences
発展方程式と非線形問題
Evolution Equations and Nonlinear Problems
1991年10月23-25日, October 23–25, 1991

Global Sinks for Planar Vector Fields

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One of the main goals in Dynamics is to find the asymptotic behaviour of the motions. Here we will be concerned with the autonomous differential equation

$$\dot{x} = f(x) \tag{1}$$

for a C^1 vector field $f\colon \mathbb{R}^n \to \mathbb{R}^n$, and the problem of determining or estimating the *basin of attraction* of its asymptotically stable equilibria, that is, the set of initial data for which the trajectory converges to the equilibrium as $t \to +\infty$. It is well-known from Liapunov Stability Theory that an equilibrium point attracts a whole neighbourhood whenever the eigenvalues of the Jacobian matrix $f'(x)$ of f have strictly negative real parts in that point. In this case one says that the equilibrium point is a *"sink"*. A sink for equation (1) which attracts the whole space $\mathbb{R}^n$ can be called a *"global sink"*.

* Visiting from Udine University, Italy, supported by the *"European Communities Scientific Training Programme in Japan"*.
** Supported by the CNR (Consiglio Nazionale delle Ricerche).

Beyond the elementary local result, the planar case $n = 2$ is still field of research. In other words, the dimension 2 is already difficult for the current state of the art. In particular, it is an open problem *whether strictly negative real parts of the eigenvalues of $f'(x)$ for all $x \in \mathbb{R}^2$* (that we call here *global Jacobian condition*) *imply that the equilibrium point is a global sink.* The conjecture goes back to Krasovskij (1959) and Markus and Yamabe (1960), and has been answered positively only with additional assumptions in those papers and in some later works.

The paper Olech (1963), which is perhaps the most important contribution to the problem, proved in particular that if a given f satisfies the global Jacobian condition and is one-to-one, then its equilibrium point (there cannot be more than one, of course) is globally attractive. Conversely, if f were not one-to-one, i.e., $f(x_1) = f(x_2)$ for some $x_1 \neq x_2$, it is clear that the vector field $x \mapsto f(x) - f(x_1)$ would have the same Jacobian matrix as f, but two equilibrium points, neither of which attracts the other. The global attractivity planar conjecture is then equivalent to the following *global injectivity* conjecture: does the global Jacobian condition imply that f is one-to-one?

Meisters and Olech (1988) proved the conjecture in the polynomial case using Olech (1963).

Among the other relevant papers on this and on related topics such as the Jacobian conjecture of Algebraic Geometry for polynomial mappings, we refer the reader, with no claim to a complete list, to Hartman (1961), Olech (1964), Meisters (1982), Gasull et al. (1991), Gasull & Sotomayor (1990), Gutierrez (1992), Druzkowski (1991).

Proving global asymptotic stability for arbitrary dimension $n \geq 2$ is of great importance for the applications. Some results of this kind are proved in Hartman & Olech (1962), which generalizes the results of Olech (1963) and also gives other results related to Borg (1960). Hartman's book (1982) devotes part of the last chapter to global asymptotic stability. In the sequel we consider the planar case only.

In their (1992) paper, the authors introduce the 2×2 matrix function

$$g(x) := I + \frac{f'(x)^T f'(x)}{\det f'(x)}, \tag{2}$$

(I is the identity 2×2 matrix, the symbol ^{T} means transposition and det means determinant) and contribute a new approach to the global injectivity side of the conjecture, based on the remarkable properties that g enjoys when f is a planar vector field satisfying the global Jacobian condition. The main result can be described as follows: if a function f satisfies the global Jacobian condition and if the norm of the associated matrix $g(x)$ is bounded or, at least, grows slowly (for instance, linearly) as $|x| \to +\infty$, then f is one-to-one and, by the theorem in Olech (1963), its equilibrium point (if it exists) is globally attractive. The exact statement is:

Theorem. *Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be a C^2 map with $f(0) = 0$ and such that the real parts of the eigenvalues of $f'(x)$ are strictly negative at every $x \in \mathbb{R}^2$. Moreover, let us assume that there exist a point $\bar{x} \in \mathbb{R}^2$ and a function $K: [0, +\infty[\to [0, +\infty[$ such that, for the g given by formula (2),*

(i) $\|g(x)\| \leq K(|x - \bar{x}|)$;

(ii) K *is weakly increasing;*

(iii) $\displaystyle\int_0^{+\infty} \frac{1}{K(r)}\, dr = +\infty.$

Then f is one-to-one, and $x = 0$ is globally asymptotically stable for $\dot{x} = f(x)$.

The theorem is obtained by introducing an auxiliary boundary value problem, and by using a simple topological degree argument. We hope that this strategy can lead to more general results.

No new positive result is presented in this note, but we conclude with an example that addresses the side issue of surjectivity (or, rather, *non-surjectivity*), of f. This is partly because the examples accompanying the theorem in the original paper happen to be all onto $\mathbb{R}^2$ and because the statement itself does vaguely remind of Hadamard's celebrated theorem, according to which $f: \mathbb{R}^n \to \mathbb{R}^n$ is bijective whenever $f'(x)$ is always nonsingular and $\|f'(x)^{-1}\| \leq K(|x|)$ for a function K as above. To dispel the doubt that surjectivity may be implied by our theorem, we show a function $\mathbb{R}^2 \to \mathbb{R}^2$ which satisfies the hypotheses but is not onto.

Example. Define the function $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x) := -\varphi(r)x\,, \qquad \text{where } x = \begin{pmatrix}\xi\\ \eta\end{pmatrix} \in \mathbb{R}^2\,, \quad r = \sqrt{\xi^2+\eta^2}\,,$$

$$\varphi(r) := \begin{cases} r^{-1}\arctan r & \text{for } r \neq 0, \\ 1 & \text{for } r = 0. \end{cases}$$

It is easily seen that φ and f are analytic and that f is one-to-one. It is not onto $\mathbb{R}^2$ because $|f(x)| < \pi/2$. The Jacobian matrix $f'(x)$ is

$$f'(x) = \begin{pmatrix} -\varphi'(r)\dfrac{\xi^2}{r} - \varphi(r) & -\varphi'(r)\dfrac{\xi\eta}{r} \\ -\varphi'(r)\dfrac{\xi\eta}{r} & -\varphi'(r)\dfrac{\eta^2}{r} - \varphi(r) \end{pmatrix},$$

whose trace and determinant are

$$\operatorname{tr} f'(x) = -2\varphi(r) - r\varphi'(r) = -\varphi(r) - \frac{1}{1+r^2} < 0\,,$$

$$\det f'(x) = \varphi(r)(r\varphi'(r) + \varphi(r)) = \frac{\varphi(r)}{1+r^2} > 0\,.$$

If R is an orthogonal 2×2 matrix, we have $f(Rx) = Rf(x)$, whence

$$f'(x) = R^T f'(Rx)R\,, \quad g(x) = R^T g(Rx)R\,,$$

where $g(x)$ is the matrix in formula (2). Thus the eigenvalues and the operator norm of $g(x)$ are unaffected by a rotation of x. Then we can limit ourselves to compute them for $x = (\xi, 0)$, $\xi \geq 0$, where g is diagonal:

$$g(\xi,0) = \begin{pmatrix} 1 + \dfrac{r}{(1+r^2)\arctan r} & 0 \\ 0 & 1 + \dfrac{1+r^2}{r}\arctan r \end{pmatrix}.$$

The function $r \mapsto r^{-1}(1+r^2)\arctan r$ is nondecreasing on $[0,+\infty[$ and it is ≥ 1, so that the norm of g, which coincides with its greater eigenvalue, is:

$$\|g(x)\| = 1 + \frac{1+r^2}{r}\arctan r\,,$$

and it is obvious that, if $\bar{x} = 0$, the best possible choice for the function K of the theorem above is

$$K(r) = 1 + \frac{1+r^2}{r}\arctan r\,,$$

for which the integral condition (iii) is verified:

$$\int_0^{+\infty} \frac{1}{K(r)}\,dr = \int_0^{+\infty} \frac{r}{r + (1+r^2)\arctan r}\,dr = +\infty\,.$$

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